Some Infinite Series of Weighing Matrices From Hadamard Matrices

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Absract: *In this paper we have constructed four infinite series of weighing matrices from Hadamard matrices with the help of compound matrices.*
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I. INTRODUCTION

A matrix W of order n is called weighing matrix with entries 1, -1, 0 if $WW^T = k\, I_n$, where k is an integer > 0 defined as weight of the weighing matrix and the weighing matrix is denoted by *W(k, n)*.Weighing matrices have been studied because of their use in weighing experiments as first studied by H.Hotelling [10] and letter by Raghavarao [15]. In 1995 Gysin and Seberry [6] constructed *W(4n,4n-2)* and *W(4n, 2n-1)* using conference matrices and cyclotomy, and in 1996 Gysin and Seberry [7] constructed weighing matrices by linear combination of generalized cosets, In 2006 Arasu, et al [1] constructed circulant weighing matrices of weights 2^{2t}. Recently weighing matrices has been found to be applied at various fields especially on network and digital communication. Singh, et. al. [18] constructed weighing matrices of order 4n and weight 2n that has application to network security, information technology and electronics and telecommunication engineering. Resent advances in optical quantum computing created an interest in Hankel block Weighing matrices and related block weighing matrices ([3],[4],[19]). For more on construction and applications of weighing matrices vide Koukouvinos ,et. al. [13], Anthony, et. al. [5],Singh, et.al.[17]. In this paper we constructed four infinite series of weighing matrices from Hadamard matrix.

We begin with definitions of some terms.

1.1Hadamard Matrix [8]
Let H be an $n \times n$ matrix with entries +1,-1 then H is called a Hadamard matrix if $HH^T = n\, I_n$.

1.2 r- Compound Matrix **(** [14], [9])
Let A be an $n \times n$ matrix over *Z*. For non- empty subsets S and $T \subseteq \{1,2......n\}$. We denote by *A(S/T)* the sub matrix of A whose rows are indexed by S and whose columns are indexed by T in their natural order (lexicographic order). Let r be positive integer $r \leq n$. We denote by $C_r(A)$ the r^{th} compound of the matrix A, that is the $\binom{n}{r} \times \binom{n}{r}$ matrix whose element are the minors det *A(S/T)*, for all possible $S, T \subseteq \{1,2,.......n\}$ with cardinality $|S|=|T| = r$.
1.3 m x 4t Weighing Matrix **(**[2], [11])
An $m \times 4t$ matrix W with elements 0, ±1 will be called weighing matrix if $WW^T = kI_m$ where k is an integer > 0, k is called the weight of W.

II. CONSTRUCTION OF WEIGHING MATRICES FROM HADAMARD MATRIX OF ORDER $m \times 4t$

First we take a Hadamard matrix H of order $m \times 4t$, then we construct a matrix $C_r(H)$, the r^{th} compound of the matrix H that is the $\binom{m}{r} \times \binom{4t}{r}$ matrix whose element are minor det*(H(S/T))* for all possible $S,T \in \{1,2....n\}$ in natural order with cardinally $|S|=|T| = r$ We note that if we take $r = 2, 3$ then we get matrices whose elements are minors det$(H(S/T))$. Hence we get an $\binom{m}{2} \times \binom{4t}{2}$, $\binom{m}{3} \times \binom{4t}{3}$ matrix $2W$ and $4W$ respectively where W is a weighing matrix.

2.1 Theorem: The 2- compound matrix of an $\mathrm{m} \times 4\mathrm{t}$ Hadamard matrix is an integral multiple of a weighing matrix. i.e., $C_2\left(H_{m,\ 4t}\right) = 2W\left\{4t^2, \binom{m}{2} \times \binom{4t}{2}\right\}$ Where $m \leq 4t$ and $W\left\{4t^2, \binom{m}{2} \times \binom{4t}{2}\right\}$ is a weighing matrix.
Proof: We consider 2×2 determinants which are minors of the matrix M formed by the juxta position of any two rows R_i and R_j. We calculate the non zero minors of rank 2 and order 2

$$M = \begin{bmatrix} R_i \\ R_j \end{bmatrix}$$

Multiplying by -1 the column of M having 1st elt. -1, we make all the entries of R_i +1. Also 2t elements of R_j are +1 and remaining 2t are -1, By suitable permutation of columns of M can be put into the form

+1 +1……+1 +1 +1……+1
+1 +1 ……+1 -1 -1 …….-1

This process preserves the rank of M and so the number of zero minors of M which equals
$2\times \binom{2t}{2}$ $=2t(2t-1)$.
Hence the number of non zero minors of M =The total number of minors of M – number of zero minors of M

$$= \binom{4t}{2} - 2t\,(2t-1)$$
$$= 4t^2$$

The non zero minors are of types $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$ or $\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$

Hence $C_2\,(H_{m\,,4t}) = 2W$ is a matrix with entries 0 , +2,-2.

Finally we show that W is a weighing matrix in what follows W stands for $\mathrm{W}\left\{4\mathrm{t}^2, \binom{\mathrm{m}}{2} \times \binom{4\mathrm{t}}{2}\right\}$.

It is sufficient to show that rows of W are orthogonal.

We have $H\,H^T = 4t\;I_{\;m}$
Therefore
$C_2(H\,H^T) = C_2(4t\,I_m)$ [Vide Horn and Johnson (p. 21)]
$C_2(H)\;C_2(H\,)^{\,T} = (4t)^2\;(I_{\binom{m}{2}})$ [Vide Horn and Johnson (p. 21)]
$\Rightarrow\; WW^T = 4t^2 I_{\binom{m}{2}}$

Corollary: When m = 4t we have a series of integral multiple of weighing matrices $W\left\{4t^2, \binom{4t}{2}\right\}$.

2.2 Theorem: The 3- compound matrix of an $\mathrm{m} \times 4\mathrm{t}$ Hadamard matrix is an integral multiple of a weighing matrix, i.e., $\mathrm{C}_3\left(\mathrm{H}_{\mathrm{m}\,,\;4\mathrm{t}}\right) = 4\mathrm{W}\left\{4\mathrm{t}^3, \binom{\mathrm{m}}{3} \times \binom{4\mathrm{t}}{3}\right\}$ Where $\mathrm{m} \le 4\mathrm{t}$, $\mathrm{t} > 1$ and $\mathrm{W}\left\{4\mathrm{t}^3, \binom{\mathrm{m}}{3} \times \binom{4\mathrm{t}}{3}\right\}$ is a weighing matri**x.**

Proof: We consider 3×3 determinants which are minors of the matrix M formed by the juxta position of any two rows R_i , R_j and R_k. We calculate the non zero minors of rank 3 and order 3.

$$M = \begin{bmatrix} R_i \\ R_j \\ R_k \end{bmatrix}$$

Multiplying by -1 the column of M having 1st elt. -1, we make all the entries of R_i +1. Also $2t$ elements of R_j and R_k are +1 and remaining $2t$ elements of R_j and R_k are -1, By suitable permutation of columns of M can be put into the form

+ +…+	+ +…+	+ +…+	+ +…+
+ +…+	+ +…+	- -… -	- - … -
+ +…+	- -… -	+ +…+	- - … -

This process preserves the rank of M and so the number of zero minors of M which is $= \left[4\binom{t}{3} + 4\left\{\binom{t}{2} \times \binom{3t}{1}\right\}\right]$.

Hence the number of non zero minors =The total number of minors of M – number of zero minors

$$=\binom{4t}{3} - \left[4\binom{t}{3} + 4\left\{\binom{t}{2} \times \binom{3t}{1}\right\}\right].$$

$$= 4t^3$$

The non zero minors are of types

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 4\ , \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = -4, \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = -4 \quad \ldots\text{etc}$$

Hence $C_3\ (H_{m,4t}) = 4W$ is a matrix with entries 0 , +4, -4.

Finally we show that W is a weighing matrix in what follows W stands for $\left\{4t^3, \binom{m}{3} \times \binom{4t}{3}\right\}$.

It is sufficient to show that rows of W are orthogonal.

We have

$H H^T = 4t\, I_m$

Therefore

$C_3(H H^T) = C_3(4t\, I_m)$ [Vide Horn and Johnson (p. 21)]

$C_3(H) C_3(H)^T = (4t)^3\ (I_{\binom{m}{3}})$ [Vide Horn and Johnson (p. 21)]

$\Rightarrow\ W W^T = 4t^3 I_{\binom{m}{3}}$

Corollary: When m = 4t we have a series of integral multiple of weighing matrices $W\left\{4t^3, \binom{4t}{3}\right\}$.

III. CONSTRUCTION OF WEIGHING MATRICES FROM HADAMARD MATRIX OF ORDER 4t

3.1. Definition and known families of square weighing matrices

Definition of square weighing matrix:

Let W be a matrix of order n with entries 1, -1, 0 satisfying $WW^T = kI_n$, then we call W a square weighing matrix of order n and weight k, denoted by $W(k, n)$.

Some Known Conjecture of Families of square weighing matrices

(1)There exists a weighing matrix $W(k, 4t)$, for k = 1, 2 …..$4t$. [13]

(2) If n = 0(mod4), there exist $n\times n$ weighing matrices of every degree $k \le n$,and has been proved for n is power of 2 if n is not power of 2 we find an integer $t<n$ for which there are weighing matrices of every degree $\le t$.[5]

Our result partially solved the conjecture (2)

Each of the four families we have constructed is different from known families of weighing matrices.

(1) Circulant weighing matrices of weight 2^{2t}. [1]

(2) The weighing matrix of order $4n$ and weight $4n$-2 and $2n$-1. [6]

(3) Construction of weighing matrix of weight $2n$ and order $4n$ from a Hadamard matrix of order $4n$. [18]

(4)A weighing matrix can be obtained from any complex Hadamard matrix with entries +1,-1, +i , -i. [20]

(5)New construction of quaternary Hadamard matrix [12]

(i) For any positive integer n, we are able to construct quaternary Hadamard matrix of order 2^n.

(ii) A quaternary Hadamard matrix of order 2^n from a binary extended sequence of period 2^n -1, where n is a composite number.

3.2.Theorem: The (4t-2) - compound matrix of an $4t \times 4t$ Hadamard matrix H_{4t} is an integral multiple of a weighing matrix i.e., $C_{4t-2}\ (H_{4t}) = 2(4t)^{2t-2} W\left\{4t^2, \binom{4t}{2}\right\}$.

Proof: Let H be Hadamard matrix of order $4t$ and let $C_r(H)$ be the r- compound matrix of H .

It is known that ($4t$-2)×($4t$-2) minors of an Hadamard matrix of order 4t are zero or $\mathbf{2(4t)^{2t-2}}$ .Vide [16]

Hence H is a Hadamard matrix of order 4t then (4t-2)-compound matrix $\boldsymbol{C_{4t-2}\ (H_{4t})}$ of order

$\binom{4t}{2}$ have entries 0 or $\pm 2(4t)^{2t-2}$.

Finally we show that W is weighing matrix in what follows stands for $W\left\{4t^2, \binom{4t}{2}\right\}$. It is sufficient to show that rows of W are orthogonal.

We have

$H\,H^T = 4t\, I_{4t}$

Therefore

$C_{4t-2}(H\,H^T) = C_{4t-2}(4t\, I_{4t})$ [Vide Horn and Johnson (p. 21)]

$C_{4t-2}(H)C_{4t-2}(H)^T = (4t)^{4t-2}\,(I_{\binom{4t}{2}})$ [Vide Horn and Johnson (p. 21)]

$\Rightarrow\; WW^T = 4t^2 I_{\binom{4t}{2}}$

3.3. Theorem: The (4t-3)- compound matrix of an $4t \times 4t$ Hadamard matrix H_{4t} is an integral multiple of a weighing matrix i.e., $C_{4t-3}\,(H_{4t}) = 4(4t)^{2t-3} W\left\{4t^3, \binom{4t}{3}\right\}$ where $t > 1$.

Proof: Let H be Hadamard matrix if of order 4t and let $C_r(H)$ be the r- compound matrix of H .It is known that (4t-3)×(4t-3) minors of a Hadamard matrix of order 4t are zero or $\mathbf{4(4t)^{2t-3}}$ Vide[16]

Hence H is a Hadamard matrix of order 4t then (4t-3)-compound matrix $\boldsymbol{C_{4t-3}}\,(\boldsymbol{H_{4t}})$ of order $\binom{4t}{3}$ have entries 0 or $\pm 4(4t)^{2t-3}$.

Finally we show that W is weighing matrix in what follows W stands for $W\left\{4t^3, \binom{4t}{3}\right\}$.It is sufficient to show that rows of W are orthogonal.

We have

$H\,H^T = 4t\, I_{4t}$

Therefore

$C_{4t-3}(HH^T) = C_{4t-3}(4t\, I_{4t})$ [Vide Horn and Johnson (p. 21)]

$C_{4t-3}(H)C_{4t-3}(H)^T = (4t)^{4t-3}\,(I_{\binom{4t}{3}})$ [Vide Horn and Johnson (p. 21)]

$\Rightarrow\; WW^T = 4t^3 I_{\binom{4t}{3}}$

Proposition -1: If H is symmetric Hadamard matrix then W is symmetric weighing matrix.

Proposition -2*:* If H is Skew symmetric Hadamard matrix then W is skew symmetric weighing matrix.

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