Right (θ, θ)-4-Derivations on Prime Near-Rings

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Abstract

Let N be a near –ring and θ is a mapping on N .In this paper, we define the concept of right (θ, θ)-4-derivation in near-ring N and we explore the commutativity of addition and multiplication of prime near-ring N satisfying certain identities involving right (θ, θ)-4-derivation on N .

Keywords : *near-ring , prime near-ring ,right-4-derivation ,right (θ, θ)-4-derivation .*

I. Introduction

Suppose that N is a near –ring and θ is a mapping on N . This paper consists of two sections . In section one , we recall some basic definitions and other concepts , which will be used later in our paper , we explain these concepts by examples and remarks . In section two , we introduce the notion of right (θ, θ)-4-derivation in near-ring N and we determine some conditions of right (θ, θ)-4-derivation which make prime near-ring commutative ring .

II. Basic Concepts

Definition 2.1:[1] A ring R is called a prime ring if for any a,b $\in$R, aRb ={0} implies that either a=0 or b=0 .

Example 2.2:[1] The ring of real numbers with the usual operation of addition and multiplication is prime ring .

Definition 2.3:[1] A ring R is said to be n-torsion free whenever na=0 with a $\in$R, then a=0 .

Definition 2.4:[1] Let R be a ring . Define a Lie product [,] on R as follows

[x,y] = xy – yx , for all x,y $\in$R .

Properties 2.5:[1] Let R be a ring , then for all x,y,z $\in$R, we have :

1-[x,yz] = y[x,z] + [x,y]z
2-[xy,z] = x[y,z] + [x,z]y
3-[x+y,z] = [x,z] + [y,z]
4-[x,y+z] = [x,y] + [x,z]

Definition 2.6:[2] A right near-ring (resp. a left near-ring) is a nonempty set N equipped with two binary operations + and . such that

(i)(N, +) is a group (not necessarily abelian)
(ii)(N, .) is a semigroup .
(iii) For all x,y,z $\in$N , we have
(x+y)z = xz + yz (resp. z(x+y) = zx + zy)

Example 2.7:[2] Let G be a group (not necessarily abelian) then all mapping of G into itself from a right near-ring M(G) with regard to point wise addition and multiplication by composite .

Lemma 2.8:[2] Let N be left (resp. right) near-ring , then

(i)x0 = 0 (resp. 0x = 0) for all x $\in$N .
(ii) – (xy) = x(-y) (resp. – (xy) = (-x)y) for all x,y $\in$N .

Definition 2.9:[2] A right near-ring (resp. left near-ring) is called zero symmetric right near-ring (resp. zero symmetric left near-ring) if x0 = 0 (resp. 0x = 0) , for all x $\in$N .

Definition 2.10:[2] Let {N_i } be a family of near-rings (i$\in$I , I is an indexing set) . $N = N_1 \times N_2 \times \ldots \times N_n$ with regard to component wise addition and multiplication , N is called the direct product of near-rings N_i .

Definition 2.11:[3] A near-ring N is called a prime near-ring if aNb = {0}, where a,b $\in$N, implies that either a = 0 or b = 0 .

Definition. **2.12:[3]** Let N be a near-ring . The symbol Z will denote the multiplicative center of N , that is Z = {x$\in$N /xy = yx for all y $\in$N } .

Definition 2.13:[3] Let R be a ring . Define a Jordan product on R as follows :
$a \circ b = ab + ba$, for all a,b $\in$R .

Lemma 2.14:[4] Let N be a near-ring. If there exists a non-zero element z of Z such that z+z $\in$Z , then (N ,+) is abelian .

Lemma 2.15:[4] Let N be a prime near-ring. If z $\in Z\setminus\{0\}$ and x is an element of N such that xz $\in$Z or zx $\in$Z , then x $\in$Z .

Definition 2.16:[4] Let N be a near-ring . An additive mapping d: N $\rightarrow$ N is said to be right derivation of N if d(xy) = d(x)y + d(y)x , for all x,y $\in$N .

Definition 2.17:[4] Let N be a near-ring . An 4-additive mapping d: N x N x N x N $\rightarrow$ N is said to be right 4-derivation if the relations .

$d(x_1x_1', x_2, x_3, x_4) = d(x_1,x_2,x_3,x_4)\, x_1' + d(x_1',x_2,x_3,x_4)x_1$

$d(x_1,x_2x_2',x_3,x_4) = d(x_1,x_2,x_3,x_4)\, x_2' + d(x_1,x_2',x_3,x_4)\, x_2$

$d(x_1,x_2,x_3x_3', x_4) = d(x_1,x_2,x_3,x_4)\, x_3' + d(x_1,x_2,x_3', x_4)\, x_3$

$d(x_1,x_2,x_3,x_4 x_4') = d(x_1,x_2,x_3,x_4)\, x_4' + d(x_1,x_2,x_3, x_4')\, x_4$

hold for $x_1, x_1', x_2, x_2', x_3, x_3', x_4, , x_4' \in N$.

Example 2.18:[4] Let S be a 2-torsion free zero-symmetric left near-ring .
Let us define :

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x, y, 0 \in S \right\}$$

It is clear that N is a zero symmetric near-ring with respect to matrix addition and matrix multiplication .
Define d: N x N x N x N $\rightarrow$ N by

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_3 & y_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_4 & y_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1\,x_2\,x_3\,x_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It can be easily seen that d is a nonzero right 4-derivation of near-ring N .

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III. RIGHT (θ,θ) – 4 - DERIVATIONS

First we introduce the basic definition in this paper

Definition 3.1: Let N be a near-ring and θ is a mapping on N .An 4-additive mapping d: N x N x N x N $\rightarrow$ N is said to be right (θ, θ)- 4-derivation if the relations :

$d(x_1x_1^{/},x_2,x_3,x_4) = d(x_1,x_2,x_3,x_4)\,\theta(x_1^{/}) + d(x_1^{/},x_2,x_3,x_4)\,\theta(x_1)$
$d(x_1,x_2x_2^{/},x_3,x_4) = d(x_1,x_2,x_3,x_4)\,\theta(x_2^{/}) + d(x_1,x_2^{/},x_3,x_4)\,\theta(x_2)$
$d(x_1,x_2,x_3x_3^{/},x_4) = d(x_1,x_2,x_3,x_4)\,\theta(x_3^{/}) + d(x_1,x_2,x_3^{/},x_4)\,\theta(x_3)$
$d(x_1,x_2,x_3,x_4x_4^{/}) = d(x_1,x_2,x_3,x_4)\,\theta(x_4^{/}) + d(x_1,x_2,x_3,x_4^{/})\,\theta(x_4)$
hold for $x_1, x_1^{/}, x_2, x_2^{/}, x_3, x_3^{/}, x_4, x_4^{/} \in N$.

we now explain this definition by the following example
Example 3.2 : Let S be a 2-torsion free zero symmetric commutative near-ring .
Let us define

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} : x, y, z, 0 \in S \right\}.$$

It can be easily see that is a non commutative 2-torsion free zero symmetric left near-ring with respect to matrix addition and matrix multiplication .

Define d: N x N x N x N → N by

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \begin{pmatrix} 0 & x_3 & y_3 \\ 0 & 0 & 0 \\ 0 & 0 & z_3 \end{pmatrix}, \begin{pmatrix} 0 & x_4 & y_4 \\ 0 & 0 & 0 \\ 0 & 0 & z_4 \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1x_2x_3x_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And $\theta : N \rightarrow N$ such that

$$\theta\left(\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \right) = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$$

It is easy to see that d is a right (θ, θ)- 4-derivation of a near-ring N .

Now , we will prove the main results :
Theorem 3.3 : Let N be a prime near-ring and d be a nonzero right (θ, θ)- 4-derivation of N , where θ is an automorphism on N . If $d(N,N,N,N) \subseteq Z$, then N is a commutative ring .

Proof : Since $d(N,N,N,N) \subseteq Z$ and d is a nonzero right (θ, θ)- 4-derivation , there exist nonzero elements $x_1, x_2, x_3, x_4 \in N$, such that $d(x_1, x_2, x_3, x_4) \in Z\setminus\{0\}$, we have $d(x_1+x_1, x_2, x_3, x_4) = d(x_1, x_2, x_3, x_4) + d(x_1, x_2, x_3, x_4) \in Z$.
By Lemma 2.14 we obtain that (N,+) is abelian .
By hypothesis we get
$d(y_1,y_2,y_3,y_4)\,\theta(y) = \theta(y)\, d(y_1,y_2,y_3,y_4)$ for all $y, y_1,y_2,y_3,y_4 \in N$. (3.1)
Now replacing y_1 by $y_1 y_1^{/}$ where $y_1^{/} \in N$ in (3.1) , we get
$(d(y_1,y_2,y_3,y_4)\,\theta(y_1^{/}) + d(y_1^{/},y_2,y_3,y_4)\,\theta(y_1))\,\theta(y) = \theta(y)\,(d(y_1,y_2,y_3,y_4)\,\theta(y_1^{/}) + d(y_1^{/},y_2,y_3,y_4)\,\theta(y_1))$
for all $y, y_1, y_1^{/}, y_2,y_3,y_4 \in N$. (3.2)
By definition of d we get
$d(y_1 y_1^{/},y_2,y_3,y_4) = d(y_1,y_2,y_3,y_4)\,\theta(y_1^{/}) + d(y_1^{/},y_2,y_3,y_4)\,\theta(y_1)$
for all $y_1, y_1^{/}, y_2,y_3,y_4 \in N$. (3.3) and
$d(y_1^{/} y_1,y_2,y_3,y_4) = d(y_1^{/},y_2\, y_3,y_4)\,\theta(y_1) + d(y_1,y_2,y_3,y_4)\,\theta(y_1^{/})$ (3.4)
Since (N, +) is abelian , from (3.3) and (3.4) we conclude that
$d(y_1 y_1^{/},y_2,y_3,y_4) = d(y_1^{/} y_1,y_2,y_3,y_4)$ for all $y_1, y_1^{/}, y_2,y_3,y_4 \in N$. So we get
$d([y_1, y_1^{/}],y_2,y_3,y_4) = 0$ for all $y_1, y_1^{/}, y_2,y_3,y_4 \in N$. (3.5)
Replacing $y_1^{/}$ by $y_1 y_1^{/}$ in (3.5) and using it again , we get
$0 = d([y_1, y_1 y_1^{/}],y_2,y_3,y_4)$
$= d(y_1 [y_1, y_1^{/}],y_2,y_3,y_4)$
$= d(y_1,y_2,y_3,y_4)\,\theta[y_1,y_1^{/}] + d([y_1,y_1^{/}],y_2,y_3,y_4)\,\theta(y_1)$
$= d(y_1,y_2,y_3,y_4)\,\theta[y_1,y_1^{/}]$

We conclude that
$d(y_1,y_2,y_3,y_4)\, N\, \theta[y_1,y_1'] = \{0\}$ for all $y_1, y_1', y_2, y_3, y_4 \in N$.
Primeness of N implies that for each $y_1 \in N$ either $d(y_1,y_2,y_3,y_4) = 0$ for all $y_2, y_3, y_4 \in N$ or $y_1 \in Z$.
If $d(y_1,y_2,y_3,y_4) = 0$, then equation (3.2) takes the form
$d(y_1',y_2,y_3,y_4)\, N\, \theta[y,y_1] = \{0\}$. Since $d \neq 0$, primeness of N implies that $y_1 \in Z$.
Hence we find that $N = Z$, we conclude that N is a commutative ring .

Theorem 3.4 : Let N be a prime near-ring admitting a nonzero right (θ, θ)- 4-derivation d , where θ is an automorphism on N . If $d([x,y], x_2, x_3, x_4) = 0$ for all $x, y, x_2, x_3, x_4 \in N$, then N is a commutative ring .

Proof : By hypothesis , we have
$d([x,y], x_2, x_3, x_4) = 0$ for all $x, y, x_2, x_3, x_4 \in N$. (3.6)
Replace y by xy in (3.6) to get
$d([x, xy], x_2, x_3, x_4) = 0$ for all $x, y, x_2, x_3, x_4 \in N$.
Which implies that $d(x[x,y], x_2, x_3, x_4) = 0$ for all $x, y, x_2, x_3, x_4 \in N$.
Therefore
$d(x, x_2, x_3, x_4)\, \theta[x,y] + d([x,y], x_2, x_3, x_4)\, \theta(x) = 0$ for all $x, y, x_2, x_3, x_4 \in N$.
Using (3.6) in previous equation implies that
$d(x, x_2, x_3, x_4)\, \theta[x,y] = 0$ for all $x, y, x_2, x_3, x_4 \in N$, or equivalently
$d(x, x_2, x_3, x_4)\, \theta(x)\, \theta(y) = d(x, x_2, x_3, x_4)\, \theta(y)\, \theta(x)$
for all $x, y, x_2, x_3, x_4 \in N$. (3.7)
Replacing $\theta(y)$ by $\theta(y)\, \theta(z)$ in (3.7) and using it again , we get
$d(x, x_2, x_3, x_4)\, \theta(y)\, [\theta(x), \theta(z)] = 0$ for all $x, y, z, x_2, x_3, x_4 \in N$.
Hence we get
$d(x, x_2, x_3, x_4)\, N\, \theta[x, z] = \{0\}$ for all $x, z, x_2, x_3, x_4 \in N$. (3.8)
This yields that for each fixed $x \in N$
Either $d(x, x_2, x_3, x_4) = 0$ for all $x_2, x_3, x_4 \in N$ or $x \in Z$. (3.9)
If $d(x, x_2, x_3, x_4) = 0$ for all $x_2, x_3, x_4 \in N$ and for each fixed $x \in N$. We get $d = 0$, leading to a contradiction as d is a nonzero right (θ, θ)- 4-derivation of N .
Therefore there exist $x_1, x_2, x_3, x_4 \in N$, all being nonzero such that $d(x_1, x_2, x_3, x_4) \neq 0$ such that $x_1 \in Z$.
Thus , we conclude that $[x_1 y, z] = x_1[y, z]$, where $y, z \in N$, by hypothesis we get
$d([x_1 y, z], x_2, x_3, x_4) = 0$. This implies that
$0 = d(x_1[y,z], x_2, x_3, x_4)$
$= d(x_1, x_2, x_3, x_4)\, \theta[y, z] + d([y,z], x_2, x_3, x_4)\, \theta(x_1)$
$= d(x_1, x_2, x_3, x_4)\, \theta[y, z]$ for all $y, z \in N$.
Which implies that
$d(x_1, x_2, x_3, x_4)\, \theta(y)\, \theta(z) = d(x_1, x_2, x_3, x_4)\, \theta(z)\, \theta(y)$ for all $y, z \in N$.
Replacing $\theta(z)$ by $\theta(z)\, \theta(t)$, where $t \in N$, in last equation , and using it again , we get
$d(x_1, x_2, x_3, x_4)\, \theta(z)\, [\theta(y), \theta(t)] = 0$ for all $y, t, z \in N$, i.e.;
$d(x_1, x_2, x_3, x_4)\, N\, [\theta(y), \theta(t)] = \{0\}$ for all $y, t \in N$.
Since $d(x_1, x_2, x_3, x_4) \neq 0$, primeness of N implies that $N = Z$. By Lemma 2.14 ,
We conclude that N is a commutative ring .

Theorem 3.5 : Let N be a prime near-ring admitting a nonzero right (θ, θ)- 4-derivation d , where θ is an automorphism on N . If $[d(x, x_2, x_3, x_4), \theta(y)] \in Z$ for all $x, y, x_2, x_3, x_4 \in N$, then N is a commutative ring .
Proof : Assume that
$[d(x, x_2, x_3, x_4), \theta(y)] \in Z$ for all $x, y, x_2, x_3, x_4 \in N$. (3.10)
Therefore
$[[d(x, x_2, x_3, x_4), \theta(y)], \theta(t)] = 0$ for all $x, y, t, x_2, x_3, x_4 \in N$. (3.11)
Replacing $\theta(y)$ by $d(x, x_2, x_3, x_4)\, \theta(y)$ in (3.11) we get
$[d(x, x_2, x_3, x_4)\, [d(x, x_2, x_3, x_4), \theta(y)], \theta(t)] = 0$
for all $x, y, t, x_2, x_3, x_4 \in N$. (3.12)

In view of (3.10) , equation (3.12) assures that

$[d(x , x_2 , x_3 ,x_4) , \theta(y)]\, N\, [d(x , x_2 , x_3 ,x_4) , \theta(t)] = \{0\}$

for all $x , y, t , x_2 , x_3 ,x_4 \in N$. (3.13)

Primeness of N implies that

$[d(x , x_2 , x_3 ,x_4) , \theta(y)] = 0$ for all $x , y, x_2 , x_3 ,x_4 \in N$.

Hence $d(N,N,N,N) \subseteq Z$ and application of Theorem 3.3 assures that N is a commutative ring .

Theorem 3.6 : Let N be a 2-torsion free prime near-ring admitting a nonzero right (θ, θ)- 4-derivation d , where θ is an automorphism on N . If $d(x , x_2 , x_3 ,x_4) \circ \theta(y) \in Z$ for all $x , y, x_2 , x_3 ,x_4 \in N$, then N is a commutative ring .

Proof : Assume that

$d(x , x_2 , x_3 ,x_4) \circ \theta(y) \in Z$ for all $x , y, x_2 , x_3 ,x_4 \in N$. (3.14)

(a) If $Z = 0$, then equation (3.14) reduces to

$\theta(y)\, d(x , x_2 , x_3 ,x_4) = - d(x , x_2 , x_3 ,x_4)\, \theta(y)$ for all $x , y, x_2 , x_3 ,x_4 \in N$. (3.15)

Substituting $\theta(z)\, \theta(y)$ for $\theta(y)$ in (3.15) , we obtain

$$\begin{aligned}\theta(z)\, \theta(y)\, d(x , x_2 , x_3 ,x_4) &= - d(x , x_2 , x_3 ,x_4)\, \theta(z)\, \theta(y)\\ &= d(x , x_2 , x_3 ,x_4)\, \theta(z)(- \theta(y))\\ &= \theta(z)\, d(-x , x_2 , x_3 ,x_4)\, (- \theta(y))\end{aligned}$$

for all $x , y ,z , x_2 , x_3 ,x_4 \in N$.

Using the fact $- \theta(z)\, \theta(y)\, d(x , x_2 , x_3 ,x_4) = \theta(z)\, \theta(y)\, d(-x , x_2 , x_3 ,x_4)$, implies that

$\theta(z)\, \theta(y)\, d(-x , x_2 , x_3 ,x_4) = \theta(z)\, d(-x , x_2 , x_3 ,x_4)\, \theta(y)$ for all $x , y, z ,x_2 , x_3 ,x_4 \in N$.

Which implies that

$\theta(z)(\theta(y)\, d(-x , x_2 , x_3 ,x_4) - d(-x , x_2 , x_3 ,x_4)\, \theta(y)) = 0$

for all $x , y ,z , x_2 , x_3 ,x_4 \in N$. (3.16)

Taking $-x$ instead of x in (3.16) , we get

$\theta(z)\, N\, (\theta(y)\, d(x , x_2 , x_3 ,x_4) - d(x , x_2 , x_3 ,x_4)\, \theta(y)) = \{0\}$

for all $x , y ,z , x_2 , x_3 ,x_4 \in N$. (3.17)

Primeness of N implies that

$d(N,N,N,N) \subseteq Z$ and application of Theorem 3.3 assures that N is a commutative ring .

(b) Suppose that $Z \neq 0$, then there exists $z \neq 0 \in Z$ and by hypothesis we have

$d(x , x_2 , x_3 ,x_4) \circ \theta(z) \in Z$ for all $x , x_2 , x_3 ,x_4 \in N$.

Thus we get $d(x , x_2 , x_3 ,x_4)\, \theta(z) + \theta(z)\, d(x , x_2 , x_3 ,x_4) \in Z$ for all $x , x_2 , x_3 ,x_4 \in N$.

Since $z \in Z$, we get

$\theta(z)\, (d(x , x_2 , x_3 ,x_4) + d(x , x_2 , x_3 ,x_4)) \in Z$ for all $x , x_2 , x_3 ,x_4 \in N$.

By Lemma 2.15 we conclude that

$d(x , x_2 , x_3 ,x_4) + d(x , x_2 , x_3 ,x_4) \in Z$ for all $x , x_2 , x_3 ,x_4 \in N$. (3.18)

By (3.14) , we get

$d(x+x , x_2 , x_3 ,x_4)\, \theta(y) + \theta(y)\, d(x+x , x_2 , x_3 ,x_4) \in Z$ for all $x , y , x_2 , x_3 ,x_4 \in N$. (3.19)

Using equation (3.18) in (3.19) we conclude that

$\theta(y)(d(x+x , x_2 , x_3 ,x_4) + d(x+x , x_2 , x_3 ,x_4)) \in Z$ for all $x , y , x_2 , x_3 ,x_4 \in N$. (3.20)

From (3.18) and (3.20) we obtain

$(d(x+x , x_2 , x_3 ,x_4) + d(x+x , x_2 , x_3 ,x_4))\, \theta(ty) = \theta(ty)\, (d(x+x , x_2 , x_3 ,x_4) + d(x+x , x_2 , x_3 ,x_4)) =$
$\theta(y)(d(x+x , x_2 , x_3 ,x_4) + d(x+x , x_2 , x_3 ,x_4))\, \theta(t) =$
$(d(x+x , x_2 , x_3 ,x_4) + d(x+x , x_2 , x_3 ,x_4))\, \theta(yt)$

for all $x , y , t , x_2 , x_3 ,x_4 \in N$. Which implies that

$(d(x+x , x_2 , x_3 ,x_4) + d(x+x , x_2 , x_3 ,x_4))\, N\, [\theta(t) , \theta(y)] = \{0\}$

for all $x , y , t , x_2 , x_3 ,x_4 \in N$. (3.21)

Primeness of N implies that

Either $d(x+x , x_2 , x_3 ,x_4) + d(x+x , x_2 , x_3 ,x_4) =0$ and thus $d = 0$, a contradiction , or $N = Z$, hence $d(N,N,N,N) \subseteq Z$ and application of Theorem 3.3 assures that N is a commutative ring .

IV. Conclusions

In present paper we introduce the notion of right (θ, θ)- 4-derivation on near-ring and we see that a prime near-ring can be make commutative with the help of right (θ, θ)- 4-derivation and other conditions

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