New Extension for Pythagoras Theorem and Its Application

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Abstract *- Present paper deals with a new extension to Pythagoras Theorem. Theorem has been derived from the fundamental relationship between hypotenuse and other two sides of a right triangle. Main use of the proposed new extension theorem is in finding Pythagorean triplets.*

Keywords - *Pythagoras Theorem, Extension Theorem, Pythagorean triads, Pythagorean triplets.*

I. INTRODUCTION

Relationship between the sides of a right-angled triangle is amazing. This property was recognized and it has been studied and practically applied for setting right-angled corners of buildings and bricks from time immemorial. This relationship was known in different names in different cultures[1;] which is presently known worldwide as 'Pythagoras theorem' after the well-known Ionian (Greek) philosopher Pythagoras (c. 570–495 BC). As triplets or triads like (3, 4 and 5) which have practical use, these were also studied and general rules were formulated to find these triplets from ancient times[2].Famous Indian Mathematician-Astronomer Nilakanta[3] (1444-1554 CE) in his notable commentary on Aryabhateeyam[4], has made a remarkable observation about the Pythagoras theorem stating that the relationship of sides of a right-angled triangle and rule of three (Ratio and Proportion or interpolation in modern parlance) are the two vital mathematical ideas that form the foundation for almost all the other discoveries in Mathematics. This statement brings out the prominence of the theorem of Pythagoras in Mathematics.

Present paper deals with a new extension to Pythagoras theorem and its application in finding Pythagorean triplets or Pythagorean triads.

II. NEW EXTENSION TO PYTHAGORAS THEOREM

Proposed Extension Theorem:

"If on the hypotenuse of a right angled triangle, segments are cutoff equal to the adjacent sides from the respective vertices and thus when the hypotenuse is divided into three segments by two overlapping arcs, the square of the middle segment will be the twice the rectangle contained by the extreme segments."

Let ΔABC is right-angled at A. (See Figure: 1)
Draw an arc with B as centre and radius equal to BA cutting hypotenuse BC at D.
Similarly draw an arc with C as centre and radius equal to CA cutting hypotenuse CB at E.
Now hypotenuse CE is divided into three segments BE, CD as extreme segments and DE as middle segment.

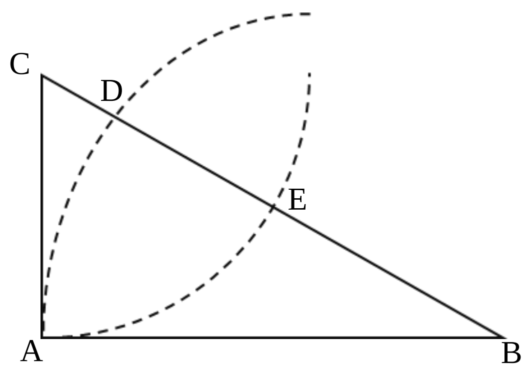

Figure 1: Right-angled Triangle ABC with arc cutting the hypotenuse

Postulation:

$$DE^2 = 2.BE.CD$$

Proof:
Data:
$\angle A = 90°$
Using Pythagoras Theorem
$BA^2 + CA^2 = BC^2$ ………… (1)
From the above figure;
$BA = BD = BE + DE$

$CA = CE = CD + DE$

$BC = BE + DE + CD$
Hence using Equation (1)
$(BE + DE)^2 + (CD + DE)^2 = (BE + DE + CD)^2$

$\therefore DE^2 = 2.BE.CD$
Thus, postulation is proved

III. Application of the proposed Theorem

Using the above extension theorem Pythagorean triplets can be found as follows.

Let

$$\begin{cases} CD = l \\ DE = m \\ BE = n \end{cases}$$

Then by proposed theorem
$\because DE^2 = 2.BE.CD$
$m^2 = 2.l.n$
Assign even integers for the value 'm', so that m^2 is also even and the product *ln* will be a whole number. Now factorize the product *l.n* to two; by assigning factors to *l and n*, number of Pythagorean triplets could be generated as follows.

$(l + m), (m + n) \, \& \, (l + m + n)$.

Particular cases can also be generalized.[5.]

A few examples are worked out below to demonstrate the above.

Case: 1
Let
$m = 2$
Then
$2.l.n = m^2 = 2^2 = 4$

$\therefore l.n = 2$
Possible integer factors (1,2)
Let l=2 & n=1;

Then Triplets;
$(l + m), (m + n) \, \& \, (l + m + n) = (2 + 2), (2 + 1) \, \& \, (2 + 2 + 1) = 4,3,5$
Case: 2
Let $m = 10$

$$\therefore l.n = \frac{m^2}{2} = \frac{100}{2} = 50$$

Possible factors are (50,1),(25,2),(10,5)

Thus three sets of Triplets are possible:

$(l+m),(m+n)\ \&\ (l+m+n) =$ (50+10).(10+1) & (50+10+1)=60,11 &61

$(l+m),(m+n)\ \&\ (l+m+n) =$ (25+10).(10+2) & (25+10+2)=35,12 & 37

$(l+m),(m+n)\ \&\ (l+m+n) =$ (10+10).(10+5) & (10+10+5)=20,15 & 25

Assigning even numbers 2, 4, 6 etc. up to 10 for the value of m, following Table I is prepared.

TABLE I

TABLE OF TRIPLETS DERIVED USING THE ABOVE THEOREM

m	m^2	$l.n = \frac{m^2}{2}$	Factors	l	n	TRIPLETS		
						c	b	a
						(l+m)	(n+m)	(l+m+n)
2	4	2	2,1	2	1	**4**	**3**	**5**
4	16	8	8,1	8	1	**12**	**5**	**13**
4	16	8	4,2	4	2	**8**	**6**	**10**
6	36	18	18,1	18	1	**24**	**7**	**25**
6	36	18	9,2	9	2	**15**	**8**	**17**
6	36	18	6,3	6	3	**12**	**9**	**15**
8	64	32	32,1	32	1	**40**	**9**	**41**
8	64	32	16,2	16	2	**24**	**10**	**26**
8	64	32	8.4	8	4	**16**	**12**	**20**
10	100	50	50,1	50	1	**60**	**11**	**61**
10	100	50	25,2	25	2	**35**	**12**	**37**
10	100	50	10,5	10	5	**20**	**15**	**25**

IV. CONCLUSIONS

Use and applicability of the proposed extension theorem have been established by demonstrating the method for arriving at Pythagorean triplets. Based on the above, triplets could be classified as derivatives of each even number commencing from number two (2).

ACKNOWLEDGMENT

Proposed theorem is based on an unpublished work titled "Mathematical Tit-Bits", written by the author's Father, Late Sri. Parameswara Aiyer N (Vaikom), compiling his own research works on various mathematical topics. Correspondence with various research and educational institutions including NCERT, New Delhi is also available with the MS proving the authenticity of the work.

NOTES

1. Earliest recorded statement of Pythagoras Theorem is contained in Baudhāyana Sulbasūtra (c.700 BCE), India. The statement in Sanskrit is as follows: *"dīrghachatursrasyākṣaṇayā rajjuḥ pārśvamānī, tiryagmānī, cha yatpṛthagbhūte kurutastadubhayāṅ karoti"*
   Above could roughly be translated as square on the diagonal of a rectangle will be equal to the sum of the areas of the squares on the sides of the rectangle.
2. Several rules for finding triplets are given in chapter-I, Book-2 Leelavathi (1159 CE) written by Bhaskaracharya.
3. Nilakanta Somayajin (1444-1544 BCE), the doyen of Kerala School of Mathematics which flourished in southernmost tip of peninsular India during 14^{th} to 18^{th} Cent. He has authored many original works including famous Tantrasangraha (1500CE)
4. Aryabhateeya Bhashya is a remarkable Sanskrit commentary on the famous Indian Astronomy text, Aryabhateeyam (499CE). This commentary is written by Nilakanta Somayajin.
5. A typical set of Triplets that can be derived using the above theorem is depicted below.
   Let m=2N
   Then: $m^2 = 4N^2$
   $l.n = 2.N^2$

If $l = 2.N^2$, then $n = 1$

Thus $(l+m),(m+n)\ \&\ (l+m+n) = (2N^2+1),(2N+1)\ \&\ (2N^2+2N+1)$

will always represent a Pythagorean triplet.

Thus, all odd numbers will form a triplet, If number is K is any odd number, K will be one side

then other side will be $\frac{K^2-1}{2}$ and Hypotenuse will be $\frac{K^2+1}{2}$

REFERENCES

[1] Bernard.S & Child.J.M. *Elements of Geometry. MacMillan and Co, London ,1948*

[2] Todhunter.L. *The Elements of EUCLID.* , *MacMillan and Co, London ,1889*

[3] John Taylor MD , LILAWATI, A Treatise on Arithmetic and Geometry by Bhascara Acharya, Courier Books, Bombay, 1816..

[4] Sharma.K V. *Tantrasangraha of Nilakanta Somayaji with Yukti Deepika and Laghuvivruthi* , Panjab University ,1977.

[5] Sen SN & Bag AK.,*The Sulbasutras of Boudhayana, Apasthamba, Katyana and Manava with English translation* , Indian Science Academy , New Delhi 1983

[6] Sambavasastri, Aryabhateeya Bhashya of Nilakantasomayajin : Part I&II, TSS.101, Trivandrum