Eigenvalue Approach on Generalized Thermoelastic Interactions of a Layer

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Abstract: *The unified form of the basic equations of generalized thermoelastic interactions for Lord Shulman (LS) and Green Lindsay (GL) models for a layer have been written in the form of a vector matrix differential equation and solved by the eigen value approach technique in the Laplace transform domain in a closed form. The inversions of the physical variables from the transformed domain have been made by using Zakian algorithm for the numerical inversion from the Laplace transform domain. Graphs for the physical variables have been presented for different cases and the results are compared with the existing literature.*

Keywords*: Generalized thermoelasticity, relaxation time, unified form of equation, eigen value approach, Zakian algorithm.*

I. INTRODUCTION

Biot [1] formulated the coupled problem of thermoelasticity (CTE) by introducing the strain rate term in the uncoupled heat conduction equation which remained parabolic in nature and as such propagate thermal wave of infinite speed. The equation of motion considered in this case is hyperbolic in nature and produce finite speed of propagation of mechanical wave.

Lord and Shulman [2] presented the coupled dynamic theory in thermoelastic continuum media where the concept of wave type heat propagation has been formulated with an additional term in the Fourier law of heat conduction .They modified the law by inserting a heat flux rate term and relaxation time parameter vide, [3].

Since then the researchers became interested in the generalized thermoelasticity theory and published a bulk of research papers. Later, Green and Lindsay [4], Green and Naghdi [5] contributed the theoretical modeling of this generalized theory. Thus the researchers become involved in solving and analyzing a number of problems in this generalized theory in presence of the second sound. The review articles of Chandrasekharaiah [6, 7] may be mentioned in this context.

Following the models of Lord and Shulman (LS), Green and Lindsay (GL) and Green and Naghdi (GN), a number of research papers, mainly linear, have been published during the last few years. A few of them [10-20] may be mentioned. In these research articles following linear theory, the first law of thermodynamics is linearized and the temperature change in coupling term is ignored in comparison to the initial (reference) temperature. Such assumption may be considered only when the change of temperature during the thermoelastic interaction is small in comparison to the reference temperature. By introducing some parameters, the theories on models of LS, GL and GN have been combined and a unified set of equations has been rendered by Bagri A.et.al [9]. These equations are then solved to study the one dimensional thermoelastic disturbances in a Layer. Taheri et al [8] studied the problem of a layer based on GN theory. Most of the recent publications on layer are concerned with the nonlinear theory and solutions are made using numerical methods. Based on the thermodynamical framework, Ignaczak [21], Hetnarski and Innaczak [22] presented a nonlinear generalized thermoelastic model. To mention a few of the recent studies on nonlinear thermoelasticity, the papers [23-27] may be cited. However, it may be noted that in the purview of non-linear investigations there are cases where neither the temperature field nor the deformed field and their interactions are nonlinear, the dependency of the material properties on the field variables have been considered [28] as the source of nonlinear analysis.

In this paper, we propose to present the basic equations on LS and GL models of generalized thermoelasticity in the Laplace transform domain and then write down the transformed equations in the form of a vector-matrix differential equation and later decoupled the equations to determine the physical variables. The equations are then solved in closed form in the Laplace transform domain and finally present in the space-time domain by Zakian algorithm [29].The results have been compared with the existing literature. The theory for decoupling the resulting equations and their solutions are presented in the Appendix.

II. BASIC EQUATIONS IN ONE DIMENSIONAL FORMULATION

We assume all the field variables as functions of x and t. Let us consider an infinite isotropic homogeneous layer between $x=0$ to $x=a$.

The stress-strain relation in this case becomes

$$\tau_{xx}=(\lambda+2\mu)e_{xx}-\beta T \tag{1}$$

where λ, μ are the Lame constants, and the thermoelastic parameter $\beta=\alpha(3\lambda+2\mu)$, where α is the coefficient of thermal expansion. T is the absolute temperature within the layer. The strain e_{xx} and displacement u are related by the equation

$$e_{xx}=\frac{\partial u}{\partial x} \tag{2}$$

The equation of motion in displacement term can be written as

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-\beta\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2} \tag{3}$$

Lord and Shulman introduced a relaxation time parameter in the conventional Fourier law of heat conduction. This relaxation time parameter represents the time-lag which is needed to establish a steady state conduction of heat in a volume element of the body when the temperature gradient is suddenly imposed [3]. This relation can be written as

$$q+t_0\dot{q}=-k\frac{\partial T}{\partial x} \tag{4}$$

where q is the heat flux and k is the thermal conductivity and t_0 is the relaxation time parameter. Clearly, when $t_0=0$, this equation simplifies to the classical Fourier Law. Noda [30] pointed out that the heat balance for an element of a body is concerned with the relation

$$\dot{Q}=-\frac{\partial q}{\partial x} \tag{5}$$

where Q is the specific heat in flux and q the axial heat flux .From the second law of thermodynamics in the Clausius-Duhem sense, the thermal power may also be written as

$$\delta Q=TdS \tag{6}$$

where S is the entropy of the system. The above equation when written in rate form, takes the form

$$\dot{Q} = T\dot{S} = T\left(\frac{\partial S}{\partial e_{xx}}\dot{e}_{xx} + \frac{\partial S}{\partial T}\dot{T}\right) \tag{7}$$

which simplifies to vide, [23]

$$\dot{Q} = T\beta\dot{e}_{xx} + c_e\rho\dot{T} \tag{8}$$

Differentiating (4) w.r.t x and using (5), we write

$$\left(1 + t_0\frac{\partial}{\partial t}\right)\dot{Q} = k\frac{\partial^2 T}{\partial x^2}$$

which after using (8) takes the form

$$k\frac{\partial^2 T}{\partial x^2} = \left(1 + t_0\frac{\partial}{\partial t}\right)\left(T\beta\dot{e}_{xx} + c_e\rho\dot{T}\right)$$

Rearranging this equation after using (2), we get

$$k\frac{\partial^2 T}{\partial x^2} = \beta\left(T\frac{\partial^2 u}{\partial x\partial t} + t_0\frac{\partial T}{\partial t}\frac{\partial u}{\partial x} + t_0 T\frac{\partial^3 u}{\partial x\partial t^2}\right) + c_\epsilon\rho\frac{\partial T}{\partial t} + t_0 c_\epsilon\rho\frac{\partial^2 T}{\partial t^2} \tag{9}$$

The above equation (9) is the energy equation in thermally nonlinear form for the LS model. Setting the relation $t_0 = 0$ in (9) we get the nonlinear energy equation for the classical coupled thermoelasticity as

$$k\frac{\partial^2 T}{\partial x^2} = \beta T\frac{\partial^2 u}{\partial x\partial t} + c_\epsilon\rho\frac{\partial T}{\partial t} \tag{10}$$

We note that all the terms written within the bracket in equation (9) and the second term in equation (10) are nonlinear. The usual procedure to linearize these nonlinear terms is to assume that the temperature change may be ignored in comparison with reference temperature. But this simplification, as expected, is valid in the range where the variations of temperature may be neglected in comparison with the reference temperature T_0.

Thus the linearized version of the energy equation in LS one parameter theory can be written as

$$k\frac{\partial^2 T}{\partial x^2} = \beta T_0\left(1 + t_0\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial x\partial t} + \rho c_\epsilon\frac{\partial}{\partial t}\left(1 + t_0\frac{\partial}{\partial t}\right)T \tag{11}$$

In this theory, there is no change in the constitutive equation.

The corresponding linearized version in CTE becomes

$$k\frac{\partial^2 T}{\partial x^2} = \beta T_0\frac{\partial^2 u}{\partial x\partial t} + \rho c_\epsilon\frac{\partial T}{\partial t} \tag{12}$$

However, in GL-theory, both the energy equation and Duhamel-Neuman constitutive equations are modified. It admits two relaxation time parameters α_0 and α. In isotropic elastic material and in one dimensional form, these equations may be written as

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \beta\left(1 + \alpha\frac{\partial}{\partial t}\right)(T - T_0) \tag{13}$$

$$k\frac{\partial^2 T}{\partial x^2} = \beta T_0 \frac{\partial^2 u}{\partial x \partial t} + \rho c_\epsilon \frac{\partial}{\partial t}\left(1 + \alpha_0 \frac{\partial}{\partial t}\right)T \tag{14}$$

These equations in unified form can be written as

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial}{\partial x}\left(1 + \alpha\frac{\partial}{\partial t}\right)T = \rho\frac{\partial^2 u}{\partial t^2} \tag{15}$$

$$k\frac{\partial^2 T}{\partial x^2} - \beta T_0\left(1 + \tau\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial x \partial t} - \rho c_\epsilon \frac{\partial}{\partial t}\left(1 + \alpha_0\frac{\partial}{\partial t}\right)T = 0 \tag{16}$$

Along with these equations the constitutive equation (13) need to be considered. We now classify the models into three classes:

(i)The system of equations will reduce to CTE if $\alpha_0 = \alpha = \tau = 0$

(ii)When $\alpha = 0, \alpha_0 = \tau \neq 0$; the equations reduce to LS model.

(iii)If $\alpha \neq 0, \alpha_0 \neq 0$, but $\tau = 0$, then these equations reduce to GL model.

For Convenience, the following dimensionless variables are introduced

$$x' = \frac{x}{l},\ t' = \frac{c_e t}{l},\ \theta' = \frac{T - T_0}{T_0} = \frac{\theta}{T_0},\ \tau'_{xx} = \frac{\tau_{xx}}{\beta T_0},\ u' = \frac{(\lambda + 2\mu)u}{l\beta T_0},\ \alpha' = \frac{c_e\alpha}{l},\ \alpha'_0 = \frac{c_e\alpha_0}{l},$$
$$\tau' = \frac{c_e\tau}{l},\ c_e^{\ 2} = \frac{\lambda + 2\mu}{\rho},\ l = \frac{k}{\rho c_\epsilon c_e} \tag{17}$$

Suppressing the primes the equations (13), (15), and (16) now take the non-dimensional form

$$\tau_{xx} = \frac{\partial u}{\partial x} - \left(1 + \alpha\frac{\partial}{\partial t}\right)\theta \tag{18}$$

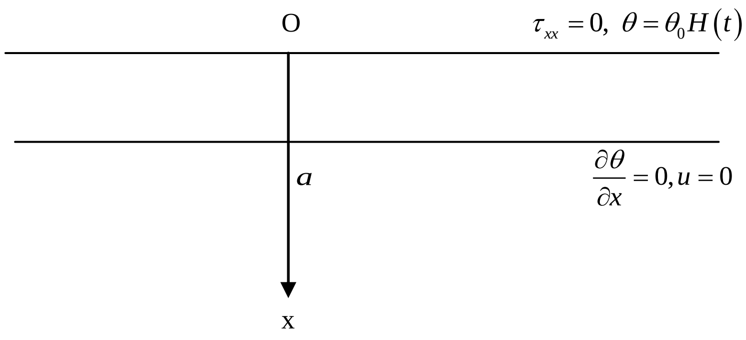
$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x} - \alpha\frac{\partial^2 \theta}{\partial x \partial t} = \frac{\partial^2 u}{\partial t^2} \tag{19}$$

$$\frac{\partial^2 \theta}{\partial x^2} - c\left(1 + \tau\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial}{\partial t}\left(1 + \alpha_0\frac{\partial}{\partial t}\right)\theta = 0 \tag{20}$$

where c is the non-dimensional thermoelastic coupling parameter.

III. WAVE PROPAGATION IN A LAYER

We now consider a one dimensional (x- direction) wave propagation in an infinite layer between $x=0$ and $x=a$, where a is a specified non-dimensional length.



In order to investigate the propagation of thermoelastic waves, we need to decouple the equations (19) and (20). As such we transform the equations in the space time domain to a Laplace transform domain of parameter s defined by

$$L\left[f(t)\right]=\bar{f}(s)=\int_0^\infty e^{-st} f(t)\,dt$$

Taking Laplace transform of equations (18), (19) and (20) and assuming initial condition as

$u(x,0)=\frac{du}{dt}(x,0)=\theta(x,0)=\frac{d\theta}{dt}(x,0)=0$, we get

$$\bar{\tau}_{xx}=\frac{d\bar{u}}{dx}-(1+\alpha s)\bar{\theta} \tag{21}$$

$$\frac{d^2\bar{u}}{dx^2}=(1+\alpha s)\frac{d\bar{\theta}}{dx}+s^2\bar{u} \tag{22}$$

$$\frac{d^2\bar{\theta}}{dx^2}=cs(1+\tau s)\frac{d\bar{u}}{dx}+s(1+\alpha_0 s)\bar{\theta} \tag{23}$$

The equations (22) and (23) may be written in the form of a vector-matrix differential equation as

$$\frac{d}{dx}\begin{bmatrix}\frac{d\bar{u}}{dx}\\ \frac{d\bar{\theta}}{dx}\\ \bar{u}\\ \bar{\theta}\end{bmatrix}=\begin{bmatrix}0 & c_{12} & c_{13} & 0\\ c_{21} & 0 & 0 & c_{24}\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{d\bar{u}}{dx}\\ \frac{d\bar{\theta}}{dx}\\ \bar{u}\\ \bar{\theta}\end{bmatrix} \tag{24}$$

where $c_{12}=1+\alpha s$, $c_{13}=s^2$, $c_{21}=cs(1+\tau s)$ and $c_{24}=s(1+\alpha_0 s)$

Equation (24) may be written in abridge form as

$$\frac{d\underset{\sim}{v}}{dx}=\underset{\sim}{A}\underset{\sim}{v} \tag{25}$$

where $\underset{\sim}{v}$ the vector of the components $\bar{u}$, $\bar{\theta}$ and their derivatives, and A is the constant matrix whose elements are independent of x.

IV. SOLUTION OF THE VECTOR -MATRIX DIFFERENTIAL EQUATION

The method of solution as given in the Appendix will be followed. In the present case , the matrix A as given in (25) is of (4×4) dimension , and the vector $\underset{\sim}{v}$ whose components in the Laplace transform domain are to be determined is of the form

$$\underset{\sim}{v} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^T = \begin{bmatrix} \frac{d\bar{u}}{dx} & \frac{d\bar{\theta}}{dx} & \bar{u} & \bar{\theta} \end{bmatrix}^T \tag{26}$$

The characteristic equation of the matrix A can be determined as

$$\lambda^4 - (c_{13} + c_{24} + c_{12} + c_{21})\lambda^2 + c_{13}c_{24} = 0 \tag{27}$$

Let the four distinct eigenvalues (roots of (27)) of A be λ_1, λ_2, λ_3 and λ_4 where $\lambda_2 = -\lambda_1$ and $\lambda_4 = -\lambda_3$. The eigenvectors corresponding to the eigenvalues λ_r, $r = 1,2,3,4$ may be calculated as

$$\begin{bmatrix} \lambda^2 c_{12} \\ \lambda(\lambda^2 - c_{13}) \\ \lambda c_{12} \\ \lambda^2 - c_{13} \end{bmatrix}_{\lambda=\lambda_r} = \begin{bmatrix} x_{1r} \\ x_{2r} \\ x_{3r} \\ x_{4r} \end{bmatrix} \text{(say)} \tag{28}$$

Following (A-7), (A-8) and (A-9), we write down the solution of equation (25) in the Laplace Transform domain in explicit form as

$$\begin{bmatrix} \frac{d\bar{u}}{dx} \\ \frac{d\bar{\theta}}{dx} \\ \bar{u} \\ \bar{\theta} \end{bmatrix} = c_1 \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{bmatrix} e^{\lambda_1 x} + c_2 \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{bmatrix} e^{\lambda_2 x} + c_3 \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \\ x_{43} \end{bmatrix} e^{\lambda_3 x} + c_4 \begin{bmatrix} x_{14} \\ x_{24} \\ x_{34} \\ x_{44} \end{bmatrix} e^{\lambda_4 x} \tag{29}$$

Thus we may obtain from (29)

$$\bar{u}(x,s) = c_1 x_{31} e^{\lambda_1 x} + c_2 x_{32} e^{\lambda_2 x} + c_3 x_{33} e^{\lambda_3 x} + c_4 x_{34} e^{\lambda_4 x}$$

$$\bar{\theta}(x,s) = c_1 x_{41} e^{\lambda_1 x} + c_2 x_{42} e^{\lambda_2 x} + c_3 x_{43} e^{\lambda_3 x} + c_4 x_{44} e^{\lambda_4 x}$$

It may be noted that $\frac{d\bar{u}}{dx}$ and $\frac{d\bar{\theta}}{dx}$ may also be obtained from (29).

The constants c_1, c_2, c_3 and c_4 are to be determined from the following boundary conditions on the two sides of a layer.

$$\begin{aligned} \text{At} \quad & x = 0: \ \tau_{xx} = 0, \ \theta = \theta_0 H(t) \\ & = a: \ \frac{\partial \theta}{\partial x} = 0, \ u = 0 \end{aligned} \tag{30}$$

where H(t)is the Heaviside unit step function.

The above boundary conditions indicate a layer which is stress free but exposed to a sudden application of temperature shock at one surface, while the other one is fixed and thermally insulated.

Taking the Laplace transform on the boundary conditions (30) we write

$$\bar{\tau}_{xx}(0,s)=0\,,\ \bar{\theta}(0,s)=\frac{\theta_0}{s}\,,\ \frac{d\bar{\theta}(a,s)}{dx}=0\,,\ \bar{u}(a,s)=0 \tag{31}$$

where the nondimensional coordinate a of the layer is designated by $0 \le x \le a$.

Using (21) and (29) and applying the boundary conditions (31), we can determine the four unknown constants c_1 , c_2 , c_3 and c_4 .Thus the temperature $\bar{\theta}$, deformation $\bar{u}$ and the stress $\bar{\tau}_{xx}$ are found out in the Laplace transform domain. In order to invert these physical quantities in the space-time domain, we apply Zakian algorithm [29] for numerical evaluation of inverse Laplace transform for values of time t and the results are shown in the graphs.

V. RESULTS AND DISCUSSIONS

The nature of propagation of one-dimensional waves for each of the thermal, deformation and stress fields and for each of the models (CTE, LS and GL) as classified in terms of relaxation parameters in (16) have been presented in different figures.

Temperature vs. coordinate x

The thermo mechanical coupling factor c and the constant θ_0 are chosen as c=0.0168 (for copper material) and $\theta_0 = 1$.The relaxation parameters $\alpha = \alpha_0 = \tau = 0$ for CTE model. We choose $\alpha = 0, \alpha_0 = \tau = 0.02$ for LS model and $\alpha = \alpha_0 = 0.02$ and $\tau = 0$ for GL model. Several graphs for temperature vs. x ($0 \le x \le 1$) have been drawn for specific value of time t. It is noticed that when $0 < t < 1$, infinite number of oscillations are observed which are not recognizable for different models. At t=1, the crest and trough for CTE model occur in (-120,110) when $0 < x < 0.2$, thereafter the wave dies off very quickly and from $x \approx 0.6$, it is nearly a straight line converging to x=1.The curves for LS and GL models are almost identical and they have maximum amplitude in ($-300 < \theta < 310$) when $0 < x < 0.2$, but the peaks of the amplitudes decreases almost like an decreasing exponential curve as x increases. In order to recognize the curves for the three models we only change $\alpha = \alpha_0 = 0.03$,and $\tau = 0$ for the GL model and the graph is shown in Fig.1.Keeping the same parameters as in the previous case, several graphs are drawn for other increasing values of time t to observe the nature of the temperature variations in ($0 \le x \le 1$).It is noticed that as t increases, the wave length increases and the wavy character gradually dies off as seen in Fig. 2 and Fig. 3 for t=3 and t=4 respectively.

Temperature vs. time t

Several graphs of temperature vs. time($1 < t < 10$)have been drawn and it is noticed that in all cases rapid oscillations are observed for earlier values of time ($1 < t < 6$).It is also observed as in the earlier case, the curves when the relaxation parameters are ($\alpha = 0$, $\alpha_0 = \tau = 0.02$)for LS model and ($\alpha = \alpha_0 = 0.02$, $\tau = 0$) for GL model, the temperature curves are almost identical .As such we prefer ($\alpha = \alpha_0 = 0.03$, $\tau = 0$)for GL model for distinct visualization. The maximum value (upper peak) and the minimum value of the temperature are predicted approximately in the range (-315 to 250). In Fig. 4, we present the graph when $x = 0.5$ (middle surface of the layer).It is seen that the highest peak values for GL model is 248.6 when $t = 1.871$,for LS model it is 183.6 at $t = 1.855$ and for CTE model it is 118.4 at $t = 3.652$ .The trough values assumed in GL, LS and CTE model are respectively -311.1 at $t = 5.204$, -266.9 at $t = 5.185$ and -209.7 at $t = 5.099$ .It is also observed that when($6 < t < 10$)the nature of the curves is almost the same as shown in Fig.4.

Deformation vs. x

Several graphs for ($0<t<10$) when ($0\le x\le 1$) have been drawn to observe the nature of the deformation. At t=1,the deformation exhibits oscillatory nature with decreasing amplitude as x increases as shown in Fig. 5.The relaxation time parameters are assumed as in the previous cases. The numbers of oscillations gradually diminish abruptly as t increases and at t=3, the graphs have been shown in Fig.6.The highest and lowest values of deformations for each model are recorded as follows:

CTE: $(09.08 \text{ at } x=0.564)$, $(-20.61 \text{ at } x=0.844)$

LS: $(17.96 \text{ at } x=0.564)$, $(-32.43 \text{ at } x=0.847)$

GL: $(25.07 \text{ at } x=0.571)$, $(-41.47 \text{ at } x=0.85)$

It is interesting to note that in this range of values of x there are two points (x = 0.15 and x = 0.709) at which the deformations for all the three models are identical which may be noted as around 3.05 and 0.6 whereas at x = 0.424 the deformations for CTE and LS models are identical (≈2.1) and the same value for GL model is attained at x = 0.43.

Deformation vs. t

With the same values of the relaxation parameters as in the earlier case, several graphs are drawn for deformation vs. t($0<t<10$) when x is in the range $0<x<1$.It is observed that curves start with small oscillation at x = 0.1 in ($0<t<4$) assumes a peak at t = 5.2, and then follows a curvy path during $5.2<t<9$ towards the value of -150 where each model form a trough. Similar observations are found for x = 0.2, 0.3 and 0.4. The graph for x = 0.5 has been presented in Fig. 8.It is noticed that the waves propagate with greater amplitude as t increases. For x = 0.6 to 0.9 similar results may be observed.

Stress vs. x

In order to recognize the graphs for the stress of the three models, we choose the time relaxation parameters as mentioned earlier in case of graphs in Fig. 1.Now several graphs have been drawn for different values of time t in ($0\le x\le 1$).It is observed that at time $t=1$ the stress graphs, soon after start, exhibit oscillation in the range (-32,32), (-16,20) and (-9,10) for GL,LS and CTE model respectively, but uniformly diminish in amplitude as x increases. Stress graph for CTE looks almost like a straight line in the range (0.6, 1.0), as shown in Fig. 9. As t increases it is noticed that the graphs oscillate in each case with increasing wave-length. Fig. 10 shows the graphs when $t=2$.However it is also observed at time $t=5$,the curves for LS and GL models deviate noticeably from the curve of CTE model, but the curves are coming very close to each other at around $x=0.66$.

Stress vs. t

The stress graphs for the three models with relaxation parameters as mentioned in earlier sections show that when t varies in ($1\le t\le 10$)for specified value of x in ($0<x<1$),the graphs oscillate with greater amplitude as t increases from $t=1$ to $t=6$ and after that instant they follow a simple path as shown in the Fig.11 for $x=0.5$.

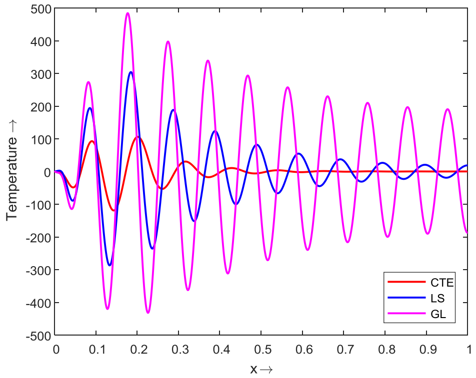


Fig.1. Variation of temperature vs. x for t=1.

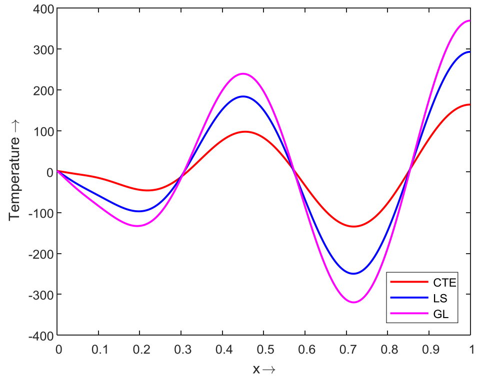


Fig.2.Variation of temperature vs. x for t=3.

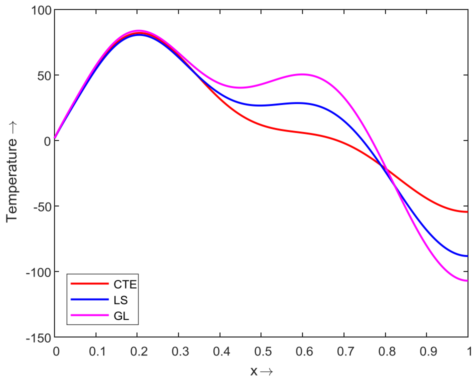


Fig.3.Variation of temperature vs. x for t=4.

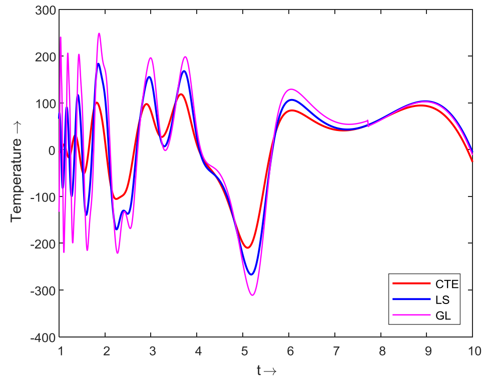


Fig.4.Variation of temperature vs. time.

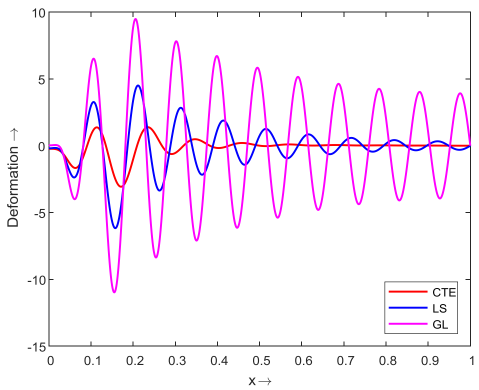


Fig.5.Variation of deformation vs. x for t=1.

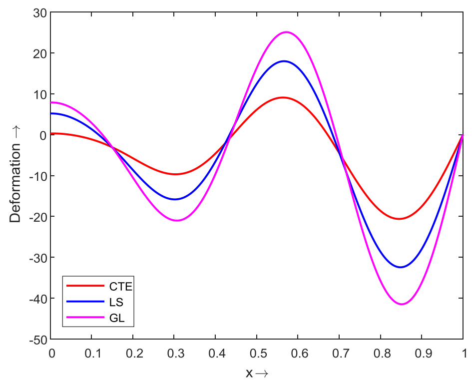


Fig.6.Variation of deformation vs. x for t=3.

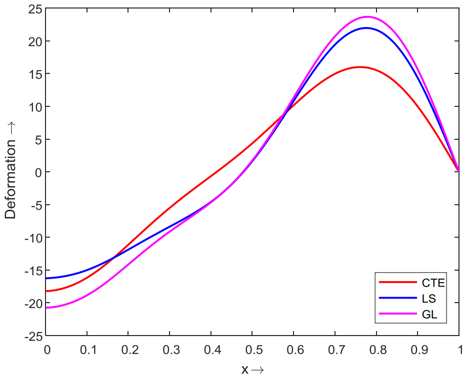


Fig.7.Variation of deformation vs. x for t=4.

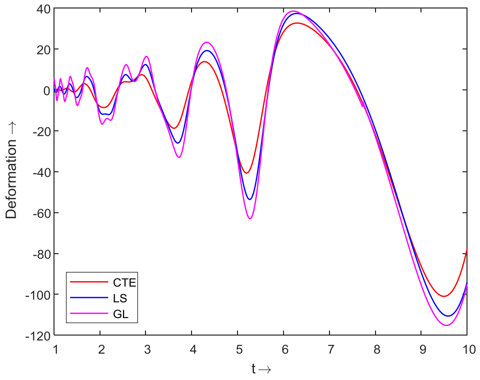


Fig.8. Variation of deformation vs. time.

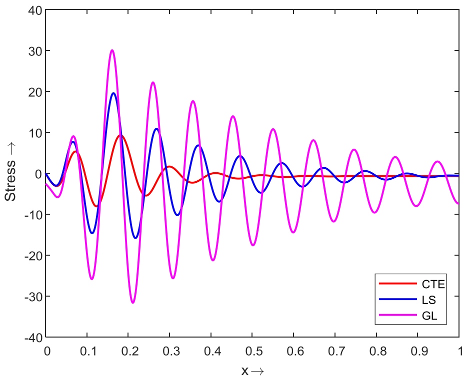


Fig.9.Variation of stress vs. x for t=1.

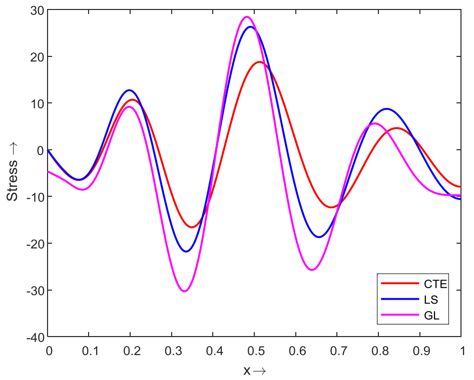


Fig.10.Variation of stress vs. x for t=2.

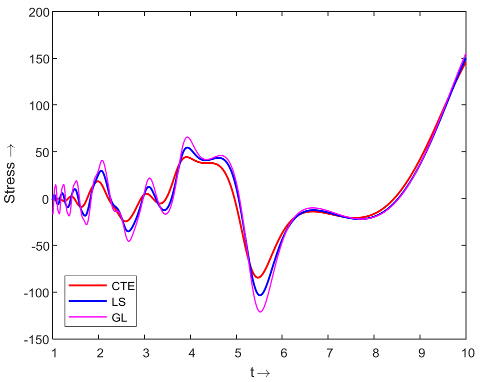


Fig.11.Variation of stress vs. time.

VI. CONCLUSION

The nature of the wave for variation of temperature for all the models shows that the thermal wave propagates with finite speed. The waves are more or less similar when t varies in $(1<t<10)$ with different specified values of x in $(0<x<1)$ however, the curves assume different shapes when x-varies and different values of t are specified.

Variation of deformation and stress when represented by curves for all the three models show similar observations as in the case of temperature graphs.

It has been further observed that if the relaxation parameters $\alpha=0$, $\alpha_0=\tau\neq 0$ for LS model and $\tau=0$, $\alpha=\alpha_0\neq 0$ for GL model are the same and the curves are nearly identical then the increase of the relaxation parameters α, α_0 amplifies the amplitude of the wave in general when compared to LS model and vice-versa. When the elastic field is coupled with other field such as i)viscous field ii)thermal field iii) magnetic field iv)electric field or the elastic field along with combination of the fields mentioned above, a set of coupled partial differential equations are formed. The decoupling of the equations are sometimes made by the method of potential function technique or by elimination process after use of integral transform technique, in order to obtain solution in closed form. It may be noted that in the eigenvalue approach method, in order to decouple the equations, as may be seen in the Appendix that we need to calculate the eigenvalues and the corresponding eigenvectors of the resulting matrix A of the vector-matrix equation. A set of decoupled scalar differential equation is formed, from which a typical equation is solved, which is generally a known solution since it is depending on the differential operator of the vector-matrix equation. Finally, this solution along with eigenvalues and eigenvectors form the complete solution of the desired equation as given in the equation (A-9). In the present problem, the use of elimination process of solution would generate eight constants to be determined from the four boundary conditions of equation (30) .Thus a set of four relations are necessary interrelating the constants. The eigenvalue approach would generate four constants to be determined from the four boundary conditions. Thus an extensive algebra can be avoided.

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APPENDIX

Solution of the vector-matrix differential equation

In this section we present the theory for decoupling a vector-matrix equation into a set of differential equations for the desired solutions.

We consider a vector-matrix differential equation as occurred in equation (25) as

$$\frac{d\underset{\sim}{v}}{dx} = A\underset{\sim}{v} \tag{A-1}$$

where $\underset{\sim}{v} = [v_1, v_2, ..., v_n]^T$ and $A = \left(a_{ij}\right), i, j = 1, 2, 3, ..., n$ (say) (A-2)

are real vector and matrix respectively.

Let $A = V\Lambda V^{-1}$ (A-3)

where $\Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ is a diagonal matrix whose elements $\lambda_1, \lambda_2, ..., \lambda_n$ are distinct eigenvalues of A. Let $\underset{\sim}{V_1}, \underset{\sim}{V_2}, \underset{\sim}{V_3}, ..., \underset{\sim}{V_n}$ be the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ respectively and

$$\begin{aligned} \underset{\sim}{V} &= [\underset{\sim}{V_1}, \underset{\sim}{V_2}, \underset{\sim}{V_3}, ...\underset{\sim}{V_n}] \\ &= x_{ij}, (say), \quad i, j = 1, 2, 3, ..., n \end{aligned} \tag{A-4}$$

Using (A-3) in (A-1) and premultiplying the resulting equations by V^{-1}, we get

$$V^{-1}\frac{d\underset{\sim}{v}}{dx} = V^{-1}(V\Lambda V^{-1})\underset{\sim}{v} = \Lambda(V^{-1}\underset{\sim}{v})$$

or, $\frac{d}{dx}(V^{-1}\underset{\sim}{v}) = \Lambda(V^{-1}\underset{\sim}{v})$

If we define $\underset{\sim}{y} = [y_1, y_2, y_3, ..., y_n]^T = V^{-1}\underset{\sim}{v}$, (A-5) we need to solve the equation

$$\frac{d\underset{\sim}{y}}{dx} = \Lambda \underset{\sim}{y}$$

In explicit form, this equation can be written as

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \qquad \text{(A-6)}$$

This is a set of n decoupled ordinary differential equations.
Consider a typical equation from this set as

$$\frac{dy_r}{dx} = \lambda_r y_r$$

The solution of the equation is

$$y_r = C_r \exp(\lambda_r x) \qquad \text{(A-7)}$$

Thus $\underset{\sim}{y}$ can be determined by taking $r = 1, 2, 3, \ldots\ldots, n$; where C_r are constants to be determined from the initial condition.

Since from (A-5), $\underset{\sim}{v} = V\underset{\sim}{y}$, we write $\underset{\sim}{v} = \sum_{r=1}^{n} V_r y_r$, which can be written in explicit form as

$$\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} x_{11} \\ x_{21} \\ \cdot \\ \cdot \\ x_{n1} \end{bmatrix} y_1 + \begin{bmatrix} x_{12} \\ x_{22} \\ \cdot \\ \cdot \\ x_{n2} \end{bmatrix} y_2 \ldots\ldots + \begin{bmatrix} x_{1n} \\ x_{2n} \\ \cdot \\ \cdot \\ x_{nn} \end{bmatrix} y_n \qquad \text{(A-8)}$$

Substituting (A-7) in (A-8) we get the complete solution in the form

$$v_r = C_1 x_{r1} \exp(\lambda_1 x) + C_2 x_{r2} \exp(\lambda_2 x) + \ldots\ldots + C_n x_{rn} \exp(\lambda_n x); \quad r = 1, 2, 3, \ldots\ldots, n \qquad \text{(A-9)}$$

The theory may be extended when the vector-matrix equation is of the form: vide [31].
As for example, we may consider equations of the form

$\frac{d\underset{\sim}{v}}{dx} = A\underset{\sim}{v} + \underset{\sim}{f}$, $L\underset{\sim}{v} = A\underset{\sim}{v}$, $L\underset{\sim}{v} = A\underset{\sim}{v} + \underset{\sim}{f}$ and others.

where $\mathrm{L} = \frac{\mathrm{d}^2}{\mathrm{dx}^2} + \frac{1}{\mathrm{x}}\frac{\mathrm{d}}{\mathrm{dx}} - \frac{\mathrm{n}^2}{\mathrm{x}^2}$ may be a Bessel operator or other form of operator.