Graphical Equations on Neighbourhood Chromatic Domination

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Abstract—*Equations connecting two parameters of a graph have already been studied. For example,* $\boldsymbol{\gamma(G)+\chi(G)=n}$ *or* $\boldsymbol{n-1}$ *or* $\boldsymbol{\Delta(G)+\chi(G)=n}$ *or* $\boldsymbol{n-1}$. *A subset S of G is called a neighbourhood chromatic dominating set if S is a dominating set and* $\boldsymbol{\chi}(<N(\boldsymbol{S})>)=\boldsymbol{\chi(G)}$. *The minimum cardinality of a neighbourhood chromatic dominating set of* $\boldsymbol{G}$ *is called the neighbourhood chromatic domination number of G and is denoted by* $\boldsymbol{\gamma_{nchd}(G)}$. *In this paper, graph equation* $\boldsymbol{\gamma_{nchd}(G)+\Delta(G)=n}$ *is solved for* $\boldsymbol{\Delta(G)=1}$ *or* $\boldsymbol{2}$ *or* $\boldsymbol{3}$ *or* $\boldsymbol{n-2}$. *Further* $\boldsymbol{\gamma_{nchd}(G)+\Delta(G)=n-1}$ *is solved for* $\boldsymbol{\Delta(G)=n-3}$.

Keywords—*Dominating set, domination number, neighbourhood chromatic dominating set, neighbourhood chromatic domination number.*

I. INTRODUCTION

Let $G=(V,E)$ be a simple, finite and undirected graph. Throughout this paper $G\neq\overline{K_n}$ and order of G is at least 2. A dominating set D of $V(G)$ is called a neighbourhood chromatic dominating set if $\chi(<N(D)>)=\chi(G)$. The minimum cardinality of a neighbourhood chromatic dominating set of G is called the neighbourhood chromatic domination number of G and is denoted by $\gamma_{nchd}(G)$. $\gamma_{nchd}(G)=n$if and only if $G=K_2\cup(n-2)K_1$. Therefore $\gamma_{nchd}(G)+\Delta(G)=n+1$ can be solved with $\Delta(G)=1$. The same equation can be solved with $\Delta(G)=2$. The solution is $C_3\cup(n-3)K_1$ or $P_3\cup(n-3)K_1$. We consider the equations in which $\gamma_{nchd}(G)+\Delta(G)=n$ when $\Delta(G)=1$ or 2 or 3 or $n-2$ and characterize the graphs satisfying the above equation. Further the equation $\gamma_{nchd}(G)+\Delta(G)=n-1$ is solved for $\Delta(G)=n-3$.

For further notations and terminology refer to [[2], [3]].

II. PRIOR RESULTS

Definition 1. [1] A subset D of V is said to be a **neighbourhood chromatic dominating set** (**nchd-set**) if D is a dominating set and $\chi(<N(D)>)=\chi(G)$. The minimum cardinality of a neighbourhood chromatic dominating set of a graph G is called the **neighbourhood chromatic domination number** (**nchd-number**) of G and it is denoted by $\gamma_{nchd}(G)$.

Theorem 1. [1] Let G be a triangle free graph. If$\gamma_{nchd}(G)=2$, then $\chi(G)=2$.

III.GRAPHS EQUATION WITH RESPECT TO $\boldsymbol{\Delta(G)}$

Proposition 1. Let G be a graph with $\Delta(G)=1$. Then $\gamma_{nchd}(G)=n-1$ if and only if $G=2K_2\cup(n-4)K_1$.

Proof. Let G be a graph of order n with $\Delta(G)=1$. Let $\mathcal{G}_1,\mathcal{G}_2,\dots,\mathcal{G}_k$ be the components of G such that $\sum_{i=1}^{k}|\mathcal{G}_i|=|G|$. Suppose $\gamma_{nchd}(G)=n-1$. Since $\Delta(G)=1$, G must have at least one non-trivial component.

Claim 1. G has at least two non-trivial components.

Suppose G has exactly one non-trivial component. Then $\gamma_{nchd}(G)=n$, a contradiction. Hence the claim 1.

Claim 2. Number of non-trivial components of G is 2 and remaining components are isolate.

Suppose G contains three non-trivial components. Since $\Delta(G)=1$, the non-trivial components are K_2. Then $S=V(G)-\{y,z\}$ is a nchd-set of G, where $y\in V(\mathcal{G}_i)$ and $z\in V(\mathcal{G}_j)$ with $|S|\leq n-2$, a contradiction. Hence G contains only two non-trivial components and the remaining vertices are isolates.

Therefore, G is isomorphic to $2K_2 \cup (n-4)K_1$. The converse is obvious.

Proposition 2. Let G be a graph with $\Delta(G)=2$. Then $\gamma_{nchd}(G)=n-2$ if and only if $G=P_3 \cup K_2 \cup (n-5)K_1$, $C_3 \cup K_2 \cup (n-5)K_1$, $C_4 \cup (n-4)K_1$, $P_4 \cup (n-4)K_1$ or $C_5 \cup (n-5)K_1$.

Proof. Let G be a graph of order n with $\Delta(G)=2$. Let $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k$ be the components of G such that $\sum_{i=1}^{k}|\mathcal{G}_i| = |G|$. Suppose $\gamma_{nchd}(G)=n-2$. Since $\Delta(G)=2$, G must have at least one non-trivial component.

Claim 1. Number of non-trivial components of G is either 1 or 2 and remaining components are isolates.

Suppose G contains three non-trivial components. Let $\mathcal{G}_i$ be a graph which contains the vertex u such that $\deg_{\mathcal{G}_i}(u)=\Delta=2$and $\chi(<\mathcal{G}_i>)=\chi(G)$. Clearly, $S_1=V(G)-\{x,y,z\}$, where $x\in\mathcal{G}_i$, $y\in\mathcal{G}_j$ and $z\in\mathcal{G}_t$, is a nchd-set of G with $|S_1|<n-2$, a contradiction. Hence G contains either one or two non-trivial components.

Claim 2. Exactly one component of G contains a Δ-vertex v.

Suppose let us assume that u_1 and u_2 are the vertices of $\mathcal{G}_i$ and $\mathcal{G}_j$ respectively, such that $\deg_{\mathcal{G}_i}(u_1)=\Delta=\deg_{\mathcal{G}_j}(u_2)$. Without loss of generality assume that $\chi(\mathcal{G}_i)\geq\chi(\mathcal{G}_j)$. Clearly, $S_2=V(G)-\{x,y,z\}$, where $x\in V(\mathcal{G}_i)$, $y,z\in V(\mathcal{G}_j)$ and $yz\notin E(\mathcal{G}_j)$, is a nchd-set of G with $|S_2|<n-2$, a contradiction. Hence the claim 2.

Case 1.G has two non-trivial components, say $\mathcal{G}_1, \mathcal{G}_2$ and the components $\mathcal{G}_i$, $3\leq i\leq k$ are isolates.

By claim 2, $\mathcal{G}_1$ or $\mathcal{G}_2$ is K_2. Let $\mathcal{G}_2=K_2$. Since $\Delta(G)=2$, it follows that $\mathcal{G}_1$ is either a path on s vertices or a cycle on s vertices. Thus, $s\geq 3$.

Claim 3.$s=3$

Suppose $s\geq 4$. Then $S_3=V(G)-\{x,y,z\}$where $x,y\in V(\mathcal{G}_1)$, $xy\in E(\mathcal{G}_1)$ and $z\in V(\mathcal{G}_2)$. Clearly, S_3 is a nchd-set of G with $|S_3|<n-2$, a contradiction. Thus $s\leq 3$. Therefore, $s=3$. Hence $\mathcal{G}_1$ is either P_3 or C_3.

If $\mathcal{G}_1\cong P_3$, then G is isomorphic to $P_3\cup K_2\cup(n-5)K_1$.
If $\mathcal{G}_1\cong C_3$, then G is isomorphic to $C_3\cup K_2\cup(n-5)K_1$.

Case 2. G has exactly one non-trivial component, say $\mathcal{G}_1$ and the components $\mathcal{G}_j$, $2\leq j\leq k$ are isolates.

Since $\Delta(G)=2$, it follows that $\mathcal{G}_1$ is either a path on s vertices or a cycle on s vertices. Thus, $s\geq 3$.

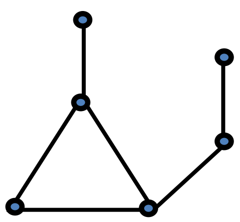
Claim 4.$s=4$or 5.

Since $\gamma_{nchd}(G)=n-2$, $\mathcal{G}_1$ is neither a path P_3 nor a cycle C_3. Let $s\geq 6$. Let $V(\mathcal{G}_1)=\{u_1,u_2,\dots,u_s\}$. Then $S_4=V(G)-\{u_1,u_4,u_5\}$, where $u_4u_5\in E(\mathcal{G}_1)$, is a nchd-set of G with $|S_4|<n-2$, a contradiction. Therefore, $4\leq s\leq 5$.

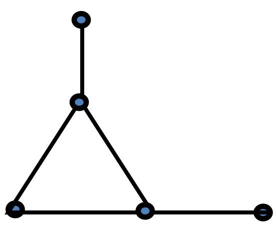
Let $\mathcal{G}_1$ be isomorphic to P_5. Since $\gamma_{nchd}(\mathcal{G}_1)=2$, it follows that $\gamma_{nchd}(G)=n-3<n-2$, a contradiction. Hence $\mathcal{G}_1$ is isomorphic toP_4, C_4 or C_5.
If $\mathcal{G}_1$ is isomorphic to P_4, then G is isomorphic to $P_4\cup(n-4)K_1$.
If $\mathcal{G}_1$ is isomorphic to C_4, then G is isomorphic to $C_4\cup(n-4)K_1$.
If $\mathcal{G}_1$ is isomorphic to C_5, then G is isomorphic to $C_5\cup(n-5)K_1$.
The converse is obvious.

Proposition 3. Let G be a graph with $\Delta(G)=3$. Then $\gamma_{nchd}(G)=n-3$ if and only if G is one of the following graphs $G_i^*=G_i\cup(n-|G_i|)K_1$, $1\leq i\leq 22$, where

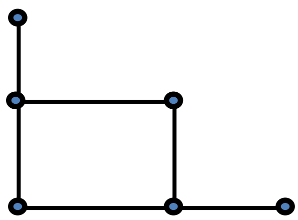
$G_1: K_4\cup K_2$; $G_2: K_{1,3}\cup K_2$;
$G_3: P_3 o K_1$; $G_4: K_3 o\, K_1$;
$G_5: (P_3+K_1)\cup K_2$; $G_6: K_{2,3}$



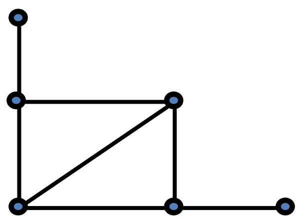
G_7



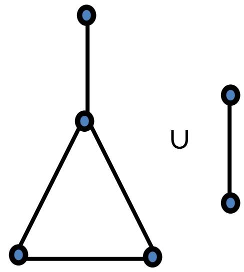
G_8



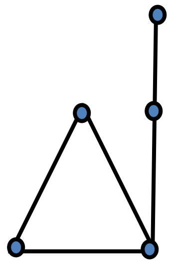
G_9



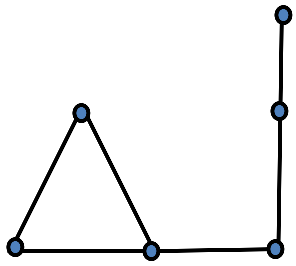
G_{10}



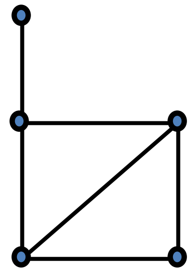
G_{11}



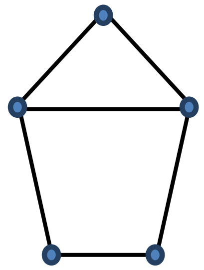
G_{12}



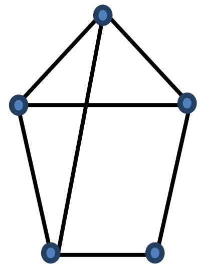
G_{13}



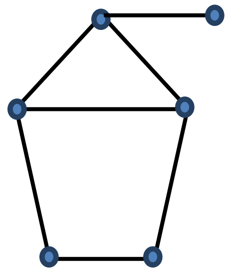
G_{14}



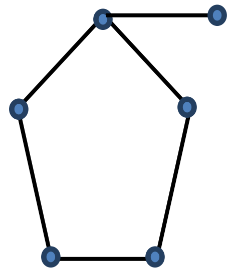
G_{15}



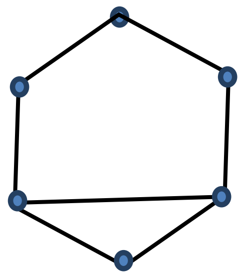
G_{16}



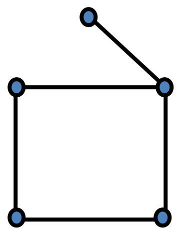
G_{17}



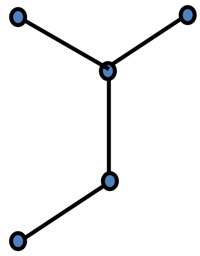
G_{18}



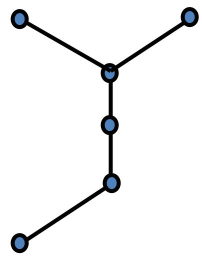
G_{19}



G_{20}



G_{21}



G_{22}

Proof. Let G be a graph of order n with $\Delta(G) = 3$. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ be the components of G such that $\sum_{i=1}^{k} |\mathcal{G}_i| = |G|$.

Suppose $\gamma_{nchd}(G) = n - 3$. Since $\Delta(G) = 3$, G must have at least one non-trivial component.

Claim 1. Number of non-trivial components of G is either 1 or 2 and remaining components are isolates.

Suppose G contains three non-trivial components. Let $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ be such components of G. Let $\mathcal{G}_1$ be a graph which contains the vertex u such that $\deg_{\mathcal{G}_1}(u) = \Delta = 3$and $\chi(<\mathcal{G}_1>) = \chi(G)$. Then$S_1 = V(G) - \{x, y, z, w\}$, where $x \in N(u)$, $xy \in E(\mathcal{G}_1)$, $z \in \mathcal{G}_2$ and $w \in \mathcal{G}_3$, is a nchd-set of G with $|S_1| < n - 3$, a contradiction. Hence the claim 1.

Claim 2. Exactly one component of G contains a Δ-vertex v.

Suppose let us assume that u_1 and u_2 are the vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, such that $\deg_{\mathcal{G}_1}(u_1) = \Delta = \deg_{\mathcal{G}_2}(u_2)$. Without loss of generality assume that $\chi(\mathcal{G}_1) \geq \chi(\mathcal{G}_2)$. Clearly, $S_2 = V(G) - \{x, y, z, w\}$, where $x \in N(u_1)$, $y \in \mathcal{G}_1$, $xy \in E(\mathcal{G}_1)$,$z \in N(u_2)$, $w \in \mathcal{G}_2$ and $zw \in E(\mathcal{G}_2)$, is a nchd-set of G with $|S_2| < n - 3$, a contradiction. Hence the claim 2.

Case 1. G has two non-trivial components, say $\mathcal{G}_1, \mathcal{G}_2$ and the components $\mathcal{G}_i$, $3 \leq i \leq k$contains isolate.

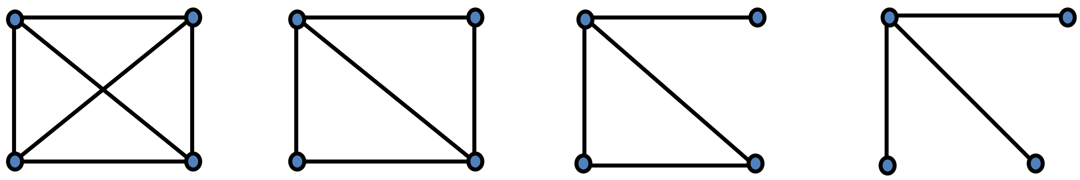
Let $\deg_{\mathcal{G}_1}(u_1) = \Delta = 3$. If there exists a vertex $v \in \mathcal{G}_2$ such that $\deg_{\mathcal{G}_2}(v) = 2$, then $S_3 = V(G) - \{x, y, z, w\}$, where $x \in N(u_1)$, $y \in \mathcal{G}_1$, $xy \in E(\mathcal{G}_1)$, $z, w \in \mathcal{G}_2$ and $zw \notin E(\mathcal{G}_2)$, is a nchd-set of G with $|S_3| < n - 3$, a contradiction.

Hence $\mathcal{G}_2 \cong K_2$. Since $\Delta = 3$, $|\mathcal{G}_1| \geq 4$.

Claim 3. $|\mathcal{G}_1| = 4$

Suppose assume that $\mathcal{G}_1$ is a graph of order at least 5. Let u be a Δ-vertex of $\mathcal{G}_1$. Then $S_4 = V(G) - \{x,y,z,w\}$, where $x,y,z \in \mathcal{G}_1$, $w \in \mathcal{G}_2$ and $< \{x,y,z\} > \ncong C_3$ and P_3, is a nchd-set of G with $|S_4| < n-3$, a contradiction. Hence the claim 3.

Since $\Delta = 3$and $|\mathcal{G}_1| = 4$, $\mathcal{G}_1$ is isomorphic to one of the graphs given below:



H_1 H_2 H_3 H_4

If $\mathcal{G}_1 \cong H_1$, then G is isomorphic to G_1^*. If $\mathcal{G}_1 \cong H_2$, then G is isomorphic to G_3^*. If $\mathcal{G}_1 \cong H_3$, then G is isomorphic to G_{11}^*. If $\mathcal{G}_1 \cong H_4$, then G is isomorphic to G_2^*.

Case 2. Ghas exactly one non-trivial component, say $\mathcal{G}_1$ and the components $\mathcal{G}_i$, $2 \le i \le k$ are isolates.

Let u be a Δ-vertex in $\mathcal{G}_1$. Since $\gamma_{nchd}(G) = n-3$and $\gamma_{nchd}(\mathcal{G}_1) \ge 2$, it follows that $|\mathcal{G}_1| \ge 5$.

Claim 4. $5 \le |\mathcal{G}_1| \le 6$.

Suppose assume that $\mathcal{G}_1$ is a graph of order at least 7. Then $S_5 = V(G) - \{x,y,z,w\}$, where $x,y \in N(u)$, $z,w \notin N[u]$ and $zw \notin E(\mathcal{G}_1)$, is a nchd-set of G with $|S_5| < n-3$, a contradiction. Hence the claim 4.

Let A be the set of all pendent vertices in $\mathcal{G}_1$.

Case 3. $|\mathcal{G}_1| = 5$.

Then A has at most three pendent vertices.

Subcase 3(a). $|A| = 3$

Let $x,y,z \in A$. Then there exist non-pendent vertices $u_1,u_2 \in \mathcal{G}_1$ such that $< \{u_1,u_2\} >$ is connected. By the hypothesis, either u_1 or u_2 is a Δ-vertex. Then the graph $\mathcal{G}_1$ is isomorphic to G_{21}. Hence G is isomorphic to G_{21}^*.

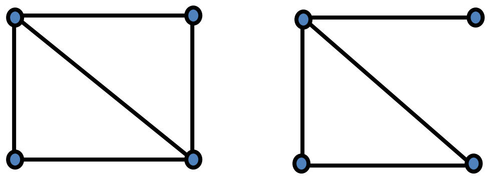
Subcase 3(b). $|A| = 2$.

Let $x,y \in A$. Then there exist non-pendent vertices $u_1,u_2,u_3 \in \mathcal{G}_1$ such that $< \{u_1,u_2,u_3\} >$ is connected.

Let $< \{u_1,u_2,u_3\} > \cong P_3$. Since $u_1,u_3 \notin A$, $xu_1 \in E(\mathcal{G}_1)$ and $yu_3 \in E(\mathcal{G}_1)$, a contradiction to the hypothesis $\Delta = 3$. Hence $< \{u_1,u_2,u_3\} > \cong C_3$. Since $\Delta = 3$, x and y adjacent to u_i and u_j, respectively, $i \neq j$, for $1 \le i,j \le 3$. Therefore G is isomorphic to G_8^*.

Subcase 3(c). $|A| = 1$

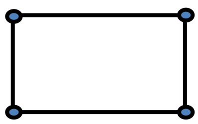
Let $x \in A$. Then there exist non-pendent vertices $u_1,u_2,u_3,u_4 \in \mathcal{G}_1$ such that $< \{u_1,u_2,u_3,u_4\} >$ is connected. If girth of $\mathcal{G}_1$is 3, then $< \{u_1,u_2,u_3,u_4\} >$ is isomorphic to one of the graphs given below:



H_5 H_6

For H_5, as $\Delta = 3$, $xu_i \in E(\mathcal{G}_1)$ for some i where $\deg_{H_5}(u_i) = 2$. Then the graph G is isomorphic to G_{14}^*. For H_6, as u_i's are non-pendent, $xu_i \in E(\mathcal{G}_1)$ where $\deg_{H_6}(u_i) = 1$. Then the graph G is isomorphic to G_{12}^*.

If girth of $\mathcal{G}_1$is 4, then $< \{u_1,u_2,u_3,u_4\} >$ is isomorphic to the graph given below:

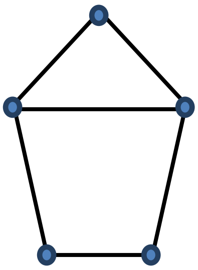


H_7

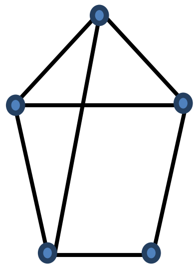
For this graph, $xu_i \in E(\mathcal{G}_1)$ for some i, as $\Delta = 3$. Then the graph G is isomorphic to G_{20}^*.

Subcase 3(d). $|A| = 0$

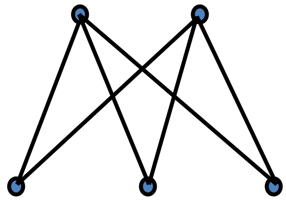
As $\Delta = 3$, $\mathcal{G}_1$ is not isomorphic to C_5. Thus the graph $\mathcal{G}_1$ is isomorphic to one of the graphs given below:



H_8



H_9



H_{10}

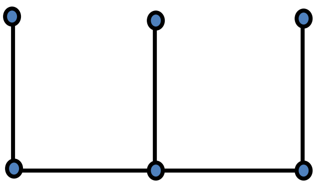
Hence G is isomorphic to G_{15}^*, G_{16}^*

Case 4. $|\mathcal{G}_1| = 6$

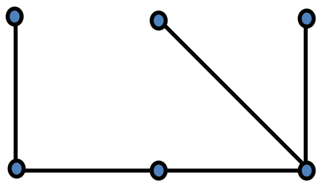
Then A has at most three pendent vertices.

Subcase 4(a). $|A| = 3$

Let $x, y, z \in A$. Then there exist non-pendent vertices $v_1, v_2, v_3 \in \mathcal{G}_1$ such that $< \{v_1, v_2, v_3\} >$ is connected. Let $< \{v_1, v_2, v_3\} > \cong P_3$. Since v_i's are non-pendent, the graph $\mathcal{G}_1$ is isomorphic to one of the graphs given below:



H_{11}



H_{12}

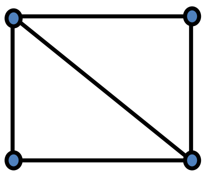
Hence G is isomorphic to G_3^* or G_{22}^*.

Let $< \{v_1, v_2, v_3\} > \cong C_3$. Since $\Delta = 3$, the graph G is isomorphic to G_4^*.

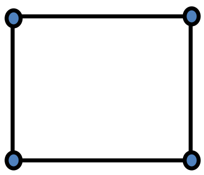
Subcase 4(b). $|A| = 2$.

Let $x, y \in A$. Then there exist non-pendent vertices $v_1, v_2, v_3, v_4 \in \mathcal{G}_1$ such that $< \{v_1, v_2, v_3, v_4\} >$ is connected.

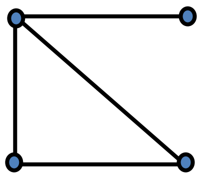
Let $< \{v_1, v_2, v_3, v_4\} > \cong P_4$. Since $v_1, v_4 \notin A$, $xv_1 \in E(\mathcal{G}_1)$ and $yv_4 \in E(\mathcal{G}_1)$, a contradiction to the hypothesis $\Delta = 3$. Since $\mathcal{G}_1$ is connected, $< \{v_1, v_2, v_3, v_4\} > \ncong K_4$. But the graph $< \{v_1, v_2, v_3, v_4\} >$ is isomorphic to one of the graphs given below:



H_{13}



H_{14}



H_{15}

For graph H_{13}, x and y must be adjacent to v_i and v_j respectively where $i \neq j$ and $\deg_{H_{13}}(v_i) = 2 = \deg_{H_{13}}(v_j)$. Thus the graph G is isomorphic to G_{10}^*.

For graph H_{14}, if x and y are adjacent to v_i and v_{i+1}, then $\gamma_{nchd}(G) \neq n - 3$, a contradiction. Hence the graph G is isomorphic to G_9^*.

For graph H_{15}, as v_i's are non-pendent, if $xv_4, yv_4 \in E(\mathcal{G}_1)$, then $\gamma_{nchd}(G) \neq n-3$, a contradiction. Hence the graph G is isomorphic to G_7^*.

Subcase 4(c). $|A| = 1$

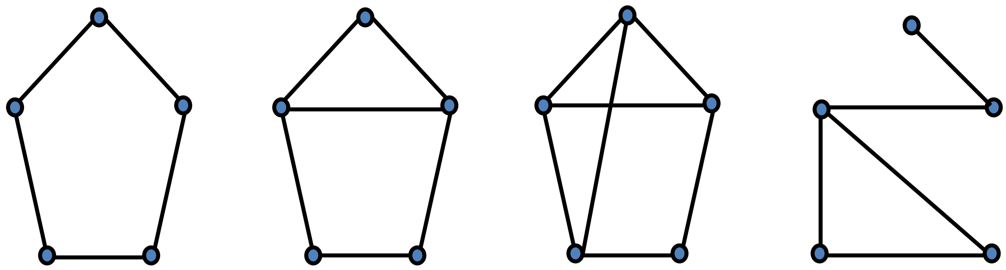
Let $x \in A$. Then there exist non-pendent vertices $v_1, v_2, v_3, v_{4,} v_5 \in \mathcal{G}_1$ such that $< \{v_1, v_2, v_3, v_4, v_5\} >$ is connected.

If the graph $< \{v_1, v_2, v_3, v_4, v_5\} >$ is isomorphic to one of the graphs given below:

H_{16} H_{17}

then $xv_5 \in$ ... where $\deg_{H_{16}}(v_5) = 1 = \deg_{H_{17}}(v_5)$, since ... non-pendent. But this implies that $\gamma_{nchd}(G) \neq n-3$, a contradiction.

Thus $< \{v_1, v_2, v_3, v_4, v_5\} >$ is isomorphic to one of the graphs given below:

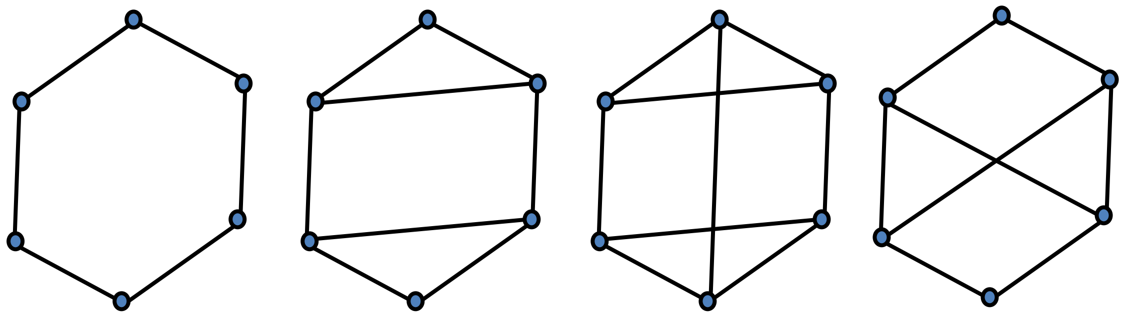


H_{18} H_{19} H_{20} H_{21}

For graph H_{20}, x must be adjacent to the vertex v_i for some i where $\deg_{H_{20}}(v_i) = 2$. This implies $\gamma_{nchd}(G) < n-3$, a contradiction. Hence the graph G is isomorphic to G_{18}^*, G_{17}^* or G_{13}^*.

Subcase 4(d). $|A| = 0$

Suppose that the graph $\mathcal{G}_1$ is isomorphic to one of the graphs given below:



H_{22} H_{23} H_{24} H_{25}

For ... graphs, $\gamma_{nchd}(G) \neq$ ... a contradiction. Thu... aph G is isomorphi... .

The converse is obvious.

Proposition 4. Let G be a graph with $\Delta(G) = n-2$. Then $\gamma_{nchd}(G) + \Delta(G) = n$ if and only if G is connected.

Proof. Let G be a graph with $\gamma_{nchd}(G) = 2$and $\Delta(G) = n-2$. Suppose that G is disconnected. Let $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k$ be the components of G, $k \geq 2$. Since $\Delta(G) = n-2$, there exist two components of G such that $\mathcal{G}_1$ contains Δ-vertex, say u, and $\mathcal{G}_2$ contains an isolate, say v.

Clearly, $D = \{u, x, v\}$ is a γ_{nchd}-set of G, as $\mathcal{G}_1$ has the vertex of degree $|\mathcal{G}_1| - 1$. Therefore $\gamma_{nchd}(G) = |D| = 3$, a contradiction. Hence G is connected.

Conversely, suppose that G is connected and $\Delta(G) = n-2$. Let u be a Δ-vertex. Then there exists a vertex $v \in G$ such that $v \notin N[u]$. Since G is connected, $vx \in E(G)$where $x \in N(u)$. Then $\{u,x\}$ is a nchd-set of G. Thus $\gamma_{nchd}(G) \leq 2$. But $\gamma_{nchd}(G) \geq 2$. Hence $\gamma_{nchd}(G) = 2$.

Proposition 5. Let G be a graph with $\Delta(G) = n-3$. Then $\gamma_{nchd}(G) + \Delta(G) = n-1$ if and only if G is connected such that one of the following conditions hold:
Let u be a Δ-vertex and $v, w \notin N[u]$.

(i). Let $N(v) \cap N(w) \neq \phi$. Then every vertex $v_i \in N(v) \cap N(w)$ is adjacent to at most $|N(u)| - 2$ vertices excluding $v$and w.

(ii). Let $N(v) \cap N(w) = \phi$.

(a). Let $vw \notin E(G)$ and let $N(u) - [N[v] \cup N[w]] = \phi$. Then $2 \leq |N(v)| \leq n-5$ and $2 \leq |N(w)| \leq n-5$. If $u_t u_k \in E(G)$ where $u_t \in N(v)$ and $u_k \in N(w)$, then either $u_t v_s \in E(G)$ and $u_k w_s \in E(G)$ for every $v_s \in N(v) - \{u_t\}$ and $w_s \in N(w) - \{u_k\}$, or, exactly one vertex, say $v_j \neq u_t \in N(v)$, and exactly one vertex, say $w_j \neq u_k \in N(w)$, which is adjacent with every other vertex of $N(v)$ and $N(w)$, respectively.

If $N(u) - [N[v] \cup N[w]] \neq \phi$ and if $v_i w_j \notin E(G)$ for every $v_i \in N(v)$ and $w_j \in N(w)$, then $|N(u)| = 4$, $|N(v)| = 2$, $|N(w)| = 1$ and $< N(v) >$ is non-independent set. Also $u_i \in N(u) - [N[v] \cup N[w]]$ is adjacent with exactly one vertex of $N(v)$and $N(w)$.

If $v_i w_j \in E(G)$ for some $v_i \in N(v)$ and some $w_j \in N(w)$, then $v_i v_k \in E(G)$ and $w_j w_k \in E(G)$ for every $v_k \in N(v)$ and $w_k \in N(w)$. Also u_i is adjacent with $v_i$or w_j.

(b). Let $vw \in E(G)$ and let $N(u) - [N[v] \cup N[w]] = \phi$. Then $2 \leq |N(v)| \leq n-5$ and $2 \leq |N(w)| \leq n-5$. Also $< N(u) >$ is a non-independent set.

If $u_i u_j \notin E(G)$ for every $u_i \in N(v)$and $u_j \in N(w)$, then $\chi(< N[u] >) = \chi(G)$.

If $u_i u_j \in E(G)$ for some $u_i \in N(v)$ and $u_j \in N(w)$ and
if $\chi(< N[u] >) = \chi(G)$, then $|N(u)| = 2$ or $|N(u)| \geq 5$.

If $\chi(< N[u] >) < \chi(G)$, then $u_i v_k \in E(G)$ and $u_j w_k \in E(G)$ for every $v_k \in N(v) - \{u_i\}$ and $w_k \in N(w) - \{u_j\}$.

If $N(u) - [N[v] \cup N[w]] \neq \phi$, then either $|N(v)| \geq 1$ or $|N(w)| \geq 1$ or both.

If $u_i u_j \in E(G)$ where $u_i \in N(v)$ and $u_j \in N(w)$, then $u_i v_k \in E(G)$ and $u_j w_k \in E(G)$ for every $v_k \in N(v) - \{u_i\}$ and $w_k \in N(w) - \{u_j\}$. Also either $u_i x_j \in E(G)$ or $u_j x_j \in E(G)$ or both where $x_j \in N(u) - [N[v] \cup N[w]]$.

Proof. Let G be a graph with $\Delta(G) = n-3$and $\gamma_{nchd}(G) = 2$. Following the argument in Proposition 4,G is connected. Let u be a Δ-vertex. Since $\Delta(G) = n-3$, there exist two vertices $v, w \in G$ such that $v, w \notin N[u]$.

Case 1.$N(v) \cap N(w) \neq \phi$

Let $N(v) \cap N(w) = \{v_1, v_2, \ldots, v_k\}$where $v_i \in N(u)$, $1 \leq i \leq k$. Since $\Delta(G) = n-3$, every vertex $v_i \in N(v) \cap N(w)$ is adjacent to at most $|N(u)| - 2$ vertices excluding v and w.

Case 2.$N(v) \cap N(w) = \phi$

Subcase 2(a).$vw \notin E(G)$

As G is connected, $|N(v)| \geq 1$and $|N(w)| \geq 1$.

Subcase 2(a)(i).$N(u) - [N[v] \cup N[w]] = \phi$

Let $|N(w)| = 1$and $|N(v)| = n-4$. Then $N(w) = \{v_1\}$. If v_1 is adjacent to any of the vertices in $N(u)$and $\chi(< N[u] - \{v_1\} >) < \chi(G)$, then $\{v_1, v\}$ is the only γ-set of G. But $\chi(< N(\{v, v_1\}) >) < \chi(G)$.

Let $N(v) = \{u_1, u_2, \ldots, u_s\}$ and $N(w) = \{u_{s+1}, u_{s+2}, \ldots, u_{n-3}\}$where $2 \leq s \leq n-5$.

If $N(v)$ does not contain a full degree vertex or $N(w)$ does not contain a full degree vertex, then $D = \{u, v, w\}$ is a γ-set of G. For this graph, $\gamma_{nchd}(G) \neq 2$, a contradiction.

Therefore, there exists a vertex of $N(v)$, say u_1, which is adjacent to every other vertex of $N(v)$ and there exists a vertex of $N(w)$, say u_{s+1}, which is adjacent to every other vertex of $N(w)$.

Suppose that no vertex of $N(v)$ is adjacent with any vertex of $N(w)$. Since $uv \notin E(G)$, $uw \notin E(G)$ and $vw \notin E(G)$, it follows that $D_1 = \{u_1, u_{s+1}\}$ is a γ-set of G.But $\chi(< N(D_1) >) < \chi(G)$. Thus $\gamma_{nchd}(G) \neq 2$, a contradiction.

Hence there exists a vertex in $N(v)$, say u_t, which is adjacent with some vertex of $N(w)$, say u_k, where $t \leq s$ and $k \geq s+1$.

Suppose that u_t is not adjacent with every other vertex of $N(v)$, or, u_k is not adjacent with every other vertex of $N(w)$. Note that $u_t u_k \in E(G)$.

Suppose that u_1 is the only vertex of $N(v)$ which is adjacent with every other vertex of $N(v)$ and u_{s+1} is the only vertex of $N(w)$ which is adjacent with every other vertex of $N(w)$. Then there is nothing to prove, as $\{u_1, u_{s+1}\}$ is a γ_{nchd}-set of G.

Suppose that either $N(v)$ has at least two vertices but not every vertex which is adjacent with every other vertex of $N(v)$, or, $N(w)$ has at least two vertices but not every vertex which is adjacent with every other vertex of $N(w)$. Then $D_2 = \{u_1, u_{s+1}\}$ is a γ-set of G. Since u_t and u_k are not a full degree vertex of $N(v)$and $N(w)$, respectively, it follows that $\chi(< N(D_2) >) < \chi(G)$. Therefore $\gamma_{nchd}(G) \neq 2$, a contradiction.

Hence for this case, $u_t \in N(v)$ must be adjacent with every other vertex of $N(v)$ and $u_k \in N(w)$ must be adjacent with every other vertex of $N(w)$.

Subcase 2(a)(ii).$N(u) - [N[v] \cup N[w]] \neq \phi$

Then there exists a vertex $u_i \in N(u)$ such that $u_i \notin N[v]$ and $u_i \notin N[w]$ for some i. Note that $|N(u)| = n - 3$. Also v is adjacent to at least one point x of $N(u)$ and w is adjacent to at least one point $y$of $N(u)$, where $x \neq y$. Therefore, $|N(u) - [N[v] \cup N[w]]| \leq n - 5$. Let $N(v) = \{u_1, u_2, \dots, u_k\}$ and $N(w) = \{u_{k+1}, u_{k+2}, \dots, u_t\}$where $1 \leq k \leq t < n - 4$. Let $N(u) - [N[v] \cup N[w]] = \{u_{t+1}, u_{t+2}, \dots, u_{n-5}\}$.

Subsubcase 2(a)(ii)(a).$v_i w_j \notin E(G)$for every $v_i \in N(v)$ and $w_j \in N(w)$

Suppose that $|N(u)| = 3$. Then $|N(v)| = 1$and $|N(w)| = 1$. Clearly $\gamma_{nchd}(G) \neq 2$, a contradiction.

Suppose that $|N(u)| \geq 5$. Then $u_i \in N(u) - [N[v] \cup N[w]]$ is adjacent with exactly one vertices of $N(v)$, say v_t, and one vertex of$N(w)$, say w_t. Also $v_t v_k \in E(G)$ and $w_t w_k \in E(G)$ for every $v_k \in N(v)$and $w_k \in N(w)$. Otherwise there always exists a γ-set of cardinality greater than or equal to 3. Now let $D_3 = \{v_t, w_t\}$ be the γ-set of G. Since v and w receives the color of u, $\chi(< N[u] >) = \chi(G)$. Therefore $G - \{v_t\}$ or $G - \{w_t\}$ has the chromatic number less than the chromatic number of G. Since $v_t w_t \notin E(G)$, D_3 is not a nchd-set of cardinality 2, a contradiction. Therefore $|N(u)| = 4$.

Let $|N(u)| = 4$. Then either $|N(v)| = 1$and $|N(w)| = 1$, or, $|N(v)| = 2$ and $|N(w)| = 1$.

Suppose that $|N(v)| = 1$and $|N(w)| = 1$. Then $\gamma_{nchd}(G) \neq 2$, a contradiction.

Let $|N(v)| = 2$and $|N(w)| = 1$. Suppose that $< N(v) >$ is an independent set.

If $u_i \in N(u) - [N[v] \cup N[w]]$ is adjacent with every vertices of $N(v)$ and $N(w)$, then again $\gamma_{nchd}(G) \neq 2$, a contradiction. If $u_i \in N(u) - [N[v] \cup N[w]]$ is not adjacent with any of the vertices of $N(v)$ or $N(w)$ or both, then again $\gamma_{nchd}(G) \neq 2$, a contradiction. Therefore $< N(v) >$ is a non-independent set.

Now, suppose that $u_i \in N(u) - [N[v] \cup N[w]]$ is adjacent with every vertex of $N(v)$and $N(w)$. Then $D_4 = \{v_i, w_j\}$ is the only γ-set of G. It is easy to verify that $\chi(< N(D_4) >) < \chi(G)$ and hence $\gamma_{nchd}(G) \neq 2$, a contradiction. Therefore $u_i \in N(u) - [N[v] \cup N[w]]$ is adjacent with exactly one vertex of $N(v)$ and exactly one vertex of $N(w)$.

Subsubcase 2(a)(ii)(b).$v_i w_j \in E(G)$for some $v_i \in N(v)$ and $w_j \in N(w)$

Then $2 \leq |N(v)| \leq n - 6$and $2 \leq |N(w)| \leq n - 6$.

Suppose that $v_i v_k \notin E(G)$ and $w_j w_k \notin E(G)$ for some $v_k \in N(v)$ and $w_k \in N(w)$. Then clearly $D_5 = \{u, v_i, w_j\}$ is a γ-set of G. Therefore $\gamma_{nchd}(G) \geq 3$, a contradiction. Therefore, $v_i v_k \in E(G)$ and $w_j w_k \in E(G)$ for every $v_k \in N(v) - \{v_i\}$and $w_k \in N(w) - \{w_j\}$.

If $u_i \in N(u) - [N[v] \cup N[w]]$ is not adjacent with $v_i$and w_j, where $v_i \in N(v)$ and $w_j \in N(w)$, then $\gamma_{nchd}(G) \geq 3$, a contradiction. Therefore $u_i \in N(u) - [N[v] \cup N[w]]$ is adjacent with $v_i \in N(v)$ or $w_j \in N(w)$.

Subcase 2(b).$vw \in E(G)$

As G is connected and $vw \in E(G)$, either $|N(v)| \geq 2$ or $|N(w)| \geq 2$.

Subcase 2(b)(i).$N(u) - [N[v] \cup N[w]] = \phi$

If $|N(w)| = 1$and $|N(v)| = n - 4$, as in subcase 2(a)(i), either v_1 is not adjacent to any of the vertices in $N(u)$ or $\chi(< N[u] - \{v_1\} >) = \chi(G)$.

Let $N(v) = \{u_1, u_2, \dots, u_s\}$ and $N(w) = \{u_{s+1}, u_{s+2}, \dots, u_{n-3}\}$where $2 \leq s \leq n - 5$.

Suppose that $< N(u) >$ is independent. Since $vw \in E(G)$, it follows that C_5 as an induced subgraph and triangle free. Therefore $\gamma_{nchd}(G) \geq 3 \neq 2$. Hence $< N(u) >$ is a non-independent set.

Subsubcase 2(b)(i)(a).$u_i u_j \notin E(G)$for every $u_i \in N(v)$ and $u_j \in N(w)$

Suppose that $\chi(< N[u] >) < \chi(G)$. Then v or w must receive different color from u. Thus $G - \{v\}$ or $G - \{w\}$ or $G - \{v, w\}$ have a chromatic number less than the chromatic number of G. Clearly, $D_6 = \{u, v\}$ or $D_7 = \{u, w\}$ or $D_8 = \{x, y\}$ are the only γ-sets of G, where $x \in N(v)$ and $y \in N(w)$. But D_6 and D_7 are independent sets of G. Since $xy \notin E(G)$, D_8 is also an independent set of G. Thus $< N(D_6) >$ has $G - \{u, v\}$ as an induced subgraph and $< N(D_7) >$ has $G - \{u, w\}$ as an induced subgraph. Therefore D_6 and D_7 are not a nchd-set of G.

Suppose that G is a triangle free. Since $\gamma_{nchd}(G) = 2$ and by Theorem 1, $\chi(G) = 2$. But this is not possible, since $\chi(< N[u] >) \geq 2$. Therefore G contains a triangle.

Suppose that $\chi(< N[v] >) < \chi(< N[w] >)$. Then v may receive some of the color from $N(w)$ and w may receives the color of u. Thus $\chi(< N[u] >) = \chi(G)$, a contradiction. Therefore $\chi(< N[v] >) = \chi(< N[w] >)$.

Without loss of generality, assume that x and y receive the unique color. Then $< N(D_8) >$ has $G - \{x, y\}$ as an induced subgraph and hence $\chi(< N(D_8) >) < \chi(G)$. Therefore no dominating set of cardinality two is a nchd-set of G and hence $\gamma_{nchd}(G) > 2$, a contradiction. Therefore $\chi(< N[u] >) = \chi(G)$.

Subsubcase 2(b)(i)(b).$u_i u_j \in E(G)$for some $u_i \in N(v)$ and some $u_j \in N(w)$.

Let $\chi(< N[u] >) = \chi(G)$. Suppose that $|N(u)| = 3$ *or* 4 and both $< N(v) >$ and $< N(w) >$ are independent. Then clearly, $\gamma_{nchd}(G) \neq 2$, a contradiction. Therefore $|N(u)| = 2$or $|N(u)| \geq 5$.

Let $\chi(< N[u] >) < \chi(G)$. Then v or w must receive different color from u. Thus $G - \{v\}$ or $G - \{w\}$ or $G - \{v, w\}$ have a chromatic number less than the chromatic number of G. Suppose that $u_i v_k \notin E(G)$ or $u_j w_k \notin E(G)$ for every $v_k \in N(v) - \{u_i\}$and $w_k \in N(w) - \{u_j\}$. Then clearly $D_9 = \{u, v\}$ or $D_{10} = \{u, w\}$ or $D_{11} = \{x, y\}$ are the γ-sets of G, where $x \in N(v)$ and $y \in N(w)$. Then this proof is analogous to the proof of Subsubcase 2(b)(i)(a). Therefore $\gamma_{nchd}(G) \neq 2$, a contradiction. Therefore $u_i v_k \in E(G)$ and $u_j w_k \in E(G)$ for every $v_k \in N(v) - \{u_i\}$and $w_k \in N(w) - \{u_j\}$.

Subcase 2(b)(ii).$N(u) - [N[v] \cup N[w]] \neq \phi$

This is analogous to the proof of subcase 2(b)(i).

Let $u_i u_j \in E(G)$ where $u_i \in N(v)$and $u_j \in N(w)$. Suppose that u_i and u_j are not a full degree vertex of $N(v)$and $N(w)$, respectively. Then $\gamma_{nchd}(G) \neq 2$, a contradiction. Therefore u_i and u_j must be a full degree vertex of $N(v)$and $N(w)$, respectively.

Suppose that u_i and u_j are not adjacent with the vertices of $N(u) - [N[v] \cup N[w]]$. Then $\{u, u_i, u_j\}$ is a γ-set of G and hence $\gamma_{nchd}(G) \neq 2$, a contradiction. Therefore u_i or u_j is adjacent with every other vertex of $N(u) - [N[v] \cup N[w]]$.

The converse is obvious.

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