Identification of Patterns of Continued Fractions of $\sqrt{S}-\left[\sqrt{S}\right]$ Where s is a Square Free Positive Number

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Abstract:
It is observed in [3] any periodic continued fraction represents a quadratic irrational and vice versa. In this paper we try to identify the patterns of continued fractions of $\sqrt{S}-\left\lfloor\sqrt{S}\right\rfloor$ where S is a square free positive number.

Keywords: Continued fractions, Simple continued fraction, Periodic continued fraction, Quadratic irrational, Euclidean algorithm, Bisection method, False Position method, Iteration method and Newton -Raphson method.
Subject Classification: MSC 11A05, 11A55, 30B70, 40A15, 65L99.

Notations:

1. $\langle a_0, a_1, a_2, a_3, \cdots a_n\rangle$ Continued fraction expansion.
2. $[x_1]$ Integral part of the rational number x_1.

I. INTRODUCTION

Continued fractions provide much insight into mathematical problems, particularly into the nature of numbers [6]. A reference to continued fractions is found in the works of the Indian mathematician Aryabhatta. John wallis used for the first time the name "continued fraction" in his book Artihmetica Infinitorium, published in 1655. Christian Huygens used continued fractions for the purpose of approximating the correct design or the toothed wheels of a planetarium. Euler, Lambert and Lagrange were prominent amongst those who developed the theory of continued fractions[9,10].

Any eventually periodic continued fraction represents a quadratic irrational [3,4]. Conversely, Lagrange's theorem asserts that the continued fraction expansion of every quadratic irrational is eventually periodic. A purely periodic continued fraction represents a quadratic irrational of a special kind called a reduced quadratic irrational. A quadratic irrational is said to be reduced if it is greater than 1 and the other root of the quadratic equation that it satisfies, lies between -1 and 0 [3,4]. Conversely, the continued fraction expansion of a reduced quadratic irrational is purely periodic.

An expression of the form $a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3+\cfrac{b_4}{a_4+\cfrac{b_5}{\ddots}}}}$

where $a_1, a_2, \ldots, a_n, \ldots b_2, b_3, \ldots, b_n, \ldots$ be a series of numbers positive or negative is called a continued fraction.

The continued fraction is usually written as $a_1+\frac{b_2}{a_2+}\frac{b_3}{a_3+}\frac{b_4}{a_4+}\cdots$. The quantities $a_1, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \frac{b_4}{a_4}, \cdots$ are called the elements of the continued fraction. The fraction obtained by stopping at any particular stage is called a convergent. Thus the successive convergence in the continued fraction are a_1 ; $a_1+\frac{b_2}{a_2}$; $a_1+\frac{b_2}{a_2+}\frac{b_3}{a_3}$; $a_1+\frac{b_2}{a_2+}\frac{b_3}{a_3+}\frac{b_4}{a_4+}\cdots$ and so on and they are denoted by $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots, \frac{p_n}{q_n}, \ldots$. When the number of quotients $a_1, a_2, a_3, \ldots$ is finite the continued fraction is said to be terminating. If the number of quotients is unlimited, the fraction is called an infinite continued fraction [5, 8].

In the case of infinite continued fraction, if the sequence $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, ..., \frac{p_n}{q_n}, ...$ converges, the value of the fraction F is defined as $\lim_{n\to\infty} \frac{p_n}{q_n}$ and we write $F = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \frac{b_4}{a_4 +} \cdots = \lim_{n\to\infty} \frac{p_n}{q_n}$.

An expression of the form $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \cdots \frac{1}{a_n} ...$ where $a_1, a_2, a_3, ..., a_n, ...$ are positive integers and a_1 may be zero, is called a simple continued fraction. Here $a_1, a_2, a_3, ..., a_n, ...$ are called the partial quotients [5, 8].

II.BASIC CONCEPTS

2.1 The Continued Fraction Algorithm:[3, 4]

Suppose we wish to find continued fraction expansion of $x \in R$.

Let $x_1 = x$ *and set* $a_1 = [x_1]$.

Define $x_2 = \frac{1}{x_1 - [x_1]}$ *and set* $a_2 = [x_2]$. and $x_3 = \frac{1}{x_2 - [x_2]} \Rightarrow a_3 = [x_3]$...., $x_k = \frac{1}{x_{k-1} - [x_{k-1}]} \Rightarrow a_k = [x_k]$.....

This process is continued infinitely or to some finite stage till an $x_i \in N$ exists such that $a_i = [x_i]$.

Examples:

1. Continued fraction expansion of 674/313 = 2.15335 is [2; 6, 1, 1, 11, 2]
2. Continued fraction expansion of $\sqrt{2}$ and $\sqrt{24}$ are [1; 2, 2 , 2, 2, 2, 2,…] and [4; 1, 8, 1, 8, 1, 8, 1, 8,…]. Which is known as periodic continued fraction. The above periodic continued fractions are also denoted by $[1; , \overline{2}]$ and $[4; \overline{1, 8}]$.

2.2 *Convergence of a continued fraction:[3, 4, 5]*

Let $x = [a_1, a_2, a_3, ... a_n]$. The reduced fractions given below are called the convergence of x and are defined by

$$C_1 = \frac{p_1}{q_1} = a_1, \quad C_2 = \frac{p_2}{q_2} = a_1 + \frac{1}{a_2}, \quad C_3 = \frac{p_3}{q_3} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}, \; \ldots, \quad C_n = \frac{p_n}{q_n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots + \cfrac{1}{a_n}}}}$$

Where $\frac{p_n}{q_n}$ denote the nth convergent of x.

2.3 Periodic Continued fraction:[1, 3, 4]

An infinite simple continued fraction is said to be periodic continued fraction if there is an integer n such that $a_r = a_{n+r}$ for sufficiently large r. Thus the periodic continued fraction can be written in the form $\langle a_0, a_1, ..., a_j, \overline{a_{j+1}, a_{j+2}, ..., a_n} \rangle$. Where the bar over $a_{j+1}, a_{j+2}, ..., a_n$ indicates that the partial quotients are repeated indefinitely.

Examples:

$\langle \overline{2, 4} \rangle$ and $\langle 4, 1, \overline{2, 4} \rangle$. If x denote $\langle \overline{2, 4} \rangle$, then $x = 2 + \cfrac{1}{4 + \cfrac{1}{x}} = 2 + \frac{x}{4x+1} = \frac{9x+2}{4x+1} \Rightarrow 2x^2 - 4x - 1 = 0.$ Thus the periodic continued fraction $\langle \overline{2, 4} \rangle$ represents the quadratic equation $2x^2 - 4x - 1 = 0.$

If we discard the negative root we get $x = \frac{2+\sqrt{6}}{2}$.

In the second example consider $\langle 4,1,\overline{2,4}\,\rangle$ as ξ.

Then $\xi = \langle 4,1,\overline{2,4}\,\rangle = 4+\cfrac{1}{1+\cfrac{1}{x}} = 4+\cfrac{x}{1+x} = 4+\cfrac{\cfrac{2+\sqrt{6}}{2}}{1+\cfrac{2+\sqrt{6}}{2}}$. On simplifying , $\xi = \dfrac{18+3\sqrt{6}}{5}$.

Theorem [3, 4]: 2.1
Any periodic simple continued fraction represents a quadratic irrational and vice versa.
Theorem [3]: 2.2
If S is any positive integer and not a perfect square, then the continued fraction of $\sqrt{S}-\left[\sqrt{S}\right]$ is of the form $\langle a_0, \overline{a_1, a_2, \ldots, 2\left[\sqrt{S}\right]}\rangle$, where $a_0 = 0$.
Based on theorem 2.2, we identify the patterns of continued fraction of square root of integers which are not perfect square.

III PREPOSITION

If the square free positive number is of the form $\sqrt{(kn+1)^2+(2n+1)}-\left[\sqrt{(kn+1)^2+(2n+1)}\right]$ where $k \le 3$ and n is any positive integer then

$$\sqrt{(kn+1)^2+(2n+1)}-\left[\sqrt{(kn+1)^2+(2n+1)}\right] = \begin{cases} \langle 0;\overline{1,n,1,2(kn+1)}\,\rangle & if\ k=1 \\ \langle 0;\overline{2,\ 2(kn+1)}\,\rangle & if\ k=2 \\ \langle 0;\overline{2,1,3n,1,2,\ 2(kn+1)}\,\rangle & if\ k=3 \end{cases}$$

Since all the above continued fractions are periodic we try to find the quadratic irrationals
$\theta_1 = \langle \overline{1,n,1,2(kn+1)}\,\rangle$, $\theta_2 = \langle \overline{2,\ 2(kn+1)}\,\rangle$, $\theta_3 = \langle \overline{2,1,3n,1,2,\ 2(kn+1)}\,\rangle$

where $k = 1,2,3$ respectively. Also the corresponding irrationals are denoted by ξ_i, $i = 1,2,3.$ and $\xi_i = \theta_i^{-1}$, $i = 1,2,3.$
Also the solution of the corresponding quadratic equations are obtained by continued fractions method. and compare this convergence with numerical methods such as Bisection method, False position method, Iteration method and Newton - Raphson method [7].

IV. THEOREMS ON THE PATTERNS OF CONTINUED FRACTIONS OF $\sqrt{S}-\left\lfloor\sqrt{S}\right\rfloor$ WHERE S IS A SQUARE FREE POSITIVE NUMBER.

Theorem 4.1
For $k = 3$ and if n is any positive integer then the periodic continued fraction of the form $\langle 0;\overline{2,1,3n,1,2,\ 2(kn+1)}\,\rangle$ represents the quadratic irrational $x^{-2}-2(kn+1)x^{-1}-(2n+1)=0$ and vice versa.

Lemma 4.1
Let n be any positive integer then the quadratic irrational $x^{-2}-2(3n+1)x^{-1}-(2n+1)=0$ represent the periodic continued fraction $\langle 0;\overline{2,1,3n,1,2,\ 2(3n+1)}\,\rangle$.

Proof:

Take $x^{-1}=y.$ Then the above quadratic irrational becomes $y^2-2(3n+1)y-(2n+1)=0$.

Solving the above quadratic, $y=\dfrac{-(6n+2)\pm\sqrt{(6n+2)^2+4(2n+1)}}{2}$.

Take $\xi_0=\dfrac{m_0+\sqrt{d}}{q_0}$ and $q_0 \mid d-m_0^2$, where $m_0=-(6n+2)$, $d=(6n+2)^2+4(2n+1)$, $q_0=2$.

Then choose $a_0=\lfloor \xi_0 \rfloor$. So that $a_0=0$.

Now $\xi_1=\dfrac{m_1+\sqrt{d}}{q_1}$, where $m_1=a_0q_0-m_0$, and $q_1=\dfrac{d-m_1^2}{q_0}$.

Therefore $\xi_1=\dfrac{(6n+2)\pm\sqrt{(6n+2)^2+4(2n+1)}}{2(2n+1)}$, where $m_1=6n+2$ and $q_1=2(2n+1)$

Hence $a_1=[\xi_1] \Rightarrow a_1=2$

Again set $\xi_2=\dfrac{m_2+\sqrt{d}}{q_2}$, where $m_2=a_1q_1-m_1$, and $q_2=\dfrac{d-m_2^2}{q_1}$.

Therefore $\xi_2=\dfrac{(2n+2)\pm\sqrt{(6n+2)^2+4(2n+1)}}{8n+2}$, where $m_2=2n+2$ and $q_2=8n+2$

Hence $a_2=[\xi_2] \Rightarrow a_2=1$

Now $\xi_3=\dfrac{m_3+\sqrt{d}}{q_3}$, where $m_3=a_2q_2-m_2$, and $q_3=\dfrac{d-m_3^2}{q_2}$.

Therefore $\xi_3=\dfrac{6n\pm\sqrt{(6n+2)^2+4(2n+1)}}{4}$, where $m_3=6n$ and $q_3=4$

Hence $a_3=[\xi_3] \Rightarrow a_3=3n$

Now $\xi_4=\dfrac{m_4+\sqrt{d}}{q_4}$, where $m_4=a_3q_3-m_3$, and $q_4=\dfrac{d-m_4^2}{q_3}$.

Therefore $\xi_4=\dfrac{6n\pm\sqrt{(6n+2)^2+4(2n+1)}}{8n+2}$, where $m_4=6n$ and $q_4=8n+2$

Hence $a_4=[\xi_4] \Rightarrow a_4=1$

Now $\xi_5=\dfrac{m_5+\sqrt{d}}{q_5}$, where $m_5=a_4q_4-m_4$, and $q_5=\dfrac{d-m_5^2}{q_4}$.

Therefore $\xi_5=\dfrac{(2n+2)\pm\sqrt{(6n+2)^2+4(2n+1)}}{4n+2}$, where $m_5=(2n+2)$ and $q_5=4n+2$

Hence $a_5=[\xi_5] \Rightarrow a_5=2$

Now $\xi_6=\dfrac{m_6+\sqrt{d}}{q_6}$, where $m_6=a_5q_5-m_5$, and $q_6=\dfrac{d-m_6^2}{q_5}$.

Therefore $\xi_6 = \dfrac{(6n+2) \pm \sqrt{(6n+2)^2 + 4(2n+1)}}{2}$, *where* $m_6 = (6n+2)$ *and* $q_6 = 2$

Hence $a_6 = [\xi_6] \Rightarrow a_6 = 6n+2 \Rightarrow a_6 = 2(3n+1)$

Now $\xi_7 = \dfrac{m_7 + \sqrt{d}}{q_7}$, *where* $m_7 = a_6 q_6 - m_6$, *and* $q_7 = \dfrac{d - m_7^2}{q_6}$.

Therefore $\xi_7 = \dfrac{(6n+2) \pm \sqrt{(6n+2)^2 + 4(2n+1)}}{2(2n+1)}$, *where* $m_7 = (6n+2)$ *and* $q_7 = 2(2n+1)$

Hence $a_7 = [\xi_7] \Rightarrow a_7 = 2 = a_1$

Proceeding in this way we get,

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= a_7 = a_{13} = \ldots = 2 \\
a_2 &= a_8 = a_{14} = \ldots = 1 \\
a_3 &= a_9 = a_{15} = \ldots = 3n \\
a_4 &= a_{10} = a_{16} = \ldots = 1 \\
a_5 &= a_{11} = a_{17} = \ldots = 2 \\
a_6 &= a_{12} = a_{18} = \ldots = 6n+2 = 2(3n+1)
\end{aligned}$$

Hence we get the periodic continued fraction of the form $\langle 0; \overline{2,1,3n,1,2,\ 2(3n+1)} \rangle$

Also if l is the length of the periodic continued fraction, then $\xi_0 = \xi_l = \xi_{2l} = \xi_{3l} \ldots$

Lemma: 4.2

Let n be any positive integer then the periodic continued fraction $\langle 0; \overline{2,1,3n,1,2,\ 2(3n+1)} \rangle$ represents the quadratic irrationals $x^{-2} - 2(3n+1)x^{-1} - (2n+1) = 0$.

Proof:

$$\langle 0; \overline{2,1,3n,1,2,\ 2(3n+1)} \rangle = 0 + x^{-1} \ \textit{where} \ \ x = \langle \overline{2,1,3n,1,2,\ 2(3n+1)} \rangle$$

$$x = \left[2,1,3n,1,2,(6n+2) + \frac{1}{x} \right]$$

$$\Rightarrow x = \left[2,1,3n,1,2,\frac{6nx+2x+1}{x} \right] \Rightarrow x = \left[2,1,3n,1,2 + \frac{x}{6nx+2x+1} \right] \Rightarrow x = \left[2,1,3n,1,\ \frac{12nx+5x+2}{6nx+2x+1} \right]$$

$$\Rightarrow x = \left[2,1,3n,1 + \frac{6nx+2x+1}{12nx+5x+2} \right] \Rightarrow x = \left[2,1,3n,\ \frac{18nx+7x+3}{12nx+5x+2} \right] \Rightarrow x = \left[2,1,3n + \frac{12nx+5x+2}{18nx+7x+3} \right]$$

$$\Rightarrow x = \left[2,1,\ \frac{54n^2x+33nx+5x+9n+2}{18nx+7x+3} \right] \Rightarrow x = \left[2,1 + \frac{18nx+7x+3}{54n^2x+33nx+5x+9n+2} \right]$$

$$\Rightarrow x = \left[2,\ \frac{54n^2x+51nx+12x+9n+5}{54n^2x+33nx+5x+9n+2} \right] \Rightarrow x = \left[2 + \frac{54n^2x+33nx+5x+9n+2}{54n^2x+51nx+12x+9n+5} \right]$$

$$\Rightarrow (54n^2+51n+12)x^2 - (162n^2+126n+24)x - (27n+12) = 0$$

Divided by $x^2(27n+12)$ we get,

$$(2n+1) - (6n+2)x^{-1} - x^{-2} = 0 \Rightarrow x^{-2} + (6n+2)x^{-1} - (2n+1) = 0 \Rightarrow x^{-2} + 2(3n+1)x^{-1} - (2n+1) = 0$$

Hence if n is any positive integer then the periodic continued fraction $\langle 0;\overline{2,1,3n,1,2,\ 2(3n+1)}\rangle$ represents the quadratic irrationals $x^{-2}-2(3n+1)x^{-1}-(2n+1)=0$.

Lemma 4.1 and 4.2 completes the proof of theorem 4.1.

Theorem: 4.2

For $k=2$ and if n is any positive integer then the periodic continued fraction of the form $\langle 0;\overline{2,\ 2(kn+1)}\rangle$ represents the quadratic irrational $x^{-2}+2(kn+1)x^{-1}-(2n+1)=0$ and vice versa.

Proof: Similar to the proof of theorem 4.1

Theorem: 4.3

For $k=1$ and if n is any positive integer then the periodic continued fraction of the form $\langle 0;\overline{1,n,1,2,\ 2(kn+1)}\rangle$ represents the quadratic irrational $x^{-2}+2(kn+1)x^{-1}-(2n+1)=0$ and vice versa.

Proof: Similar to the proof of theorem 4.1

TABLE 4.1

Illustrations based on theorems 4.1, 4.2 And 4.3

K	N	Irrational number	Continued fraction of irrational number	Quadratic irrational
1	1	$\sqrt{7}-2$	$\langle 0;\overline{1,1,1,4,}\rangle$	$x^{-2}+4x^{-1}-3=0$
	2	$\sqrt{14}-3$	$\langle 0;\overline{1,2,1,6,}\rangle$	$x^{-2}+6x^{-1}-5=0$
	3	$\sqrt{23}-4$	$\langle 0;\overline{1,3,1,8,}\rangle$	$x^{-2}+8x^{-1}-7=0$
2	1	$\sqrt{12}-3$	$\langle 0;\overline{2,6,}\rangle$	$x^{-2}+6x^{-1}-3=0$
	2	$\sqrt{30}-5$	$\langle 0;\overline{2,10,}\rangle$	$x^{-2}+10x^{-1}-5=0$
	3	$\sqrt{56}-7$	$\langle 0;\overline{2,14,}\rangle$	$x^{-2}+14x^{-1}-7=0$
3	1	$\sqrt{19}-4$	$\langle 0;\overline{2,1,3,1,2,8}\rangle$	$x^{-2}+8x^{-1}-3=0$
	2	$\sqrt{54}-7$	$\langle 0;\overline{2,1,6,1,2,14}\rangle$	$x^{-2}+14x^{-1}-5=0$
	3	$\sqrt{107}-10$	$\langle 0;\overline{2,1,9,1,2,20}\rangle$	$x^{-2}+20x^{-1}-7=0$

TABLE 4.2

Comparison of the solution of the quadratic equation $x^{-2}+4x^{-1}-3=0$ with the convergence of the continued fractions

Iterations	Bisection Method	False position Method	Iteration Method	Newton- Raphson Method	ContinuedFraction Method
1	0.5	0.6	0.5	0.5	0
2	0.75	0.64286	0.66667	0.65	1
3	0.675	0.64557	0.64286	0.64575	0.5
4	0.6875	0.64574	0.64615	0.64575	0.66667
5	0.65625	0.64575	0.64570		0.64286
6	0.64063	0.64575	0.64575		0.64706
7	0.64844		0.64575		0.64516
8	0.64453				0.64583
9	0.64648				0.64574
10	0.64551				0.64576
11	0.64600				0.64575
12	0.64600				
13	0.64575				
14	0.64575				

TABLE 4.3

Comparison of the solution of the quadratic equation $x^{-2} + 6x^{-1} - 3 = 0$ with the convergence of the continued fractions

Iterations	Bisection Method	False position Method	Iteration Method	Newton-Raphson Method	Continued Fraction Method
1	0.5	0.42857	0.5	0.5	0
2	0.25	0.46154	0.46154	0.46429	0.5
3	0.375	0.46392	0.46429	0.46410	0.46154
4	0.4375	0.46409	0.46409	0.46410	0.46429
5	0.46875	0.46410	0.46410		0.46409
6	0.45313	0.46410	0.46410		0.46410
7	0.46094				
8	0.46484				
9	0.46289				
10	0.46387				
11	0.46484				
12	0.46435				
13	0.46411				
14	0.46411				
15	0.46411				
16	0.46411				
17	0.46411				
18	0.46410				
19	0.46410				

TABLE 4.4

Comparison of the solution of the quadratic equation $x^{-2} + 8x^{-1} - 3 = 0$ with the convergence of the continued fractions

Iterations	Bisection Method	False position Method	Iteration Method	Newton-Raphson Method	Continued Fraction Method
1	0.5	0.33333	0.5	0.5	0
2	0.25	0.35714	0.35294	0.36111	0.5
3	0.625	0.35878	0.35915	0.35890	0.33333
4	0.4375	0.35889	0.35889	0.35890	0.36364
5	0.34375	0.35890	0.35890		0.35714
6	0.39063	0.35890	0.35890		0.35897
7	0.36719				0.35890
8	0.35547				
9	0.36133				
10	0.35840				
11	0.35986				
12	0.35913				
13	0.35876				
14	0.35895				
15	0.35885				
16	0.35890				
17	0.35888				
18	0.35889				
19	0.35890				
20	0.35890				

V. CONCLUSION

In this paper the patterns of continued fraction of square root of the square free number of the form $\sqrt{(kn+1)^2 + (2n+1)} - \left[\sqrt{(kn+1)^2 + (2n+1)}\right]$ where $k \le 3$ and n is any positive integer are identified. From the comparison table we find that the values of convergence of continued fractions and number of iterations are nearer to all the methods given in the above tables except the Bisection method.

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