Pre Generalized Regular Weakly-Closed (pgrw-closed) Sets in a Bitopological Space

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Absract - *In this paper pgrw-closed sets, pgrw-closure of a set in a bitopological space are studied and discussed some of their basic properties. A subset A of a bitopological space* (X,τ_1,τ_2) *is called a* (τ_i,τ_j)*-pgrw-closed set if* τ_j*-pcl(A)⊆G whenever A⊆G and G is a* τ_i*-rw-open set where* $i,j \in \{1,2\}$, $i \neq j$. *The union and the intersection of two* (τ_i,τ_j)*-pgrw-closed sets need not be* (τ_i,τ_j)*-pgrw-closed.*

Keywords - (τ_i,τ_j)-pgrw-closed set, (τ_i,τ_j)-pgrw-closure, τ_i-rw-open set

I. Introduction

In 1963 Kelly [1] initiated a systematic study of the concept of bitopological spaces. Later various researchers like Arya and Nour [2], Fukutake [3], Gnanambal [4] and Sheik Jhon [5] followed the concept of bitopological spaces. Fukutake extended the notion of generalised sets in a topological space to a bitopological space.

II. Preliminaries: A subset A of a topological space (X,τ) is called

i) a semi-open set [6] if A⊆cl(int(A)) and a semi-closed set if int(cl(A))⊆A.
ii) a pre-open set [7] if A⊆int(cl(A)) and a pre-closed set if cl(int(A))⊆A.
iii) an α-open set [8] if A⊆int(cl(int(A))) and an α-closed set if cl(int(cl(A)))⊆A.
iv) a semi-pre open set (β-open) [9] if A⊆cl(int(cl(A))) and a semi-pre closed set (β-closed) if int(cl(int(A)))⊆A.
v) a regular-open set [10] if A=int(cl(A)) and a regular closed set if A=cl(int(A)).
vi) a δ-closed set [11] if $A=cl_\delta(A)$ where $cl_\delta(A=\{x\epsilon X:int(cl(U))\cap A\neq\phi, U\epsilon\tau \text{ and } x\epsilon U\}$.
vii) a regular semi-open set (rs-open) [12] if there is a regular open set U such that U⊆A⊆cl(U).
viii) a generalized-closed set (g-closed) [13] if cl(A)⊆U whenever A⊆U and U is open in X.
ix) a strongly generalized closed set (g*-closed) [14] if cl(A)⊆U whenever A⊆U and U is g-open in X.
x) a regular w-closed set (rw-closed) [15] if cl(A)⊆U whenever A⊆U and U is regular semi-open in X.
xi) a #regular generalized-closed (#rg-closed) set [16] if cl(A)⊆U whenever A⊆U and U is rw-open.

xii) a Regular weakly generalized closed [17] (rwg - closed) if cl (int(A)) ⊆ U whenever A⊆U and U is regular open in X.

xiii) A generalised α-closed set (gα-closed) [18] if αcl(A)⊆U whenever A⊆U and U is α-open in X.

Definiion: A subset A of a bitopological space (X,τ_1,τ_2) is called a

i) (τ_i,τ_j)-g-closed [3] if τ_j-cl(A) ⊆ U whenever A⊆U and U is open in τ_i.

ii) (τ_i,τ_j)-gp-closed [19] if τ_j-pcl(A) ⊆ U whenever A ⊆ U and U is open in τ_i.

iii) (τ_i,τ_j)-rg-closed [2] if τ_j-cl(A) ⊆U whenever A⊆U and U is regular open in τ_i.

iv) (τ_i,τ_j)-gpr-closed [21] if τ_j-pcl(A)⊆U whenever A⊆U and U is regular open in τ_i.

v) (τ_i,τ_j)-αg-closed [22] if τ_j -αcl(A) ⊆ U whenever A ⊆ U and U is open in τ_i.

Here $i,j\in\{1,2\}$ and $i \neq j$.

III pgrw-closed sets in a bitopological space:

3.1 Definition: A subset A of a bitopological space (X,τ_1,τ_2) is called a (τ_i,τ_j)-pgrw-closed set if τ_j-pcl(A)⊆G whenever A⊆G and G is a τ_i-rw open set where $i,j\in\{1,2\}$, $i \neq j$.

3.2 Example: X={a,b,c}, τ_1={X,ϕ,{a}}, τ_2={X,ϕ,{a},{b},{a,b}}

τ_1-rw-open sets are X,ϕ,{a},{b},{c},{a,b},{b,c},{a,c}.

τ_2-pre-closed sets are X,ϕ,{b,c},{a,c},{c}.

{c} is a (τ_1,τ_2)-pgrw closed set.

Let A={a}. τ_2-pcl(A)={a,c}. Here {a}⊆{a}, a τ_1-rw-open set.

τ_2-pcl(A) ={a,c}⊄{a}. Hence A is not a (τ_1,τ_2)-pgrw-closed set.

The family of all (τ_i,τ_j)-pgrw-closed sets in a bitopological space (X,τ_1,τ_2) is denoted by $D_{pgrw}(X,\tau_i,\tau_j)$.

3.3 Remark: By setting $\tau_1=\tau_2=\tau$ in definition 3.1 a (τ_i,τ_j)-pgrw-closed set reduces to a pgrw-closed set in (X,τ).

3.4 Remark : The family $D_{pgrw}(X,\tau_1,\tau_2)$ is generally not equal to the family $D_{pgrw}(X,\tau_2,\tau_1)$ as seen from the following example.

3.5 Example: X={a,b,c,d}, τ_1={X,ϕ,{a},{b},{a,b},{a,b,c}}, τ_2={X,ϕ,{a},{c,d},{a,c,d}}

τ_1-rw-open sets : X,ϕ,{a,b,c},{c,d},{a,b},{a},{b},{c},{d}

τ_2-pre-closed sets : X,ϕ,{b},{c},{d},{a,b},{b,c},{b,d},{a,b,c},{b,c,d},{a,b,d}

$D_{pgrw}(X,\tau_1,\tau_2)$={X,ϕ,{b},{c},{d},{a,b},{b,c},{a,d},{b,d},{a,c},{a,b,c},{b,c,d}, {a,c,d}, {a,b,d}}

τ_2-rw-open sets : X,ϕ,{a},{b},{c},{d},{c,d},{a,d},{b,c},{b,d},{a,c},{a,c,d};

τ_1-pre-closed sets : X,ϕ,{b,c,d},{a,c,d},{c,d},{d},{c};

$D_{pgrw}(X,\tau_2,\tau_1)$ = {X,ϕ,{c},{d},{c,d},{a,b,c},{b,c,d},{a,c,d},{a,b,d}}

∴ $D_{pgrw}(X,\tau_1,\tau_2) \neq D_{pgrw}(X,\tau_2,\tau_1)$.

3.6 Remark: The union and the intersection of two (τ_i,τ_j)-pgrw-closed sets need not be (τ_i,τ_j)-pgrw-closed.

3.7 Example: Consider the space in 3.5.

1) The sets {c} and {d} are (τ_1,τ_2)-pgrw-closed sets.

But {c}∪{d}={c,d} is not (τ_1,τ_2)-pgrw-closed.

2) The sets {a,b} and {a,c} are (τ_1,τ_2)-pgrw-closed sets.

But $\{a,b\}\cap\{a,c\}=\{a\}$ is not (τ_1,τ_2)-pgrw-closed.

3.8 Theorem: In a bitopological space (X,τ_1,τ_2) if $\tau_1\subseteq\tau_2$ and $RWO(X,\tau_1)\subseteq RWO(X,\tau_2)$, then $D_{pgrw}(X,\tau_2,\tau_1)\subseteq D_{pgrw}(X,\tau_1,\tau_2)$.

Proof: (X,τ_1,τ_2) is a bitopological space in which $\tau_1\subseteq\tau_2$ and $RWO(X,\tau_1)\subseteq RWO(X,\tau_2)$. $A\in D_{pgrw}(X,\tau_2,\tau_1)$. Let $G\in RWO(X,\tau_1)$ and $A\subseteq G$. Then $G\in RWO(X,\tau_2)$ and $A\subseteq G$. As A is a (τ_2,τ_1)-pgrw-closed set, τ_1-pcl(A)$\subseteq$G. Since $\tau_1\subseteq\tau_2$, τ_2-pcl(A)$\subseteq\tau_1$-pcl(A)$\subseteq$G. Thus τ_2-pcl(A)$\subseteq$G whenever $A\subseteq G$ and G is τ_1-rw-open. Hence A is (τ_1,τ_2)-pgrw-closed. That is $A\in D_{pgrw}(X,\tau_1,\tau_2)$. $\therefore$ $D_{pgrw}(X,\tau_2,\tau_1)\subseteq D_{pgrw}(X,\tau_1,\tau_2)$.

3.9 Theorem: In a bitopological space (X,τ_1,τ_2) if A is a (τ_i,τ_j)-pgrw-closed set, then τ_j-pcl(A)–A contains no non-empty τ_i-rw-closed set.

Proof: In a bitopological space (X,τ_1,τ_2) A is a (τ_i,τ_j)-pgrw-closed set, F is a τ_i-rw-closed set and F$\subseteq\tau_j$-pcl(A)–A.

$\Rightarrow$ A is a (τ_i,τ_j)-pgrw-closed set, F^c is τ_i-rw-open, A$\subseteq F^c$ and F$\subseteq\tau_j$-pcl(A).

$\Rightarrow$ τ_j-pcl(A)$\subseteq F^c$ and F$\subseteq\tau_j$-pcl(A)

$\Rightarrow$ F$\subseteq(\tau_j$-pcl(A)$)^c$ and F$\subseteq\tau_j$-pcl(A).

$\Rightarrow$ F$\subseteq(\tau_j$-pcl(A)$)^c\cap\tau_j$-pcl(A)=ϕ

$\Rightarrow$ F= ϕ.

Hence τ_j-pcl(A)–A does not contain any non-empty τ_i-rw-closed set.

The converse is not true.

3.10 Example: w.r.t. the example 3.5 if A={b,d}, τ_1-pcl(A)–A = τ_1-pcl({b,d})–{b,d} = {b,c,d}–{b,d}={c} contains no non-empty τ_2-rw-closed set, but {b,d} is not (τ_2,τ_1)-pgrw-closed.

3.11 Corollary: If A is a (τ_i,τ_j)-pgrw-closed set in (X,τ_1,τ_2) and τ_j-pcl(A)–A is a τ_i-rw-closed set, then A is τ_j-pre-closed.

Proof: A is a (τ_i,τ_j)-pgrw-closed set and τ_j-pcl(A)–A is τ_i-rw-closed in (X,τ_1,τ_2).

$\Rightarrow$ A is a (τ_i,τ_j)-pgrw-closed set and a τ_i-rw-closed set τ_j-pcl(A)–A$\subseteq\tau_j$-pcl(A)–A

$\Rightarrow$ τ_j-pcl(A)–A=ϕ

$\Rightarrow$ τ_j-pcl(A)=A

$\Rightarrow$ A is τ_j-pre-closed.

3.12 Theorem: In any bitopological space (X,τ_1,τ_2) $\forall$ x$\in$X $\{x\}^c$ is either τ_i-rw-open or (τ_i,τ_j)-pgrw-closed .

Proof: (X,τ_1,τ_2) is a bitopological space.

$\forall$ x$\in$X, $\{x\}^c$ and X are the only two sets containing $\{x\}^c$.

So if $\{x\}^c$ is not τ_i-rw-open, then X is the only τ_i-rw-open set containing $\{x\}^c$ and so whenever $\{x\}^c\subseteq$X, a τ_i-open set τ_j-pcl($\{x\}^c$)$\subseteq$X. That is every τ_i-rw-open set containing $\{x\}^c$ contains τ_j-pcl($\{x\}^c$). Hence $\{x\}^c$ is (τ_i,τ_j)-pgrw-closed.

3.13 Theorem: In a bitopological space (X,τ_1,τ_2) if $RWO(X,\tau_i)=\{X,\phi\}$, then every subset of X is (τ_i,τ_j)-pgrw-closed.

Proof: Let $RWO(X,\tau_i)=\{X,\phi\}$ in (X,τ_1,τ_2) and A be any subset of X.

If A=ϕ, then A is a (τ_i,τ_j)-pgrw-closed set.

If A$\neq\phi$, then X is the only τ_i-rw-open set containing A and τ_j-pcl(A)$\subseteq$X.

Hence A is a (τ_i,τ_j)-pgrw-closed set.

3.14 Theorem: Every τ_j-pre-closed subset of a bitopological space (X,τ_1,τ_2) is a (τ_i,τ_j)-pgrw-closed set.

Proof: If A is a τ_j-pre-closed subset of a bitopological space (X,τ_1,τ_2), then A=τ_j-pcl(A). Therefore if A⊆G and G is τ_i-rw-open, then τ_j-pcl(A)⊆G.

So A is (τ_i,τ_j)-pgrw-closed.

The converse is not true.

3.15 Example: X={a,b,c}, τ_1={X,ϕ,{a},{b},{a,b}}, τ_2={X,ϕ,{a}}.

τ_1-rw-open sets are X,ϕ,{a},{b},{a,b}. τ_2-pre-closed sets are X,ϕ,{b,c},{c},{b}. The set {a,c} is a (τ_1,τ_2)-pgrw-closed set, but not τ_2-pre-closed.

3.16 Theorem: Every (τ_i,τ_j)-pgrw-closed and τ_i-rw-open set in a bitopological space (X,τ_1,τ_2) is τ_j-pre-closed.

Proof: In (X,τ_1,τ_2), A is a (τ_i,τ_j)-pgrw closed and τ_i-rw-open set.

⇒ A is (τ_i,τ_j)-pgrw-closed and A⊆A, a τ_i-rw-open set.

⇒ τ_j-pcl(A)⊆A

⇒ τ_j-pcl(A)=A.

⇒ A is τ_j-pre-closed.

3.17 Corollary: Every (τ_i,τ_j)-pgrw-closed and τ_i-regular-open set in a bitopological space (X,τ_1,τ_2) is τ_j-pre-closed.

Proof: A is a (τ_i,τ_j)-pgrw-closed and τ_i-regular-open set in (X,τ_1,τ_2).

⇒ A is a (τ_i,τ_j)-pgrw-closed and τ_i-rw-open set.

⇒ A is τ_j-pre-closed.

3.18 Theorem: In a bitopological space (X,τ_1,τ_2) every τ_i-rw-open set is τ_j-pre-closed if and only if every subset of (X,τ_1,τ_2) is a (τ_i,τ_j)-pgrw-closed set.

Proof: (X,τ_1,τ_2) is a bitopological space and every τ_i-rw-open set is τ_j-pre-closed.

A is a subset of X.

Now A⊆G and G is τ_i-rw-open.

⇒ τ_j-pcl(A)⊆τ_j-pcl(G) and G is τ_j-pre-closed.

⇒ τ_j-pcl(A)⊆τ_j-pcl(G) and τ_j-pcl(G)=G.

⇒ τ_j-pcl(A)⊆G

Thus τ_j-pcl(A)⊆G whenever A⊆G and G is τ_i-rw-open.

∴ A is (τ_i,τ_j)-pgrw-closed.

Conversely

Suppose that every subset of (X,τ_1,τ_2) is a (τ_i,τ_j)-pgrw-closed set. Let G be a τ_i-rw-open set in X. By the hypotheses G is (τ_i,τ_j)-pgrw-closed.

By 3.16 G is τ_j -pre-closed.

3.19 Theorem: In a bitopological space (X,τ_1,τ_2)

i) every τ_j-closed set is (τ_i,τ_j)-pgrw-closed.

ii) every τ_j-δ-closed set is (τ_i,τ_j)-pgrw-closed.

iii) every τ_j-regular closed set is (τ_i,τ_j)-pgrw-closed.

iv) every τ_j-α-closed set is (τ_i,τ_j)-pgrw-closed.

v) every τ_j-#rg-closed set is (τ_i,τ_j)-pgrw-closed.

Proof: i) A is a τ_j-closed set in (X,τ_1,τ_2).

$\Rightarrow$ A is τ_j-pre-closed .

$\Rightarrow$ A is (τ_i,τ_j)-pgrw-closed (3.14).

Other statements may be proved similarly.

The converse statements are not true.

3.20 Example: In 3.5 the set {c} is (τ_2,τ_1)-pgrw-closed, but not τ_2-closed.

3.21 Example: In 3.5 the set { c} is (τ_2,τ_1)-pgrw-closed, but not τ_2-δ-closed.

3.22 Example: X={a,b,c}, τ_1={X,ϕ,{a}}, τ_2={X,ϕ,{a},{b},{a,b}}. Here the set {c} is (τ_1,τ_2)-pgrw-closed, but not τ_2-regular-closed.

3.23 Example: Consider 3.5, {c} is (τ_2,τ_1)-pgrw-closed, but not τ_2-#rg-closed.

3.24 Theorem: Every (τ_i,τ_j)-pgrw-closed set in a bitopological space (X,τ_1,τ_2) is (τ_i,τ_j)-gp-closed.

Proof: In a bitopological space (X,τ_1,τ_2), A is a (τ_i,τ_j)-pgrw-closed set and G is a τ_i-open set containing A.

$\Rightarrow$ A is a (τ_i,τ_j)-pgrw-closed set and G is a τ_i-rw-open set containing A.

$\Rightarrow$ τ_j-pcl(A)$\subseteq$G

Thus every τ_i-open set containing A contains τ_j-pcl(A).

Hence A is (τ_i,τ_j)-gp-closed.

The converse is not true.

3.25 Example: Refer 3.5. (τ_2,τ_1)-gp-closed sets are X,ϕ,{b},{c},{d},{a,b},{b,c},{a,d},{b,d},{a,c},{c,d},{a,b,c},{b,c,d},{a,c,d},{a,b,d}. The set {b} is (τ_2,τ_1)-gp-closed, but not (τ_2,τ_1)-pgrw-closed.

3.26 Theorem: Every τ_i-open and (τ_i,τ_j)-gp-closed set in a bitopological space (X,τ_1,τ_2) is (τ_i,τ_j)-pgrw-closed.

Proof: A is a τ_i-open and (τ_i,τ_j)-gp-closed set in (X,τ_1,τ_2). Let G be a τ_i-rw-open set containing A.

Now A$\subseteq$A, τ_i-open and A is (τ_i,τ_j)-gp-closed.

$\therefore$ τ_j-pcl(A)$\subseteq$A$\subseteq$G. Thus τ_j-pcl(A)$\subseteq$G whenever A is subset of G and G is τ_i-rw-open.

$\therefore$ A is (τ_i,τ_j)-pgrw-closed.

3.27 Theorem: Every (τ_i,τ_j)-pgrw-closed set in a bitopological space (X,τ_1,τ_2) is (τ_i,τ_j)-gpr-closed.

Proof: In a bitopological space (X,τ_1,τ_2) A is a (τ_i,τ_j)-pgrw-closed set and G is a τ_i-regular open set containing A.

$\Rightarrow$ A is a (τ_i,τ_j)-pgrw-closed set and G is a τ_i-rw-open set containg A.

$\Rightarrow$ τ_j-pcl(A)$\subseteq$G.

Thus every τ_i-regular open set containing A contains τ_j-pcl(A).

$\therefore$ A is (τ_i,τ_j)-gpr-closed.

The converse is not true.

3.28 Example: X={a,b,c}, τ_1= {X,ϕ,{a},{b},{a,b}}, τ_2={X,ϕ,{a}}

τ_1-regular open sets are X,ϕ,{a},{b} and τ_2-pre closed sets: X,ϕ,{b,c},{c},{b}.

τ_1-rw-open sets are X,ϕ,{a},{b},{c},{a,b}. The set {a,b} is (τ_1,τ_2)-gpr-closed, but not (τ_1,τ_2)-pgrw-closed.

3.29 Theorem: Every τ_i-regular open and (τ_i,τ_j)-gpr-closed set in a bitopological space (X,τ_1,τ_2) is (τ_i,τ_j)-pgrw-closed.

Proof: A is a τ_i-regular open and (τ_i,τ_j)-gpr-closed set in a bitopological space (X,τ_1,τ_2).

Let G be a τ_i-rw-open set such that $A\subseteq G$.

Now $A\subseteq A$, τ_i-regular open and A is (τ_i,τ_j)-gpr-closed.

$\therefore$ τ_j-pcl(A)$\subseteq$A

$\therefore$ τ_j-pcl(A)$\subseteq$G

$\therefore$ A is (τ_i,τ_j)-pgrw-closed.

3.30 Theorem: Every τ_i-regular open and (τ_i,τ_j)-rg-closed set in a bitopological space (X,τ_1,τ_2) is (τ_i,τ_j)-pgrw-closed.

Proof: A is a τ_i-regular open and (τ_i,τ_j)-rg-closed set in (X,τ_1,τ_2).

Let G be a τ_i-rw-open set such that $A\subseteq G$. As $A\subseteq A$, τ_i-rw-open and A is

(τ_i,τ_j)-rg-closed, τ_j-cl(A)$\subseteq$A. And so τ_j-pcl(A)$\subseteq A\subseteq G$. Thus τ_j-pcl(A)$\subseteq$G whenever A is subset of G and G is τ_i-rw-open.

$\therefore$ A is (τ_i,τ_j)-pgrw-closed .

3.31 Theorem: Every τ_i-open and (τ_i,τ_j)-g-closed set in a bitopological space (X,τ_1,τ_2) is (τ_i,τ_j)-pgrw-closed.

Proof: A is τ_i-open and (τ_i,τ_j)-g-closed set in a bitopological space (X,τ_1,τ_2).

Let G be a τ_i-rw-open set containing A in (X,τ_1,τ_2).

Now $A\subseteq A$, a τ_i-open set and A is (τ_i,τ_j)-g-closed.

$\therefore$ τ_j-cl(A)$\subseteq$A. And so τ_j-pcl(A)$\subseteq$G.

$\therefore$ A is (τ_i,τ_j)-pgrw-closed.

3.32 Theorem: Every τ_i-open and (τ_i,τ_j)-αg-closed set in a bitopological space (X,τ_1,τ_2) is (τ_i,τ_j)-pgrw-closed.

Proof: A is a τ_i-open and (τ_i,τ_j)-αg-closed set in a bitopological space (X,τ_1,τ_2).

Let G be a τ_i-rw-open set such that $A\subseteq G$ in (X,τ_1,τ_2).

Now $A\subseteq A$, τ_i-open and A is (τ_i,τ_j)-αg-closed.

$\therefore$ τ_j-αcl(A)$\subseteq$A.

$\therefore$ τ_j-pcl(A)$\subseteq A\subseteq G$.

$\therefore$ A is (τ_i,τ_j)-pgrw-closed.

The following examples show that (τ_i,τ_j)-pgrw-closed sets and τ_j-rw closed sets, τ_j-g-closed sets, τ_j-semi-closed sets, τ_j-gα-closed sets, τ_j-g*-closed sets, τ_j-rwg-closed sets sets are independent.

3.33 Example:(τ_1,τ_2)-pgrw-closed sets and τ_2-rw-closed sets are independent.

i) $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\},\{b\},\{a,b\}\}$, $\tau_2=\{X,\phi,\{a\}\}$

Here {a,b} is τ_2-rw closed set, but not (τ_1,τ_2)-pgrw-closed.

ii) $X=\{a,b,c,d\}$, $\tau_1=\{X,\phi,\{a\},\{c,d\},\{a,c,d\}\}$, $\tau_2=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$

The set {c} is (τ_1,τ_2)-pgrw-closed, but not τ_2-rw closed.

Similarly we may show that (τ_2,τ_1)-pgrw-closed sets and τ_1-rw closed sets are independent.

3.34 Example: (τ_1,τ_2)-pgrw-closed sets and τ_2-g-closed sets are independent.

i) $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\},\{b\},\{a,b\}\}$, $\tau_2=\{X,\phi,\{a\}\}$

The set {a,b} is a τ_2-g-closed set, but not (τ_1,τ_2)-pgrw-closed.

ii) $X=\{a,b,c,d\}$, $\tau_1=\{X,\phi,\{a\},\{c,d\},\{a,c,d\}\}$, $\tau_2=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$

The set {c} is (τ_1,τ_2)-pgrw-closed, but not τ_2-g-closed.

3.35 Example: (τ_1,τ_2)-pgrw-closed sets and τ_2-semi-closed sets are independent.

i) X = {a,b,c}, τ_1 = {X,ϕ,{a},{b},{a,b}}, τ_2 = {X,ϕ,{a}}

The set {a,c} is (τ_1,τ_2)-pgrw-closed, but not τ_2-semi-closed.

ii) X = {a,b,c,d}, τ_1={X,ϕ,{a},{c,d},{a,c,d}}, τ_2={X,ϕ,{a},{b},{a,b},{a,b,c}} The set {b,c} is τ_2-semi-closed, but not (τ_1,τ_2)-pgrw-closed.

3.36 Example: (τ_1, τ_2)-pgrw-closed sets and τ_2-gα-closed sets are independent.

i) X = {a,b,c}, τ_1 = {X,ϕ,{a},{b},{a,b}}, τ_2 ={X,ϕ,{a}}

{a,b} is a τ_2-gα closed set, but not (τ_1,τ_2)-pgrw-closed.

ii) X = {a,b,c,d}, τ_1 = {X,ϕ,{a},{c,d},{a,c,d}}, τ_2 = {X,ϕ,{a},{b},{a,b},{a,b,c}} The set {a,b,d} is (τ_1,τ_2)-pgrw-closed, but not τ_2-gα closed.

3.37 Example: (τ_1,τ_2)-pgrw-closed sets and τ_2-g*-closed sets are independent.

i) X = {a,b,c}, τ_1={X,ϕ,{a},{b},{a,b}}, τ_2={X,ϕ,{a}}

The set {a,c} is (τ_1,τ_2)-pgrw-closed, but not τ_2-g*-closed.

ii) X ={a,b,c,d}, τ_1={X,ϕ,{a},{c,d},{a,c,d}}, τ_2={X,ϕ,{a},{b},{a,b},{a,b,c}}

The set {a,d} is a τ_2-g*-closed set, but not (τ_1,τ_2)-pgrw-closed.

3.38 Example: (τ_1,τ_2)-pgrw-closed sets and τ_2-rwg-closed sets are independent.

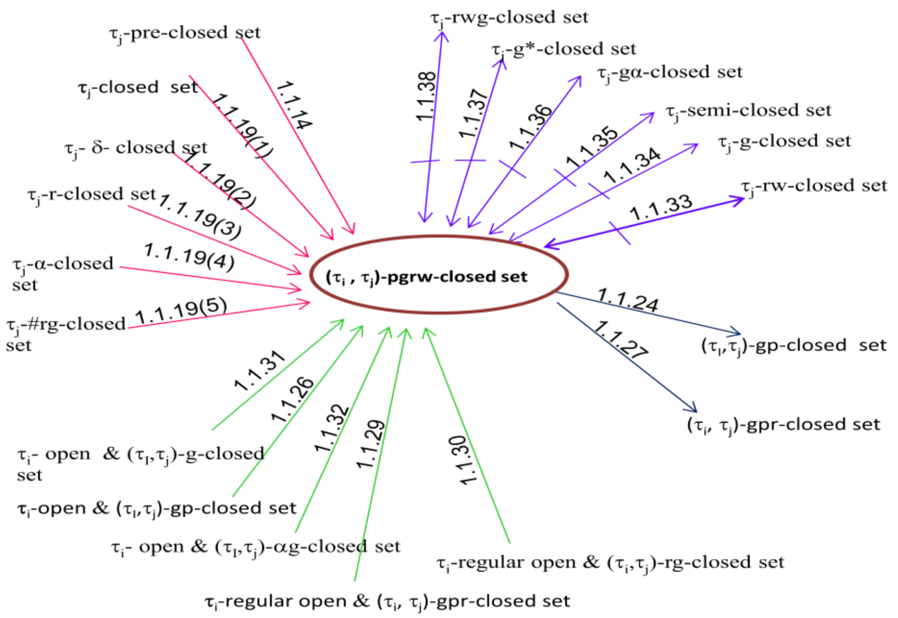
i) X ={a,b,c}, τ_1 = {X,ϕ,{a},{b},{a,b}}, τ_2 = {X,ϕ,{a}}

{a,b} is τ_2-rwg-closed, but not (τ_1,τ_2)-pgrw-closed.

ii) X = {a,b,c,d}, τ_1 = {X,ϕ,{a},{c,d},{a,c,d}}, τ_2 = {X,ϕ,{a},{b},{a,b},{a,b,c}}

The set {c,d} is (τ_1,τ_2)-pgrw-closed, but not τ_2-rwg-closed.

3.39 The following diagram shows the relation between (τ_i,τ_j)-pgrw-closed set and other sets.



IV pgrw-closure of a set in a bitopological space:

4.1 Definition: In a bitopological space (X,τ_1,τ_2) (τ_i,τ_j)-pgrw-closure of a subset A of X is the intersection of all (τ_i,τ_j)-pgrw-closed sets containing A and is denoted by (τ_i,τ_j)-pgrwcl(A). i.e. (τ_i,τ_j)-pgrwcl(A)$=\cap F_{F\epsilon \mathcal{F}}$ where $\mathcal{F}$={F:F is a (τ_i,τ_j)-pgrw-closed set containing A}.

4.2 Example: X={a,b,c,d}, τ_1={X,ϕ,{a},{c,d},{a,c,d}}, τ_2={X,ϕ,{a},{b},{a,b},{a,b,c}};

(τ_1,τ_2)-pgrw-closed sets are X,ϕ,{c},{d},{c,d},{a,b,c},{b,c,d},{a,c,d},{a,b,d}.

Let A={a,c}. (τ_1,τ_2)-pgrwcl(A)={a,c}

4.3 Example: (τ_i,τ_j)-pgrwcl(X)=X, (τ_i,τ_j)-pgrwcl(ϕ)=ϕ

4.4 Remark: (τ_i,τ_j)-pgrwcl(A) need not be (τ_i,τ_j)-pgrw-closed. For example, in 4.2 let A={a,d}, then (τ_1,τ_2)-pgrwcl(A)={a,d}=A which is not (τ_1,τ_2)-pgrw-closed.

4.5 Theorem: If A is a subset of a bitopological space (X,τ_1,τ_2), then

i) A$\subseteq(\tau_i,\tau_j)$-pgrwcl(A)

ii) (τ_i,τ_j)-pgrwcl((τ_i,τ_j)-pgrwcl(A))=(τ_i,τ_j)-pgrwcl(A)

iii) ∀ (τ_i,τ_j)-pgrw-closed set A (τ_i,τ_j)-pgrwcl(A)=A.

Proof:

i) By definition it follows that A$\subseteq(\tau_i,\tau_j)$-pgrwcl(A).

ii) A⊆X. (τ_i,τ_j)-pgrwcl(A)$=\cap F_{F\epsilon \mathcal{F}}$ where $\mathcal{F}$={F:F is a (τ_i,τ_j)-pgrw-closed set containing A}.

If F$\epsilon\mathcal{F}$, then (τ_i,τ_j)-pgrwcl(A)⊆F. Since F is a (τ_i,τ_j)-pgrw-closed set containing pgrwcl(A), (τ_i,τ_j)-pgrwcl((τ_i,τ_j)-pgrwcl(A))⊆F ∀ F$\epsilon\mathcal{F}$. Hence (τ_i,τ_j)-pgrwcl((τ_i,τ_j)-pgrwcl(A))$\subseteq\cap F_{F\epsilon \mathcal{F}}$ i.e. (τ_i,τ_j)-pgrwcl((τ_i,τ_j)-pgrwcl(A))$\subseteq(\tau_i,\tau_j)$-pgrwcl(A)........(a)

And by (1) A$\subseteq(\tau_i,\tau_j)$-pgrwcl(A) and so (τ_i,τ_j)-pgrwcl(A)$\subseteq(\tau_i,\tau_j)$-pgrwcl((τ_i,τ_j)-pgrwcl(A))..........(b)

(a) and (b) ⇒ (τ_i,τ_j)-pgrwcl((τ_i,τ_j)-pgrwcl(A)) = (τ_i,τ_j)-pgrwcl(A)

iii) By (1) A$\subseteq(\tau_i,\tau_j)$-pgrwcl(A). If A is a (τ_i,τ_j)-pgrw-closed subset of (X,τ_1,τ_2), then (τ_i,τ_j)-pgrwcl(A)⊆A.

∴ (τ_i,τ_j)-pgrwcl(A)=A.

Converse of (iii) is not true.

4.6 Example: Refer 4.4.

4.7 Theorem: A and B are subsets of a bitopological space (X,τ_1,τ_2).

i) If B is any (τ_i,τ_j)-pgrw-closed set containing A, then (τ_i,τ_j)-pgrwcl(A)⊆B.

ii) If A⊆B, then (τ_i,τ_j)-pgrwcl(A)$\subseteq(\tau_i,\tau_j)$-pgrwcl(B).

iii) (τ_i,τ_j)-pgrwcl(A)$\cup(\tau_i,\tau_j)$-pgrwcl(B)$\subseteq(\tau_i,\tau_j)$-pgrwcl(A∪B)

Proof:

i) Follows from definition.

ii) In a bitopological space (X,τ_1,τ_2) A and B are subsets of X.

(τ_i,τ_j)-pgrwcl(A)$=\cap F_{F\epsilon Ƒ}$ where $Ƒ$={F:F is a (τ_i,τ_j)-pgrw-closed set containing A} (τ_i,τ_j)-pgrwcl(B)$=\cap F_{F\epsilon \mathcal{F}}$ where $\mathcal{F}$={ F:F is a (τ_i,τ_j)-pgrw-closed set containing B}

$A\subseteq B \Rightarrow \mathcal{F}\subseteq Ƒ \Rightarrow \cap F_{F\epsilon Ƒ}\subseteq \cap F_{F\epsilon \mathcal{F}} \Rightarrow$ (τ_i,τ_j)-pgrwcl(A)$\subseteq(\tau_i,\tau_j)$-pgrwcl(B)

iii) $A\subseteq A\cup B$ and $B\subseteq A\cup B$

$\Rightarrow$ (τ_i,τ_j)-pgrwcl(A)$\subseteq(\tau_i,\tau_j)$-pgrwcl($A\cup B$), (τ_i,τ_j)-pgrwcl(B)$\subseteq(\tau_i,\tau_j)$-pgrwcl($A\cup B$)

$\Rightarrow$ (τ_i,τ_j)-pgrwcl(A)$\cup(\tau_i,\tau_j)$-pgrwcl(B)$\subseteq(\tau_i,\tau_j)$-pgrwcl($A\cup B$).

4.8 Theorem: If A and B are subsets of a bitopological space (X,τ_1,τ_2), then (τ_i,τ_j)-pgrwcl($A\cap B$)$\subseteq(\tau_i,\tau_j)$-pgrwcl(A)$\cap(\tau_i,\tau_j)$-pgrwcl(B).

Proof: A and B are subsets of (X,τ_1,τ_2). $A\cap B\subseteq A$ and $A\cap B\subseteq B$

$\therefore$ (τ_i,τ_j)-pgrwcl($A\cap B$)$\subseteq(\tau_i,\tau_j)$-pgrwcl(A), (τ_i,τ_j)-pgrwcl($A\cap B$)$\subseteq(\tau_i,\tau_j)$-pgrwcl(B).

$\therefore$ (τ_i,τ_j)-pgrwcl($A\cap B$)$\subseteq(\tau_i,\tau_j)$-pgrwcl(A)$\cap(\tau_i,\tau_j)$-pgrwcl(B)

4.9 Theorem: A is a nonempty subset of a bitopological space (X,τ_1,τ_2). $x\epsilon(\tau_i,\tau_j)$-pgrwcl(A) if and only if $A\cap V \neq \phi$ $\forall$ (τ_i,τ_j)-pgrw-open set V containing x.

Proof: A is a nonempty subset of a bitopological space (X,τ_1,τ_2) and $x\epsilon(\tau_i,\tau_j)$-pgrwcl(A)....hypothesis.

Suppose $\exists$ a (τ_i,τ_j)-pgrw-open set V containing x such that $A\cap V=\phi$. Then $A\subseteq X-V$ and $X-V$ is a (τ_i,τ_j)-pgrw-closed set and so (τ_i,τ_j)-pgrwcl(A)$\subseteq X-V$.

$\therefore$ $x \notin V$ which is a contradiction. Hence $A\cap V \neq \phi$ $\forall$ (τ_i,τ_j)-pgrw-open set V containing x.

Conversely

A is a nonempty subset of (X,τ_1,τ_2) and $x\epsilon X$ is such that $A\cap V\neq\phi$ $\forall$ (τ_i,τ_j)-pgrw-open set V containing x......hypothesis.

$x \notin (\tau_i,\tau_j)$-pgrwcl(A)

$\Rightarrow$ There exists a (τ_i,τ_j)-pgrw-closed set F such that $A\subseteq F$ and $x\notin F$.

$\Rightarrow$ There exists a (τ_i,τ_j)-pgrw-open set $X-F$ containing x and $A\cap(X-F)=\phi$ which is a contradiction. $\therefore$ $x \epsilon(\tau_i,\tau_j)$-pgrwcl(A)

4.10 Theorem: In a bitopological space (X,τ_1,τ_2) if $\tau_1\subseteq\tau_2$ and $RWO(X,\tau_1)\subseteq RWO(X,\tau_2)$, then

$\forall$ $A\subseteq X$, (τ_1,τ_2)-pgrwcl(A)$\subseteq(\tau_2,\tau_1)$-pgrwcl(A).

Proof: A is a subset of a bitopological space (X,τ_1,τ_2).

(τ_1,τ_2)-pgrwcl(A)$=\cap F_{F\epsilon Ƒ}$ where Ƒ={F:F is a (τ_1,τ_2)-pgrw-closed set and $A\subseteq F$}.

(τ_2,τ_1)-pgrwcl(A)$=\cap F_{F\epsilon \mathcal{F}}$ where $\mathcal{F}$={F:F is a (τ_2,τ_1)-pgrw-closed set and $A\subseteq F$}.

$\tau_1\subseteq\tau_2$ and $RWO(X,\tau_1)\subseteq RWO(X,\tau_2)$

$\Rightarrow$ $D_{pgrw}(X,\tau_2,\tau_1)\subseteq D_{pgrw}(X,\tau_1,\tau_2)$ (3.8).

$\Rightarrow$ $\mathcal{F}\subseteq Ƒ$

$\Rightarrow$ $\cap F_{F\epsilon Ƒ} \subseteq \cap F_{F\epsilon \mathcal{F}}$

$\Rightarrow$ (τ_1,τ_2)-pgrwcl(A)$\subseteq(\tau_2,\tau_1)$-pgrwcl(A).

4.11 Theorem: For every subset A of a bitopological space (X,τ_1,τ_2),

i) (τ_i,τ_j)-pgrwcl(A)$\subseteq\tau_j$-cl(A)

ii) (τ_i,τ_j)-pgrwcl(A)$\subseteq\tau_j$-pcl(A)

iii) (τ_i,τ_j)-pgrwcl(A)$\subseteq\tau_j$-rcl(A)

iv) (τ_i,τ_j)-pgrwcl(A)$\subseteq\tau_j$-δcl(A)

v) (τ_i,τ_j)-pgrwcl(A)$\subseteq\tau_j$-αcl(A)

vi) (τ_i,τ_j)-pgrwcl(A)$\subseteq\tau_j$-#rgcl(A).

Proof:

i) A is a subset of a bitopological space (X,τ_1,τ_2).

τ_j-cl(A)=$\cap F_{F\epsilon\mathcal{F}}$ where $\mathcal{F}$={F:A$\subseteq$F and F is τ_j-closed.}.

(τ_i,τ_j)-pgrwcl(A)=$\cap F_{F\epsilon\digamma}$ where $\digamma$ ={F:F$\supseteq$A and F is (τ_i,τ_j)-pgrw-closed.}

Every τ_j-closed set is (τ_i,τ_j)-pgrw-closed(3.19(i)).

$\Rightarrow$ $\mathcal{F}\subseteq\digamma$ $\Rightarrow$ $\cap F_{F\epsilon\digamma}\subseteq\cap F_{F\epsilon\mathcal{F}}$

$\Rightarrow$ (τ_i,τ_j)-pgrwcl(A) $\subseteq$ τ_j-cl(A).

Similarly other results may be proved.

4.12 Theorem: For every subset A of a bitopological space (X,τ_1,τ_2)

i) (τ_i,τ_j)-gpcl(A) $\subseteq$ (τ_i,τ_j)-pgrwcl(A)

ii) (τ_i,τ_j)-gprcl(A) $\subseteq$ (τ_i,τ_j)-pgrwcl(A)

Proof: i) A is a subset of a bitopological space (X,τ_1,τ_2).

(τ_i,τ_j)-gpcl(A)=$\cap F_{F\epsilon\mathcal{F}}$ where $\mathcal{F}$={F: F is a (τ_i,τ_j)-gp closed set containing A}.

(τ_i,τ_j)-pgrwcl(A)=$\cap F_{F\epsilon\digamma}$ where $\digamma$={F:F is a (τ_i,τ_j)-pgrw closed set containing A}. Every (τ_i,τ_j)-pgrw closed set is (τ_i,τ_j)-gp closed (3.24).

$\Rightarrow$ $\digamma\subseteq\mathcal{F}$

$\Rightarrow$ $\cap F_{F\epsilon\mathcal{F}}\subseteq\cap F_{F\epsilon\digamma}$

$\Rightarrow$ (τ_i,τ_j)-gpcl(A)$\subseteq(\tau_i,\tau_j)$-pgrwcl(A)

Similarly (ii) may be proved.

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