A New Bell Shape Fuzzy Number

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Abstract —*Uncertainty can be expressed by fuzzy numbers. Some fuzzy numbers are triangular and trapezoidal. Although it is easy to perform calculations on them, they are not flexible enough to show the uncertainty. Bell shape fuzzy numbers are very flexible. However, it is very difficult to conduct arithmetic operations on them. In this paper, a new bell shape fuzzy number has been developed. The new fuzzy number looks like a bell. Then, arithmetic operations have been defined for the new fuzzy number. Also, it has been proven that the ranking of the new fuzzy number can be done successfully. The above mentioned operations and the ranking method of the new fuzzy number have been described by some examples. The new fuzzy number is more flexible compared to triangular and trapezoidal ones. Also, arithmetic calculation of the new fuzzy number is much simpler than that of a bell shape fuzzy number.*

Keywords (Size 10 & Bold) —*New fuzzy number, Bell shape fuzzy number, Operations on fuzzy numbers.*

I. INTRODUCTION

Before the advent of fuzzy set theory, researches used the random variables to describe the uncertainty. It is obvious that it is difficult to perform arithmetic operations on random variables. In addition, random variables cannot describe the all cases of uncertainty completely. So, Zade in 1965 introduced promising new horizons to different scientific areas. Fuzzy theory, with presuming imprecision in deciding parameters and utilizing mental models of experts, is a useful approach to adapt scheduling models into reality [1]. (it's ok but read it again.)

In recent decades, trapezoidal fuzzy numbers have been used for describing uncertain parameters in engineering systems. In special cases, these numbers can be transformed to triangular fuzzy numbers. Although summation and subtraction can be performed on both numbers easily, they aren't flexible enough to describe the uncertain parameters. However, some researchers have used the trapezoidal fuzzy numbers to describe activity durations in GERT-type Networks [2], [3] and Metagraphs [4], [5], [6], and [7].Although bell shape fuzzy numbers are very flexible, performing summation and subtraction on these numbers is challenging.

Therefore, in this paper, a new fuzzy number have been developed which is very similar to bell shape fuzzy number. The new fuzzy number is called new bell shape fuzzy number (NBSFN). In addition to being a flexible fuzzy number in describing the uncertainty, performing summation and subtraction on this number is much easier due to the way it is defined. Also, a method has been introduced for ranking the numbers. The above mentioned topics have been clarified more, through some examples.

II. A NEW BELL SHAPE FUZZY NUMBER

The trapezoidal and triangular fuzzy numbers have been defined in reference [8].

Among the various types of fuzzy numbers, triangular and trapezoidal fuzzy numbers are the most important ones. They are especially useful in solving possibility mathematical programming problems.

The triangular fuzzy numbers can be denoted as X=(x, α, β) and Y=(y,r,δ), where x and y are the central values$(\mu_x(x) = 1 \; and \; \mu_r(y) = 1)$, α and r are the left spreads, and β and δ are the right spreads (see figure 1).

Similarly, the trapezoidal fuzzy numbers can be denoted as $X = (x_1^m, x_2^m, \alpha, \beta)$.

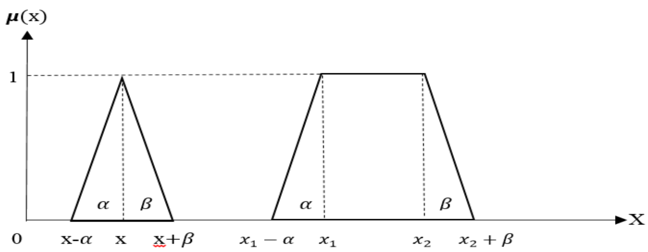

Fig. 1The triangular and trapezoidal fuzzy numbers

The newly proposed fuzzy number is similar to bell shape fuzzy number. The membership function of the new fuzzy number has eight parameters. a, b, c, d, e, f, g and h which show the parameters of the new fuzzy number such that $\mu_{\tilde{A}}(a) = \mu_{\tilde{A}}(h) = 0$ and $\mu_{\tilde{A}}(d) = \mu_{\tilde{A}}(e) = 1$.

Membership function of NBSFN is as follows:

$$\mu_{\tilde{Z}}(\boldsymbol{x}) = \begin{cases} 0 & x \le a \\ \dfrac{(\mu_{\tilde{A}}(b))(x-a)}{b-a} & a \le x \le b \\ \dfrac{(x-b)(\mu_{\tilde{A}}(c)) + (c-x)(\mu_{\tilde{A}}(b))}{c-b} & b \le x \le c \\ \dfrac{(d-x)(\mu_{\tilde{A}}(c)) + (x-c)}{d-c} & c \le x \le d \\ 1 & d \le x \le e \\ \dfrac{(x-e)(\mu_{\tilde{A}}(f)) + (f-x)}{f-e} & e \le x \le f \\ \dfrac{(x-f)(\mu_{\tilde{A}}(g)) + (g-x)(\mu_{\tilde{A}}(f))}{g-f} & f \le x \le g \\ \dfrac{(\mu_{\tilde{A}}(g))(h-x)}{h-g} & g \le x \le h \\ 0 & x \ge h \end{cases} \tag{1}$$

The shape of NBSFN has been shown in fig.2.

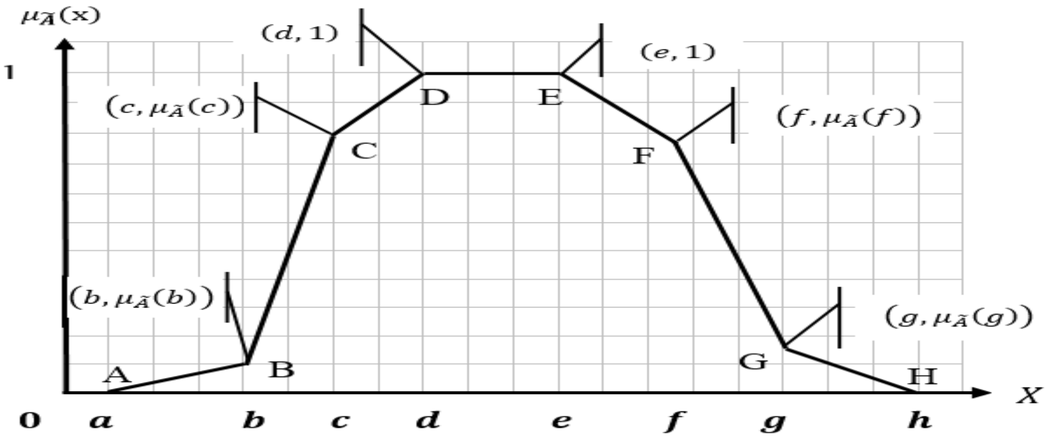

Fig. 2 The general figure of the new bell shape fuzzy number

It is obvious that $0 \le a \le b \le c \le d \le e \le f \le g \le h$. (2)

There are some conditions related to membership function:

For points A, B, C and D: $$\begin{cases} \frac{\mu_{\tilde{A}}(b)-0}{b-a} \le \frac{\mu_{\tilde{A}}(c)-\mu_{\tilde{A}}(b)}{c-b} \\ \frac{1-\mu_{\tilde{A}}(c)}{d-c} \le \frac{\mu_{\tilde{A}}(c)-\mu_{\tilde{A}}(b)}{c-b} \end{cases} \tag{3}$$

For points E, F, G and H: $\begin{cases} \frac{\mu_{\tilde{A}}(f)-1}{f-e} \geq \frac{\mu_{\tilde{A}}(g)-\mu_{\tilde{A}}(f)}{g-f} \\ \frac{0-\mu_{\tilde{A}}(g)}{h-g} \geq \frac{\mu_{\tilde{A}}(g)-\mu_{\tilde{A}}(f)}{g-f} \end{cases}$ (4)

If $\begin{bmatrix} \frac{\mu_{\tilde{A}}(b)-0}{b-a} = \frac{\mu_{\tilde{A}}(c)-\mu_{\tilde{A}}(b)}{c-b} = \frac{1-\mu_{\tilde{A}}(c)}{d-c} \\ and \\ \frac{\mu_{\tilde{A}}(f)-1}{f-e} = \frac{\mu_{\tilde{A}}(g)-\mu_{\tilde{A}}(f)}{g-f} = \frac{0-\mu_{\tilde{A}}(g)}{h-g} \end{bmatrix}$ then NBSFN is transformed to a trapezoidal fuzzy number. It is evident that trapezoidal fuzzy number is a special case of NBSFN.

The above mentioned inequalities make the shape of NBSFN similar to a bell.
For showing $(x,\ \mu_{\tilde{A}}(x))$, a matrix with 2 rows and 8 columns was used which is described as follows:

$$\tilde{A} = \begin{pmatrix} a & b & c & d & e & f & g & h \\ 0 & \mu_{\tilde{A}}(b) & \mu_{\tilde{A}}(c) & 1 & 1 & \mu_{\tilde{A}}(f) & \mu_{\tilde{A}}(g) & 0 \end{pmatrix} \qquad (5)$$

III. ARITHMETIC OPERATIONS ON THE NEW BELL SHAPE FUZZY NUMBER

Suppose that $\tilde{A}$ and $\tilde{B}$ are two bell shape fuzzy numbers:

$$\tilde{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & \mu_{\tilde{A}}(a_2) & \mu_{\tilde{A}}(a_3) & 1 & 1 & \mu_{\tilde{A}}(a_6) & \mu_{\tilde{A}}(a_7) & 0 \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ 0 & \mu_{\tilde{B}}(b_2) & \mu_{\tilde{B}}(b_3) & 1 & 1 & \mu_{\tilde{B}}(b_6) & \mu_{\tilde{B}}(b_7) & 0 \end{pmatrix}$$

Then sum of the two new bell shape fuzzy numbers can be computed as follows: $\tilde{A} \oplus \tilde{B} =$

$$\left(\frac{a_1+b_1}{0}, \frac{a_2+b_2}{\max\{\mu_{\tilde{A}}(a_2),\mu_{\tilde{B}}(b_2)\}}, \frac{a_3+b_3}{\max\{\mu_{\tilde{A}}(a_3),\mu_{\tilde{B}}(b_3)\}}, \frac{a_4+b_4}{1}, \frac{a_5+b_5}{1}, \frac{a_6+b_6}{\max\{\mu_{\tilde{A}}(a_6),\mu_{\tilde{B}}(b_6)\}}, \frac{a_7+b_7}{\max\{\mu_{\tilde{A}}(a_7),\mu_{\tilde{B}}(b_7)\}}, \frac{a_8+b_8}{0}\right) \qquad (6)$$

A subtraction operator is defined for the two positive trapezoidal fuzzy numbers such that the subtraction of them is a positive trapezoidal fuzzy number [1].

By generalizing this operator to the new bell shape fuzzy numbers, subtraction of two NBSFNs will be a positive bell shape fuzzy number as follows:

$$\tilde{B} - \tilde{A} = \tilde{C} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ 0 & \mu_{\tilde{C}}(c_2) & \mu_{\tilde{C}}(c_3) & 1 & 1 & \mu_{\tilde{C}}(c_6) & \mu_{\tilde{C}}(c_7) & 0 \end{pmatrix} \qquad (7)$$

$$c_8 = max(0\ ,min\,(b_8 - a_8)),\ \mu_{\tilde{C}}(c_8) = min\big(\mu_{\tilde{A}}(a_8), \mu_{\tilde{B}}(b_8)\big) = 0$$

$$c_7 = max(0\ ,min\,(c_8, (b_7 - a))),\ \mu_{\tilde{C}}(c_7) = min\big(\mu_{\tilde{A}}(a_7), \mu_{\tilde{B}}(b_7)\big)$$

If $c_6 = max\,(0\ ,min\,(c_7, (b_6 - a_6)))$, $\mu_{\tilde{C}}(c_6) = min\big(\mu_{\tilde{A}}(a_6), \mu_{\tilde{B}}(b_6)\big)$ can satisfy this inequality $\left(\frac{-\mu_{\tilde{C}}(c_7)}{c_8-c_7} \geq \frac{\mu_{\tilde{C}}(c_7)-\mu_{\tilde{C}}(c_6)}{c_7-c_6}\right)$ then c_6 is acceptable. Otherwise c_6 should be computed as follows:

$$c_6 = \frac{(c_8 - c_7)\big(\mu_{\tilde{C}}(c_6)\big) - c_8\big(\mu_{\tilde{C}}(c_7)\big)}{-\mu_{\tilde{C}}(c_7)}$$

If $c_5 = max\,(0\ ,min\,(c_6, (b_5 - a_5)))$, $\mu_{\tilde{C}}(c_5) = min\big(\mu_{\tilde{A}}(a_5), \mu_{\tilde{B}}(b_5)\big)$ can satisfy this inequality $\left(\frac{\mu_{\tilde{C}}(c_6)-1}{c_6-c_5} \geq \frac{\mu_{\tilde{C}}(c_7)-\mu_{\tilde{C}}(c_6)}{c_7-c_6}\right)$ then c_5 is acceptable. Otherwise c_5 should be computed as follows:

$$c_5 = \frac{(c_7 - c_6) - c_7\big(\mu_{\tilde{C}}(c_6)\big) + c_6\big(\mu_{\tilde{C}}(c_7)\big)}{\mu_{\tilde{C}}(c_7) - \mu_{\tilde{C}}(c_6)}$$

$$c_4 = max(0\ ,min\,(c_5, (b_4 - a_4))),\ \mu_{\tilde{C}}(c_4) = min\big(\mu_{\tilde{A}}(a_4), \mu_{\tilde{B}}(b_4)\big)$$

$c_3 = max(0\ ,min(c_4, (b_3 - a_3))),\ \mu_{\tilde{C}}(c_3) = min(\mu_{\tilde{A}}(a_3), \mu_{\tilde{B}}(b_3))$

If $c_2 = max\ (0\ ,min(c_3, (b_2 - a_2))),\ \mu_{\tilde{C}}(c_2) = min(\mu_{\tilde{A}}(a_2), \mu_{\tilde{B}}(b_2))$ can satisfy this inequality $\left(\frac{1-\mu_{\tilde{C}}(c_3)}{c_4-c_3} \leq \frac{\mu_{\tilde{C}}(c_3)-\mu_{\tilde{C}}(c_2)}{c_3-c_2}\right)$ then c_2 is acceptable. Otherwise c_2 should be computed as follows:

$$c_2 = \frac{(c_4 - c_3)(\mu_{\tilde{C}}(c_2)) - c_4(\mu_{\tilde{C}}(c_3)) + c_3}{1 - \mu_{\tilde{C}}(c_3)}$$

If $c_1 = max(0\ ,min(c_2, (b_1 - a_1))),\ \mu_{\tilde{C}}(c_1) = min(\mu_{\tilde{A}}(a_1), \mu_{\tilde{B}}(b_1)) = 0$ can satisfy this inequality $\left(\frac{\mu_{\tilde{C}}(c_2)}{c_2-c_1} \leq \frac{\mu_{\tilde{C}}(c_3)-\mu_{\tilde{C}}(c_2)}{c_3-c_2}\right)$ then c_1 is acceptable. Otherwise c_1 should be computed as follows:

$$c_1 = \frac{(c_2)(\mu_{\tilde{C}}(c_3)) - c_3(\mu_{\tilde{C}}(c_2))}{\mu_{\tilde{C}}(c_3) - \mu_{\tilde{C}}(c_2)}$$

3.1 Example A.

$$\tilde{A} = \begin{pmatrix} 5 & 10 & 13 & 17 & 21 & 25 & 29 & 33 \\ 0 & 0.3 & 0.8 & 1 & 1 & 0.8 & 0.2 & 0 \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} 7 & 11 & 14 & 17 & 22 & 27 & 30 & 35 \\ 0 & 0.2 & 0.8 & 1 & 1 & 0.7 & 0.3 & 0 \end{pmatrix}$$

$$\tilde{A} \oplus \tilde{B} = \begin{pmatrix} 12 & 21 & 27 & 34 & 43 & 52 & 59 & 68 \\ 0 & 0.3 & 0.8 & 1 & 1 & 0.8 & 0.3 & 0 \end{pmatrix}$$

$$\tilde{B} - \tilde{A} = \tilde{C}$$

$c_8 = max(0\ ,min(35 - 33)) = 2$, $\mu_{\tilde{C}}(c_8) = min(0, 0\ ) = 0$
$c_7 = max(0\ ,min(2, (30 - 29))) = 1$, $\mu_{\tilde{C}}(c_7) = min(0.3, 0.2\ ) = 0.2$
$c_6 = max(0\ ,min(1, (27 - 25))) = 1$, $\mu_{\tilde{C}}(c_6) = min(0.7, 0.8) = 0.7$
$c_5 = max(0\ ,min(1, (22 - 21))) = 1$, $\mu_{\tilde{C}}(c_5) = min(1, 1\ ) = 1$
$c_4 = max(0\ ,min(1, (17 - 17))) = 0$, $\mu_{\tilde{C}}(c_4) = min(1, 1\ ) = 1$
$c_3 = max(0\ ,min(0, (14 - 13))) = 0$, $\mu_{\tilde{C}}(c_3) = min(0.7, 0.8\ ) = 0.7$
$c_2 = max(0\ ,min(0, (11 - 10))) = 0$, $\mu_{\tilde{C}}(c_2) = min(0.3, 0.2\ ) = 0.2$
$c_1 = max(0\ ,min(0, (7 - 5))) = 0$, $\mu_{\tilde{C}}(c_1) = min(0, 0\ ) = 0$

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0.2 & 0.7 & 1 & 1 & 0.7 & 0.2 & 0 \end{pmatrix}$$

IV. RANKING METHOD FOR THE NBSFNS

Ranking methods of fuzzy numbers can be used for comparing the two NBSFNs. Like this method [9].
Now, Chen-Lu method is described:

For a fuzzy number A, the α-cuts (level sets) $A_a = \{x \in R\ |\mu_A(x) \geq a\}, a \in [0,1]$, are convex subsets of R. the lower and upper limits of the K α-cut for the fuzzy number A are defined as:

$l_{i,k} = inf\{x\ |\mu_A(x) \geq a_k\}, x \in R$

$r_{i,k} = sup\{x\ |\mu_A(x) \geq a_k\}, x \in R$

Where $l_{i,k}$ and $r_{i,k}$ are left and right spreads, respectively. While two fuzzy numbers A_i and A_j are compared, fig.3 illustrates their corresponding left and right spreads at a_k level.

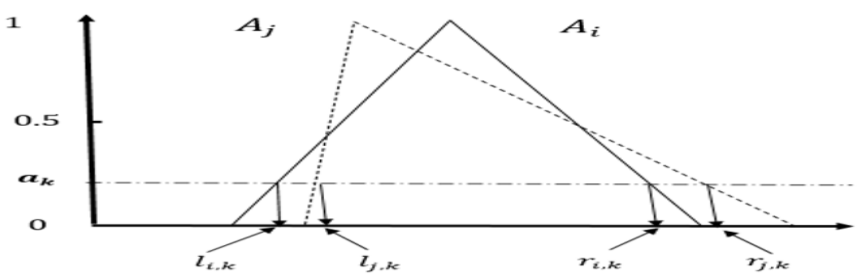

Fig. 3The left and right spreads of fuzzy numbers A_i and A_j.

The left (right) dominance $D^L_{i,j}(D^R_{i,j})$ of A_i over A_j is defined as the average difference of the left (right) spreads at some α-levels. They are formulated as:

$$D_{i,j}^{L} = \frac{1}{n+1}\sum_{k=0}^{n}\left(l_{i,k} - l_{j,k}\right) \tag{8}$$

$$D_{i,j}^{R} = \frac{1}{n+1}\sum_{k=0}^{n}\left(r_{i,k} - r_{j,k}\right) \tag{9}$$

Where n+1 α-cuts are used to calculate the dominance. Let a_k denote the K α-level, and $a_k = \frac{k}{n}, k \in \{0,1,\dots,n\}$. Therefore, the distance between each two adjacent α-level is equal; $a_k - a_{k-1} = \frac{1}{n}, k \geq 1$.

The total dominance of A_i over A_j with the index of optimism $\beta \in [0,1]$ can be defined as the convex combination of $D_{i,j}^{L}$ and $D_{i,j}^{R}$ by

$$D_{i,j}(\beta) = \beta D_{i,j}^{R} + (1-\beta) D_{i,j}^{L}$$
$$= \beta\left[\frac{1}{n+1}\sum_{k=0}^{n}\left(r_{i,k} - r_{j,k}\right)\right] + (1-\beta)\left[\frac{1}{n+1}\sum_{k=0}^{n}\left(l_{i,k} - l_{j,k}\right)\right]$$
$$= \frac{1}{n+1}\left\{\left[\beta\sum_{k=0}^{n} r_{i,k} + (1-\beta)\sum_{k=0}^{n} l_{i,k}\right] - \left[\beta\sum_{k=0}^{n} r_{j,k} + (1-\beta)\sum_{k=0}^{n} l_{j,k}\right]\right\} \tag{10}$$

The above equation indicates that the total dominance is actually a comparison function.

A decision maker can rank a pair of fuzzy numbers, A_i and A_j, using $D_{i,j}(\beta)$, based on the following rules:

(1) If $D_{i,j}(\beta) > 0$, then $A_i > A_j$;

(2) If $D_{i,j}(\beta) = 0$, then $A_i = A_j$; and

(3) If $D_{i,j}(\beta) < 0$, then $A_i < A_j$;

4.1 Example B.

Fig.4 illustrates two new bell shape fuzzy numbers,

$$\tilde{A} = \begin{pmatrix} 5 & 10 & 13 & 17 & 21 & 25 & 29 & 33 \\ 0 & 0.3 & 0.8 & 1 & 1 & 0.8 & 0.2 & 0 \end{pmatrix}$$
$$\tilde{B} = \begin{pmatrix} 7 & 11 & 14 & 17 & 22 & 27 & 30 & 35 \\ 0 & 0.2 & 0.8 & 1 & 1 & 0.7 & 0.3 & 0 \end{pmatrix}$$

Equation (10) with $\beta = 0.5$ and n=4 was used

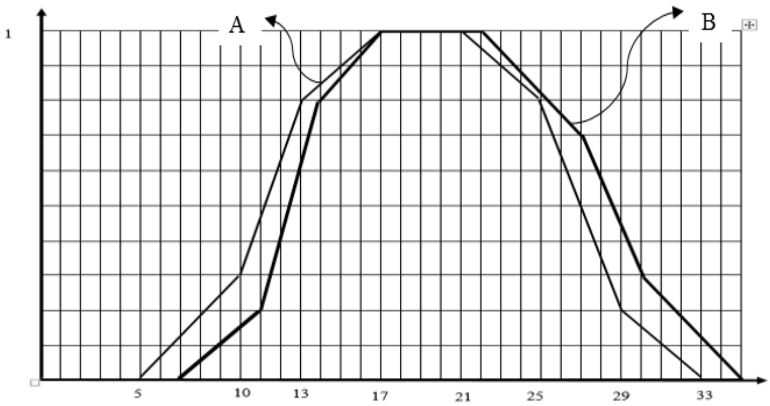

Fig. 4 Graph of fuzzy numbers $\tilde{A}$ and $\tilde{B}$

Table 1. The upper and lower limit of K^{th} α-cut for fuzzy numbers $\tilde{A}$ and $\tilde{B}$

a	0	0.25	0.5	0.75	1
$L_{A,k}$	5	9.2	11.2	12.7	17
$R_{A,k}$	33	28.7	27	25.3	21
$L_{B,k}$	7	11.2	12.5	13.75	17
$R_{B,k}$	35	30.8	28.5	26.2	22

$$D_{A,B}^{L} = \frac{1}{n+1}\sum_{k=0}^{n}\left(l_{A,k} - l_{B,k}\right)$$

$$= \frac{1}{5}\left((5-7) + (9.2-11.2) + (11.2-12.5) + (12.7-13.75) + (17-17)\right) = -1.37$$

$$D_{A,B}^{R} = \frac{1}{n+1}\sum_{k=0}^{n}\left(r_{A,k} - r_{B,k}\right)$$

$$= \frac{1}{5}\left((33-35) + (28.7-30.8) + (27-28.5) + (25.3-26.2) + (21-22)\right) = -1.5$$

$D_{i,j}(\beta) = \beta D_{i,j}^{R} + (1-\beta) D_{i,j}^{L}$
$D_{A,B}(0.5) = 0.5(-1.5) + 0.5(-1.37) = -1.43$
We conclude that: $D_{A,B}(0.5) < 0 \Rightarrow \tilde{B} > \tilde{A}$.

V. CONCLUSIONS AND RECOMENDATIONS

The NBSFN is more flexible than triangular and trapezoidal fuzzy numbers. The arithmetic calculations performed on NBSFNs are simpler than those performed on the already existing bell shape fuzzy numbers.

A. In the future, researchers can develop suitable operations to recognize the maximum and minimum of NBSFNs.

B. A method can be developed for fuzzification of the new fuzzy number.

In project planning and control, new bell shape fuzzy numbers can show activity durations. In addition, the forward and backward computations can be generalized for fuzzy metagraphs.

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