The Axioms of Spheres in geometry of Kaehler Norden Manifold with CR-Submanifold and Lightlike Submanifold

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Abstract — In this paper, we prove that if an indefinite Kaehler Norden manifold $\overline{M}$ with CR-submanifold (M, g) and lightlike submanifold $(M, \tilde{g})$ satisfies the axioms of transversal r-spheres and r-planes, then $\overline{M}$ is an indefinite complex space form.

Keywords — *Kaehler Norden manifold, Lightlike, CR, submanifolds.*

I. Introduction

Cartan [3] initiated the study of axioms of planes on Riemannian manifolds which was generalized by Yano and Mogi [16] to holomorphic planes on Kaehler manifolds. Leung and Nomizu extended this idea to the axioms of spheres on Riemannian manifolds. Further, Chen and Ogiue [4] proved that the same concept holds true on a Kaehler manifold satisfying the axioms of anti-holomorphic planes. In [10], Goldberg and Moskal generalized the notion to the axioms of holomorphic spheres and anti-holomorphic spheres on a Kaehler manifold, respectively. In [11], Kumar et. al extended the study for lightlike submanifolds on semi-Riemannian manifolds and proved the axioms of r-spheres and r-planes.

The aim of this paper is to study the axioms of transversal r-planes and r-spheres, to the setting of indefinite Kaehler Norden manifolds having CR-submanifold and radical transversal lightlike submanifold.

II. Preliminaries

Kaehler Norden manifolds[8] : Let $(\overline{M}^{2n}, \bar{J}, \bar{g})$ be an almost complex manifold with an almost complex structure $\bar{J}$ and metric $\bar{g}$ on it. The metric $\bar{g}$ is known as a Norden metric on $\overline{M}$ if

$$\bar{g}(\bar{J}X, \bar{J}Y) = -\bar{g}(X, Y),$$

for all vector fields X and Y on $\overline{M}$. Further, the metric $\bar{\tilde{g}}$ on $\overline{M}$ is defined by

$$\bar{\tilde{g}}(X, Y) = \bar{g}(\bar{J}X, Y), \qquad (1)$$

for arbitrary vector fields X and Y on $\overline{M}$ and $\bar{\tilde{g}}$ is a Norden metric and also known as an associated metric. Moreover, the metrics $\bar{g}$ and $\bar{\tilde{g}}$ are indefinite of neutral signature (n, n).

Let $\bar{\nabla}$ and $\bar{\tilde{\nabla}}$ denote the Levi-Civita connection for the metric $\bar{g}$ and $\bar{\tilde{g}}$, respectively then

$$\varphi(X,Y) = \bar{\tilde{\nabla}}_X Y - \bar{\nabla}_X Y \qquad (2)$$

is a symmetric tensor field of type $(1,2)$ on $\overline{M}$.

The tensor field F of type $(0,3)$ on $\overline{M}$ is defined as $F(X, Y, Z) = \bar{g}((\bar{\nabla}_X \bar{J})Y, Z)$ and satisfies the following property

$$F(X, Y, Z) = F(X, Z, Y) = F(X, \bar{J}Y, \bar{J}Z)$$

for all vector fields X, Y and Z on $\overline{M}$.

In [8], Ganchev and Borisov characterized eight different classes of almost complex manifolds with Norden metric by imposing conditions on the tensor F. Moreover, the following relations between the tensor F and φ were provided in [9].

$$\varphi(X,Y,Z) = \frac{1}{2}[F(\bar{J}Z,X,Y) - F(X,Y,\bar{J}Z) - F(Y,\bar{J}Z,X)] \qquad (3)$$

and

$$F(X,Y,Z) = \varphi(X,Y,\bar{J}Z) + \varphi(X,Z,\bar{J}Y)]$$

for all tangent vector fields X, Y and Z on $\bar{M}$, where $\varphi(X,Y,Z) = \bar{g}(\varphi(X,Y),Z)]$.
For a Kaehler Norden manifold $\bar{M}$, the characterization condition $F(X,Y,Z) = 0$ is equivalent to $(\bar{\nabla}_X \bar{J})Y = 0$.
Therefore from (3), for a Kaehler Norden manifold we have

$$\varphi = 0.$$

Throughout this paper, we will call $\bar{M}$ as Kaehler Norden manifolds.

CR-submanifolds of Kaehler Norden Manifold:

Let (M, g) be an m-dimensional CR-submanifold of a $2n$-dimensional Kaehler Norden manifold $\bar{M}$. Then there exist two complementary orthogonal distributions D of dimension $2p$ and $D^{\perp}$ of dimension r, where $1 \leq r < \min\{m, 2n-m\}$, of (M, g) such that D and $D^{\perp}$ are invariant and anti-invariant distributions with respect to an almost complex structure $\bar{J}$ respectively, that is, $\bar{J}D = D$ and $\bar{J}D^{\perp} \subset TM^{\perp}$.

Then the tangent bundle $T\bar{M}$ of $\bar{M}$ has the following decomposition ([2,5])

$$T\bar{M} = TM \perp TM^{\perp} = D \perp D^{\perp} \perp \bar{J}D^{\perp} \perp (\bar{J}D^{\perp})^{\perp}$$

Let P and Q be the projection morphisms on radical distribution and screen distribution, respectively. Then for any $X \in \Gamma(TM)$, we have

$$X = PX + QX \qquad (4)$$

where $PX \in \Gamma(D)$ and $QX \in \Gamma(D^{\perp})$. Applying $\bar{J}$ to (4), we obtain

$$\bar{J}X = TX + FX$$

where TX and FX are the tangential and normal components of $\bar{J}X$, respectively.
Similarly, for any $U \in \Gamma(TM^{\perp})$,

$$\bar{J}U = tU + fU \qquad (5)$$

where tU and fU denote the tangent and normal sections of $\bar{J}U$, respectively.

Let $\bar{\nabla}$ and ∇ denote the Levi-Civita connection of $\bar{g}$ and g on $\bar{M}$ and M. Then the Gauss and Weingarten formulae are given as :

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X U = -A_U X + D_X U, \qquad (6)$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma((TM)^{\perp})$, where h is a symmetric bilinear form on $\Gamma(TM)$ and is known as second fundamental form, A_U is a shape operator on M and D is the normal connection on $(TM)^{\perp}$ which is a metric connection.
Let P_1 and P_2 be the projection morphisms of $(TM)^{\perp}$ on $\bar{J}D^{\perp}$and $(\bar{J}D^{\perp})^{\perp}$ respectively, then (6) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^1(X,Y) + h^2(X,Y), \quad \bar{\nabla}_X U = -A_U X + D^1_X U + D^2_X U \qquad (7)$$

where we put

$$h^1(X,Y) = P_1(h(X,Y)), \quad h^2(X,Y) = P_2(h(X,Y)),$$

$$D^1_X U = P_1(D_X U), \quad D^2_X U = P_2(D_X U).$$

Infact D^1 and D^2 are not linear connections on $TM^{\perp}$ but are Otsuki connections with respect to the vector bundle morphisms P_1 and P_2 respectively. Thus equation (7) can be written as

$$\bar{\nabla}_X Y = \nabla_X Y + h^1(X,Y) + h^2(X,Y), \qquad (8)$$

$$\bar{\nabla}_X N = -A_N X + \nabla^1_X N + D^2(X,N), \qquad (9)$$

$$\overline{\nabla}_X W = -A_W X + D^1(X, W) + \nabla^2_X W, \tag{10}$$

where $\nabla^1_X N = D^1_X N$ and $\nabla^2_X W = D^2_X W$ are metric connections on $\bar{J}D^{\perp}$ and $(\bar{J}D^{\perp})^{\perp}$, respectively and $D^1(X, N) = D^1_X N$ and $D^2(X, W) = D^2_X W$ are bilinear mappings. Using equations (8)-(10), we obtain

$$\overline{g}(h^1(X, Y)N) = \overline{g}(Y, A_N X), \tag{11}$$

$$\overline{g}(h^2(X, Y)W) = \overline{g}(Y, A_W X), \tag{12}$$

$$\overline{g}(D^2(X, N)W) = -\overline{g}(D^1(X, W)N),$$

where $X, Y, Z \in \Gamma(TM)$, $N \in \Gamma(\overline{J}D^{\perp})$ and $W \in \Gamma(\overline{J}D^{\perp})^{\perp}$.

Assuming that the curvature tensors of $\overline{\nabla}$ and ∇ be denoted by $\bar{R}$ and R, respectively, and by making direct calculations, we have

$$\begin{aligned}\overline{R}(X, Y, Z) &= R(X, Y, Z) + A_{h^1(X,Z)}Y - A_{h^1(Y,Z)}X + A_{h^2(X,Z)}Y - A_{h^2(Y,Z)}X \\ &+ (\nabla_X h^1)(Y, Z) - (\nabla_Y h^1)(X, Z) + D^1(X, h^2(Y, Z)) - D^1(Y, h^2(X, Z)) \\ &+ (\nabla_X h^2)(Y, Z) - (\nabla_Y h^2)(X, Z) + D^2(X, h^1(Y, Z)) - D^2(Y, h^1(X, Z))\end{aligned} \tag{13}$$

and

$$\begin{aligned}(\overline{R}(X, Y)Z)^{\perp} &= (\nabla_X h^1)(Y, Z) - (\nabla_Y h^1)(X, Z) + D^1(X, h^2(Y, Z)) - D^1(Y, h^2(X, Z)) \\ &+ (\nabla_X h^2)(Y, Z) - (\nabla_Y h^2)(X, Z) + D^2(X, h^1(Y, Z)) - D^2(Y, h^1(X, Z))\end{aligned} \tag{14}$$

where $$(\nabla_X h^1)(Y, Z) = \nabla^1{}_X h^1(Y, Z) - h^1(\nabla_X Y, Z) - h^1(Y, \nabla_X Z), \tag{15}$$

and

$$(\nabla_X h^2)(Y, Z) = \nabla^2{}_X h^2(Y, Z) - h^2(\nabla_X Y, Z) - h^2(Y, \nabla_X Z), \tag{16}$$

For any $X, Y, Z \in \Gamma(TM)$.

Lightlike Submanifolds of Semi-Riemannian Manifolds :

Let $(\overline{M}, \overline{\tilde{g}})$ be a real $(m+n)$-dimensional semi-Riemannian manifold having constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and $(M, \tilde{g})$ be an m-dimensional submanifold and $\tilde{g}$ be the induced metric of $\overline{\tilde{g}}$ on M. Then M is known as a lightlike submanifold of $\overline{M}$ if $\overline{\tilde{g}}$ is degenerate metric on the tangent bundle TM of M. For a degenerate metric $\tilde{g}$ on M, $T_x M^{\perp}$ is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus both $T_x M$ and $T_x M^{\perp}$ are no longer complementary but degenerate orthogonal subspaces of $T\overline{M}$. So, there exists a subspace known as the radical or null subspace, that is,

$$Rad\,(T_x M) = T_x M \cap T_x M^{\perp}.$$

Further, if the mapping $Rad\,(TM): x \in M \rightarrow RadT_x M$, defines a smooth distribution of rank $r > 0$ on M then the submanifold M is known as an r-lightlike submanifold of $\overline{M}$ ([7]) and $Rad\,(TM)$ is known as the radical distribution on M and a semi-Riemannian complementary distribution $S(TM)$ of $Rad\,(TM)$ in TM is known as the Screen distribution, that is,

$$TM = Rad\,(TM) \perp S(TM), \tag{17}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{18}$$

$$\begin{aligned} T\overline{M}|_M &= TM \oplus tr(TM) \\ &= (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \end{aligned} \tag{19}$$

We have studied the following possible four cases with respect to the dimension m and codimension n of M and rank r of $Rad(TM)$:

1. *r-lightlike*, if $0 < r < \min(m,n)$;
2. *coisotropic*, if $1 \le r = n < m$, $S(TM^{\perp}) = \{0\}$;
3. *isotropic*, if $1 < r = m < n$, $S(TM) = \{0\}$;
4. *totally lightlike*, if $1 < r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$.

For any quasi-orthonormal fields of frames, we have the following theorem :

Theorem 1.([7]) *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)|_{U})$ consisting of smooth section $\{N_a\}$ of $S(TM^{\perp})^{\perp}|_{U}$, where U is a coordinate neighborhood of M, such that*

$$\overline{g}(N_a, \xi_b) = \delta_{ab}, \quad \overline{g}(N_a, N_b) = 0,$$

for any $a, b \in \{1,2,.., r\}$, where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\overline{\tilde{\nabla}}$ denote the Levi-Civita connection on $\overline{M}$, then using the decomposition (19), the Gauss and Weingarten formulae are given as :

$$\overline{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \tilde{h}(X,Y), \quad \overline{\tilde{\nabla}}_X U = -\tilde{A}_U X + \nabla^{\perp}_X U, \tag{20}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\tilde{\nabla}_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here $\tilde{\nabla}$ is a torsion-free linear connection on M, $\tilde{h}$ is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, $\tilde{A}_U$ is a linear a operator on M and known as shape operator.

According to decompositon (18), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then equation (20) becomes

$$\overline{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \tilde{h}^l(X,Y) + \tilde{h}^s(X,Y), \tag{21}$$

$$\overline{\tilde{\nabla}}_X N = -\tilde{A}_N X + \tilde{\nabla}^l_X N + D^s(X,N), \tag{22}$$

$$\overline{\tilde{\nabla}}_X W = -\tilde{A}_W X + \tilde{\nabla}^s_X W + D^l(X,W), \tag{23}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$ and we put

$$\tilde{h}^l(X,Y) = L(\tilde{h}(X,Y)), \quad \tilde{h}^s(X,Y) = S(\tilde{h}(X,Y)),$$

$$D^l_X U = L(\nabla^{\perp}_X U), \quad D^s_X U = S(\nabla^{\perp}_X U).$$

As h^l and h^s are $\Gamma(ltr(TM))$ -valued and $\Gamma(S(TM^{\perp}))$ -valued respectively, therefore these are known as the lightlike second fundamental form and the screen second fundamental form on M.

It is well known From the geometry of non degenerate submanifolds that the induced connection $\tilde{\nabla}$ is a metric connection. But this is not true for lightlike submanifolds (degenerate submanifolds). Indeed, considering $\overline{\tilde{\nabla}}$ a metric connection, we have

$$(\tilde{\nabla}_X g)(Y,Z) = \bar{g}(\tilde{h}^l(X,Y),Z) + \bar{g}(\tilde{h}^l(X,Z),Y),$$

for any $X, Y, Z \in \Gamma(TM)$.

The submanifold (M, g) is said to be totally umbilical submanifold if the first and second fundamental forms are proportional, that is,

$$h(X,Y) = g(X,Y)H$$

for any $X, Y \in \Gamma(TM)$, where H is called the mean curvature vector of M. Using Gauss and Weingarten formulae, it is clear that (M, g) is totally umbilical if and only if on each coordinate neighborhood u there exists smooth vector fields $H^1 \in \Gamma(\bar{J}D^{\perp})$ and $H^2 \in \Gamma(\bar{J}D^{\perp})^{\perp}$, such that

$$h^1(X,Y) = H^1 g(X,Y), h^2(X,Y) = H^2 g(X,Y), \tag{24}$$

Similarly, in case of lightlike submanifold, we have the following definition.

Definition 2. [7] A lightlike submanifold $(M, \tilde{g})$ of a semi-Riemannian manifold $(\bar{M}, \bar{J}, \bar{g}, \bar{\tilde{g}})$ is said to be totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, known as the transversal curvature tensor field of M, such that, for any $X, Y \in \Gamma(TM)$,

$$\tilde{h}(X,Y) = \tilde{g}(X,Y)H.$$

Definition 3. [13] Let $(M, \tilde{g}, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of an almost complex manifold with Norden metric $(\bar{M}, \bar{J}, \bar{g}, \bar{\tilde{g}})$. Then $(M, \tilde{g})$ is called radical transversal lightlike submanifold of $\bar{M}$ if

$$\bar{J}(Rad(TM)) = ltr(TM), \tag{25}$$

$$\bar{J}(S(TM)) = S(TM). \tag{26}$$

In [13], Nakova studied the almost complex manifold with Norden metric with non-degenerate CR-submanifold (M, g) and degenerate submanifold $(M, \tilde{g})$ and provided the mutual relationship between the geometric objects of these two submanifolds.

Theorem 4. [13] Let $(\bar{M}, \bar{J}, \bar{g}, \bar{\tilde{g}})$ be a $2n$-dimensional almost complex manifold with Norden metric and M be an m −dimensional submanifold of $\bar{M}$. The submanifold (M, g) is a CR-submanifold with a r −dimensional totally real distribution D if and only if $(M, \tilde{g})$ is a r −lightlike radical transversal lightlike submanifold of $\bar{M}$.

As a consequence of above Theorem, we have

$$S(TM) = D, Rad(TM) = D^{\perp},\ S(TM)^{\perp} = (\bar{J}D^{\perp})^{\perp} \text{ and } ltr(TM) = \bar{J}D^{\perp}.$$

Since $\bar{\tilde{\nabla}}$ is the Levi-Civita connection with respect to the Norden metric $\tilde{g}$ on $\bar{M}$, therefore (21) can be written as

$$\bar{\tilde{\nabla}}_X Y = \bar{\tilde{\nabla}}_X Y + \varphi'(X,Y) + \varphi^1(X,Y) + \varphi^2(X,Y) \tag{27}$$

where $\varphi'(X,Y), \varphi^1(X,Y)$ and $\varphi^2(X,Y)$ denote the components of $\varphi(X,Y)$ belong to TM, $\bar{J}D^{\perp}$ and $(\bar{J}D^{\perp})^{\perp}$ respectively. Since for a Kaehler Norden manifold $\varphi(X,Y) = 0$, therefore by making use of equations (3) and (27), implies the following relations between the induced geometric objects of the submanifolds $(M, \tilde{g})$ and (M, g), respectively.

$$\bar{\tilde{\nabla}}_X Y = \bar{\nabla}_X Y,\quad \tilde{\nabla}_X Y = \nabla_X Y$$

$$\tilde{h}^l(X,Y) = h^1(X,Y),\ \tilde{h}^s(X,Y) = h^2(X,Y),$$

$$\tilde{A}_N X = A_N X,\ \tilde{\nabla}^l_X N = \nabla^1_X N,\ D^s(X,N) = D^2(X,N),$$

$$\tilde{\nabla}^s_X W = \nabla^2_X W,\ \tilde{A}_W X = A_W X,\ D^l(X,N) = D^1(X,N). \tag{28}$$

Now we prove the equivalent conditions for a CR-submanifold to be totally geodesic.

Theorem 5. Let (M, g) be a CR-submanifold and $(M, \tilde{g})$ be a radical transversal lightlike submanifold of a Kaehler Norden manifold $(\bar{M}, \bar{J}, \bar{g}, \bar{\tilde{g}})$. Then the following assertions are equivalent

(i) (M, g) is totally geodesic.

(ii) $(M, \tilde{g})$ is totally geodesic.
(iii) D^1 is a metric Ostuki connection on $TM^{\perp}$.
(iv) D^2 is a metric Ostuki connection on $TM^{\perp}$.
(v) ∇^t is a metric linear connection on $tr(TM)$.
(vi) D^s is a metric Ostuki connection on $tr(TM)$

Proof. By virtue of Theorems 3.10 and 3.11 in [6], it is clear that the assertions $(i), (iii)$ and (iv) are equivalent to the condition $D^2(X, N) = 0$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(D^{\perp})$. Moreover from [7], (pp. 159 and 166), it is clear that the assertions $(ii), (v)$ and (vi) are equivalent to $D^s(X, N) = 0$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$, and $\tilde{h}^l, \tilde{h}^s$ vanish identically on M. Also from (28) we have

$$D^s(X, N) = D^2(X, N) = 0,$$
$$\tilde{h}^l(X, Y) = h^1(X, Y) = 0, \tilde{h}^s(X, Y) = h^2(X, Y) = 0.$$

Also, the conditions $D^2(X, N) = 0$ and h^1, h^2 vanishes identically on M are equivalent by use of Theorem 3.10 from [6]. Thus, completes the proof.

In [6], we have proved the following result.

Theorem 6. If (M, g) is a totally umbilical CR-submanifold of a Kaehler Norden manifold $\bar{M}$ then the induced connection $\tilde{\nabla}$of radical transversal lightlike submanifold $(M, \tilde{g})$ is a metric connection.

III. MAIN RESULTS

Axioms of transversal r-planes: An indefinite Kaehler Norden manifold $\bar{M}$ having complex dimension $d > 1$ satisfies the axioms of transversal r −planes if for each $m \in \bar{M}$ and r −dimensional transversal subspace T of $T_m(\overline{M}) = T, 1 \leq r < d$, then there exists a totally geodesic radical transversal lightlike submanifold M satisfying $m \in \bar{M}$ and $T_m(\overline{M}) = T$.

Axioms of transversal r-spheres: An indefinite Kaehler Norden manifold $\bar{M}$ having complex dimension $d > 1$ satisfies the axioms of transversal $r -$ spheres if for each $m \in \bar{M}$ and r −dimensional transversal subspace T of $T_m(\overline{M}) = T, 1 \leq r < d$, then there exists a totally umbilical radical transversal lightlike submanifold M with parallel transversal curvature vector field satisfying $m \in \bar{M}$ and $T_m(\overline{M}) = T$.

Lemma III. 1. Let (M, g) be a totally umbilical CR-submanifold and $(M, \tilde{g})$ be a radical transversal lightlike submanifold of an indefinite Kaehler Norden manifold $\bar{M}$. Then the following conditions hold true.
(i) $\nabla_U^1 H^1 = 0$ if and only if $\nabla_U h^1 = 0$ and
(ii) $\nabla_U^2 H^2 = 0$ if and only if $\nabla_U h^2 = 0$.

Proof : Suppose (M, g) be a totally umbilical CR-submanifold of an indefinite Kaehler Norden manifold $\bar{M}$. For any tangent vector fields U, V, W and use of equations (15) and (24), yields

$$\nabla_U h^1(V, W) = \nabla_U^1 (g(V, W)H^1) - g(\nabla_U V, W)H^1 - g(V, \nabla_U W)H^1$$
$$= U\left(g(V, W)H^1\right) + \left(g(V, W)\nabla_U^1 H^1\right) - g(\nabla_U V, W)H^1 - g(V, \nabla_U W)H^1$$
$$= (\nabla_U g)(V, W)H^1 + (g(V, W)\nabla_U^1 H^1)$$
$$= (\tilde{\nabla}_U \tilde{g})(V, W)\tilde{H}^l + (\tilde{g}(V, \bar{J}W)\nabla_U^1 \tilde{H}^l) = (\tilde{g}(V, \bar{J}W)\tilde{\nabla}_U^1 \tilde{H}^l),$$

using the Theorem 6, we have $\tilde{\nabla}$ is a metric connection for a totally umbilical CR-submanifold and thus implies that

$$\nabla_U h^1(V, W) = \left(\tilde{g}(V, \bar{J}W)\tilde{\nabla}_U^1 \tilde{H}^l\right) = \left(g(V, W)\nabla_U^1 H^1\right) \tag{29}$$

Similarly,

$$\nabla_U h^2(V, W) = \left(g(V, W)\nabla_U^2 H^2\right) \tag{30}$$

Thus, the result follows from above equations (29) and (30).

Theorem III.2. [1] Let $\bar{M}$ be an indefinite Kaehler manifold with complex dimension ≥2. Then $\bar{M}$ is an indefinite complex space form if and only if $\bar{g}(\bar{R}(X, Y)\bar{J}X, X) = 0$, for every orthonormal vector $X, Y, \bar{J}X \in \Gamma(T\overline{M})$.

Theorem III.3. Let $(\overline{M}, \bar{J}, \bar{g}, \bar{\tilde{g}})$ be a $2n$ −dimensional Kaehler Norden manifold. Let $(M, \tilde{g})$ be a radical transversal lightlike submanifold and (M, g) be a totally umbilical CR-submanifold of $\overline{M}$. If $\overline{M}$ satisfies the axiom of transversal r-spheres for some fixed r, $1 \leq r < 2n$, then $\bar{M}$ has a constant holomorphic curvature.

Proof. For any arbitrary point $p \in \bar{M}$, let U, V and W be orthonormal vector fields such that

$$\bar{g}(U, \bar{J}V) = \bar{g}(U, \bar{J}W)\, \bar{g}(V, \bar{J}W) = 0.$$

Let T denote an r −dimensional transversal subspace of $T_p(\bar{M})$ containing U and V transversal to $\bar{J}U$. If $\bar{M}$ satisfies the axioms of transversal r −spheres then there exists an $2r$ −dimensional totally umbilical transversal CR-submanifold (M, g) with parallel transversal curvature vector field H and an induced metric connection $\tilde{\nabla}$ such that $T_p(\bar{M}) = T$. Since the transversal curvature vector field is parallel, that is, $\nabla_U^{\perp} H = 0$, using Theorem 3.10 of [6] and LemmaIII.1, we have $\nabla_U^{\perp} h^1 = 0$ and $\nabla_U^{\perp} h^2 = 0$. Since (M, g) is a totally umbilical CR-submanifold such that the distribution D defines a totally geodesic foliation in (M, g), therefore using Theorems 3.10 of [6] and Theorem **6** with (24), we have $H^1 = 0$. Hence we have

$$\nabla_U h^1 = 0 \text{ and } \nabla_U h^2 = 0 \text{ or, in particular,}$$
$$\nabla_U h^1(\bar{J}V, U) = 0, \nabla_{\bar{J}V} h^1(U, U) = 0,$$
$$\nabla_U h^2(\bar{J}V, U) = 0, \nabla_{\bar{J}V} h^2(U, U) = 0. \quad (31)$$

Using equation (13), the transversal form of $(\bar{R}(U, V)U)^N$ is given by

$$(\bar{R}(U,V)U)^N = (\nabla_U h^1)(V,U) - (\nabla_V h^1)(U,U) + D^1\left(U, h^2(V,U)\right) - D^1\left(V, h^2(U,U)\right)$$
$$+(\nabla_U h^2)(V,U) - (\nabla_V h^2)(U,U) + D^2\left(U, h^1(V,U)\right) - D^2\left(V, h^1(U,U)\right), \quad (32)$$

For any $U, V \in \Gamma(TM)$.

Since M is totally umbilical CR-submanifold, therefore using Theorem 3.10 of [6] and by use of equations (23) and (31) in (32) gives

$$(\bar{R}(U, V)U)N = 0.$$

Hence

$$\bar{g}(\bar{R}(U, V)U, \bar{J}U) = 0.$$

Thus the assertion follows from Theorem III.2.

Corollary III.4. Let $(\overline{M}, \bar{J}, \bar{g}, \bar{\tilde{g}})$ be a $2n$ −dimensional Kaehler Norden manifold. Let $(M, \tilde{g})$ be a radical transversal lightlike submanifold and (M, g) be a totally umbilical CR-submanifold of $\overline{M}$. If $\overline{M}$ satisfies the axiom of transversal r-planes for some fixed r, $1 \leq r < 2n$, then $\bar{M}$ has a constant holomorphic curvature.

III. CONCLUSIONS

Thus, we have concluded that in the setting of Kaehler manifold with Norden manifolds, the Axioms of spheres(planes) hold true.

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