Hesitant Fuzzy Soft Group

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Abstract: Maji et. al. [2] firstly presented the concept of fuzzy soft set by combining the theories of fuzzy set and soft set together. Rosenfeld [3] proposed the concept of fuzzy groups in order to establish the algebraic structures of fuzzy sets.Aktas and Çagman [6] introduced the notion of soft groups, which extends the notion of group to include the algebraic structures of soft sets. Recently, in order to tackle the difficulty in establishing the degree of membership of an element in a set, Torra and Narukawa [7] and Torra [8] proposed the concept of a hesitant fuzzy set.**The purpose of this paper** is to extend the soft groups to the hesitant fuzzy set, and, thus, we establish a new soft group model named hesitant fuzzy soft group. The rest of this paper is organized as follows. We first review some background on soft set, fuzzy soft set, and hesitant fuzzy set, hesitant fuzzy soft set in Section 2. In Section 3, the concepts and operations of hesitant fuzzy soft groups are proposed and their properties are discussed in detail.In Section 4, we define the image and pre-image of a hesitant soft set under a soft function, and then define soft homomorphism between hesitant fuzzy soft groups. We prove the image and pre-image of a hesitant fuzzy soft group are also hesitant fuzzy soft groups. In Section 5, we give the definition of a normal hesitant fuzzy soft group and study some of their properties.

Section 1 - Introduction: In the real world, there are many complicated problems in economics, engineering, environment, social science, and management science. Molodtsov [1] pointed out all these theories had their own limitations. Moreover, in order to overcome these difficulties, Molodtsov [1] firstly proposed a new mathematical tool named soft set theory to deal with uncertainty and imprecision. The soft set model can be combined with other mathematical models.

Section 2 – Definitions and Preliminaries

Definition 2.1(Soft set): Suppose that U is an initial universal set. E is set of parametersP (U) is the power set of U and A⊆E.

Definition 2.2: A pair (f,A) is a soft set over U,where f is a mapgiven by F:A→ P (U).

Example 2.3: Suppose that U={h_1, h_2, h_3, h_4, h_5 ,h_6} is a set of houses and A={e_1, e_2, e_3, e_4, e_5} is a set of parameters ,which stands for the parameters "cheap," "beautiful," "size," "location," and "surrounding environment," respectively .In this case, a soft set (f, A) can be defined as a mapping from parameter set A to the set of all subsets of U.Assume that F(e_1)={ h_2, h_4},F(e_2)={h_1, h_3}, F(e_3)={ h_3, h_4, h_5},F(e_4)={ h_1, h_3, h_5} and F(e_5)={h_2}.Then we can view the soft set(f, A) as consisting of the following collection of approximations:
(F, A)={(e_1,{ h_2, h_4}) , (e_2,{h_1, h_3}), (e_3,{ h_3, h_4, h_5}), (e_4, { h_1, h_3, h_5}), (e_5, {h_2})}.

Fuzzy soft set:

Definition 2.4: Let P' (U) be the set of all fuzzy subsets of U, a pair (F$U, a\ pair\ (\tilde{F}, A)$ is called a fuzzy soft set over U , where $\tilde{F} : A \rightarrow \tilde{P}(U).$

Example 2.5: Reconsider example (2.2.).Then, fuzzy soft sets $(\tilde{F}, A)$ can describe the characteristics of the house under the fuzzy information.

$$\tilde{F}(e_1)=\left\{\frac{h_1}{0.2},\frac{h_2}{0.5},\frac{h_3}{0.3},\frac{h_4}{0.3},\frac{h_5}{0.4},\frac{h_6}{0.6}\right\},\ \tilde{F}(e_2)=\left\{\frac{h_1}{0.6},\frac{h_2}{0.5},\frac{h_3}{0.6},\frac{h_4}{0.7},\frac{h_5}{0.4},\frac{h_6}{0.3}\right\},$$
$$\tilde{F}(e_3)=\left\{\frac{h_1}{0.4},\frac{h_2}{0.6},\frac{h_3}{0.8},\frac{h_4}{0.3},\frac{h_5}{0.4},\frac{h_6}{0.7}\right\},\ \tilde{F}(e_4)=\left\{\frac{h_1}{0.3},\frac{h_2}{0.2},\frac{h_3}{0.5},\frac{h_4}{0.7},\frac{h_5}{0.5},\frac{h_6}{0.8}\right\},$$

Hesitant fuzzy set:

Definition 2.6: Ahesitant fuzzy set (HFS) on U is in terms of a function that when applied to U returns a subset of [0,1],which can be represented as the following mathematical symbol: $\tilde{A} = \{\langle u, h_{\tilde{A}}(u)\rangle / u \in U\},$

where $h_{\tilde{A}}(u)$ is a set of values in [0,1]denoting the possible membership degrees of the element $u \in U$ to the set $\tilde{A}$.

Definition 2.7: Given hesitant fuzzy set $\tilde{A}$.,if $h(u) = \{0\}$ *for all* u *in* U ,then $\tilde{A}$. is called the null hesitant fuzzy set denoted by Φ. *If* $h(u) = \{1\}$ *for all* u *in* U, then $\tilde{A}$. is called the full hesitant fuzzy set ,denoted by $\tilde{1}$.

Definition 2.8: Given a hesitant fuzzy set represented by its membership function h we define its complement as $h^c(x) = \bigcup_{\gamma \in h(x)} \{1-\gamma\}$.

Definition 2.9: Giventwo hesitant fuzzy sets represented by their membership functions h_1 *and* h_2 , we define their union represented by $h_1 \bigcup h_2$ as

$$h_1 \bigcup h_2(x) = \{h \in (h_1(x) \bigcup h_2(x) / h \geq \max(h_1^-, h_2^-)\}.$$

Definition 2.10: Giventwo hesitant fuzzy sets represented by their membership functions t

$$h_1 \bigcap h_2(x) = \{h \in (h_1(x) \bigcap h_2(x) / h \leq \min(h_1^+, h_2^+)\}.$$

Section 3 - Hesitant fuzzy soft set:

Definition 3.1: Let $\tilde{H}(U)$ be the set of all hesitant fuzzy sets in U ,a pair $(\tilde{F}, A)$ is called a HFSS over U, where $\tilde{F}$ is defined by $\tilde{F}: A \to \tilde{H}(U)$. A hesitant fuzzy soft set is a mapping from parameters to $\tilde{H}(U)$. It is a parameterised family of hesitant fuzzy subsets of U .

Example 3.2: Continue to consider example (2.5). Mr.X evaluates the optional six houses under various attributes with hesitant fuzzy element;then, hesitant soft set $(\tilde{F}, A)$ can describe the characteristics of the house under the hesitant fuzzy information.

Table 1: The tabular representation of the hesitant fuzzy soft set $(\tilde{F}, A)$

U	**"cheap"**	**"Beautiful"**	**"size"**	**"Location"**	**"Green Surrounding Environment"**
h_1	{0.2,0.3}	{0.4,0.6,0.7}	{0.2,0.4}	{0.3,0.5,0.6}	{0.6}
h_2	{0.5,0.6}	{0.5,0.7,0.8}	{0.6,0.7}	{0.2}	{0.2,0.3,0.5}
h_3	{0.3}	{0.6,0.8}	{0.8,0.9}	{0.5}	{0.5,0.7}
h_4	{0.3,0.5}	{0.7,0.9}	{0.3,0.5}	{0.6,0.7}	{0.2,0.4}
h_5	{0.4,0.5}	{0.3,0.4,0.5}	{0.4,0.6}	{0.5,0.6}	{0.5,0.7}
h_6	{0.6,0.7}	{0.3}	{0.7}	{0.8}	{0.3,0.5}

Definition 3.3: Let $A, B \in E$. $(\tilde{F}, A)$ *and* $(\tilde{G}, B)$ are two hesitant fuzzy soft sets over U . $(\tilde{F}, A)$ is said to be a hesitant fuzzy soft subset of $(\tilde{G}, B)$ *if*

(1) $A \subseteq B$

(2) *For all* $e \in A, \tilde{F}(e) \tilde{\subseteq} \tilde{G}(e)$.

In this case,we write $(\tilde{F}, A) \tilde{\subseteq} (\tilde{G}, B)$.

Definition 3.4: The complement of a hesitant fuzzy soft set $(\tilde{F}, A)$ is denoted by $(\tilde{F}, A)^c$ and is defined by $(\tilde{F}, A)^c = (\tilde{F}^c, A)$ where $\tilde{F}^c : A \to \tilde{H}(U)$ is a mapping given by $\tilde{F}^c(e) = (\tilde{F}(e))^c$ for all $e \in A$.

Example 3.5: Continue to consider example (3.3). Mr.X evaluates the optional six houses under various attributes with hesitant fuzzy element; then, the complement of hesitant soft set $(\tilde{F}^c, A)$ can describe the characteristics of the house under the hesitant fuzzy information.

Table 2: The tabular representation of the complement of hesitant fuzzy soft set $(\tilde{F}^c, A)$

U	**"cheap"**	**"Beautiful"**	**"size"**	**"Location"**	**"Green Surrounding Environment"**
h_1	{0.7,0.8}	{0.3,0.4,0.6}	{0.6,0.8}	{0.4,0.5,0.7}	{0.4}
h_2	{0.4,0.5}	{0.2,0.3,0.5}	{0.3,0.4}	{0.8}	{0.5,0.7,0.8 }
h_3	{0.7}	{0.2,0.4}	{0.1,0.2}	{0.5}	{0.3,0.5}
h_4	{0.5,0.7}	{0.1,0.3}	{0.5,0.7}	{0.3,0.4}	{0.6,0.8}
h_5	{0.5,0.6}	{0.5,0.6,0.7}	{0.6,0.4}	{0.4,0.5}	{0.3,0.5}
h_6	{0.3,0.4}	{0.7}	{0.3}	{0.2}	{0.5,0.7}

Definition 3.6: The AND operation on two hesitant fuzzy soft sets $(\tilde{F}, A)$ *and* $(\tilde{G}, B)$ which is denoted by $(\tilde{F}, A) \wedge (\tilde{G}, B)$ *is defined by*

$$(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{J}, A \times B), \text{ where } \tilde{J}(\alpha,\beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta), \text{ for all } (\alpha,\beta) \in A \times B.$$

Definition 3.7: The OR operation on two hesitant fuzzy soft sets $(\tilde{F}, A)$ *and* $(\tilde{G}, B)$ which is denoted by $(\tilde{F}, A) \vee (\tilde{G}, B)$ *is defined by*

$$(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{O}, A \times B), \text{ where } O(\alpha,\beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta), \text{ for all } (\alpha,\beta) \in A \times B.$$

Definition 3.8: Union of two hesitant fuzzy Soft sets $(\tilde{F}, A)$ *and* $(\tilde{G}, B)$ over U, is the hesitant soft set $(\tilde{J}, C)$, where $C = A \cup B,$ *and for all* $e \in C,$

$$\begin{aligned} \tilde{J}(e) &= \tilde{F}(e), \text{ if } e \in A - B, \\ &= \tilde{G}(e), \text{ if } e \in B - A, \\ &= \tilde{F}(e) \cup \tilde{G}(e), \text{ if } e \in A \cap B. \end{aligned}$$

We write $(\tilde{F}, A) \tilde{\cup} (\tilde{G}, B) = (\tilde{J}, C).$

Definition 3.9: Intersection of two hesitant fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ with $A \cap B \neq \Phi$ *over* U is the hesitant fuzzy soft set $(\tilde{J}, C),$ *where* $C = A \cap B,$ *and for all* $e \in C, \tilde{J}(e) = \tilde{F}(e) \cap \tilde{G}(e).$

We write $(\tilde{F}, A) \tilde{\cap} (\tilde{G}, B) = (\tilde{J}, C).$

Definition 3.10: Let $(\tilde{F}, A)$ be a hesitant soft set over U .Then soft set $(\tilde{F}, A)_\alpha = \{(\tilde{F}_a)_\alpha : a \in A\}$,for each $\alpha \in (0,1]$, is called the $\alpha-$*level set* of the hesitant soft set $(\tilde{F}, A)$

Definition 3.11: The Cartesian product of hesitant soft sets $(\tilde{F}_1, A), (\tilde{F}_2, A), (\tilde{F}_3, A), \ldots, (\tilde{F}_n, A)$ be n hesitant soft sets defined by $\Re : [0,1]^n \to [0,1]$ given by

$$\Re = \bigcup_{\gamma \in \tilde{F}_1(e) \times \tilde{F}_1(e) \times \tilde{F}_1(e) \ldots \times \tilde{F}_1(e)} \{Median(\gamma)\}, \; e \in A.$$

Example 3.12: Let $\tilde{F}_1(e) = \{0.5, 0.4, 0.7\}$, $\tilde{F}_2(e) = \{0.9, 0.6\}$, $\tilde{F}_3(e) = \{0.2, 0.4\}$ be the hesitant soft sets. Then the Cartesian product is given by

$$\begin{aligned}\Re &= \bigcup_{\gamma \in \tilde{F}_1(e)\times\tilde{F}_2(e)\times\tilde{F}_3(e)} \{Median(\gamma)\}\ ,\ e \in A.\\ &= \{Median(0.5,0.9,0.2) \cup Median(0.5,0.9,0.4) \cup Median(0.5,0.6,0.2) \cup\\ &\quad Median(0.5,0.6,0.4) \cup Median(0.4,0.9,0.2) \cup Median(0.4,0.9,0.4) \cup\\ &\quad Median(0.4,0.6,0.2) \cup Median(0.4,0.6,0.4) \cup Median(0.7,0.9,0.2) \cup\\ &\quad Median(0.7,0.9,0.4) \cup Median(0.7,0.6,0.2) \cup Median(0.7,0.6,0.4)\}\\ &= \{0.5, 0.5,\ 0.5, 0.5, 0.4, 0.4, 0.4, 0.4, 07, 0.7, 0.6, 0.6\}.\end{aligned}$$

Section 4-Hesitant fuzzy soft group:

In this section, we introduce the concept of hesitant fuzzy soft groups and investigate some related properties. Throughout the paper G will denote a classical group.

Definition 4.1: Let G be a group and $(\tilde{F}, A)$ be a soft set over G. Then $(\tilde{F}, A)$ is said to be a soft group over $X\ iff\ \ F(a) < G,\ for\ each\ a \in A.$

Definition 4.2: Let G be a group and $(\tilde{F}, A)$ be a hesitant fuzzy soft set over G .Then $(\tilde{F}, A) = \{\tilde{F}(e)\ ,\ for\ all\ e \in A\}$ is said to be a hesitant fuzzy soft groupover $G, where\ \tilde{F}(e)\ is\ a\ mapping\ \ from\ G\ to\ subsets\ of\ [0,1].$ If is satisfies the

(1) $\tilde{f}_a(x.y) \geq \min(\tilde{f}_a(x), \tilde{f}_a(y))$

(2) $\tilde{f}_a(x^{-1}) \geq \tilde{f}_a(x).\qquad for\ all\ \ x, y \in G\ and\ a \in A.$

$Here\ \tilde{f}_a\ is\ a\ \ fuzzy\, subgroup\ in\ Rosenfeld's\ sense[3].$

Theorem 4.3: Let $(\tilde{F}, A)$ be a hesitant fuzzy soft set.Then $(\tilde{F}, A)$ is a hesitant fuzzy soft group $iff\ \ a \in A\ and\ \ x, y \in G, \tilde{f}_a(x.y^{-1}) \geq \min(\tilde{f}_a(x), \tilde{f}_a(y)).$

Proof: For each $a \in A\ and\ \ x, y \in G,$

$$\tilde{f}_a(x.y^{-1}) \geq \min(\tilde{f}_a(x), \tilde{f}_a(y))\ \ by\ (2).$$

Conversely, first we have

$$\begin{aligned}\tilde{f}_a(e) &= \tilde{f}_a(x.x^{-1})\\ &\geq \min(\tilde{f}_a(x), \tilde{f}_a(x^{-1}))\\ &\geq \min(\tilde{f}_a(x), \tilde{f}_a(x))\\ &\geq \tilde{f}_a(x) \qquad for\ each\ \ x \in G\ and\ e\ is\ the\ unit\ element\ of\ G.\end{aligned}$$

Further more,

$$\begin{aligned}\tilde{f}_a(x^{-1}) &= \tilde{f}_a(e.x^{-1})\\ &\geq \min(\tilde{f}_a(e), \tilde{f}_a(x))\\ &\geq \min(\tilde{f}_a(x), \tilde{f}_a(x))\\ &= \tilde{f}_a(x).\end{aligned}$$

On the otherhand,

$$for\ each\ a \in A\ and\ x, y\ G,$$
$$\tilde{f}_a(xy) = \tilde{f}_a(x.(y^{-1})^{-1}) \geq \min(\tilde{f}_a(x^{-1}), \tilde{f}_a(y^{-1})) \geq \min(\tilde{f}_a(x), \tilde{f}_a(y)).$$

This completes the proof.

Theorem 4.4: Let $(\tilde{f}, A)\ and\ (\tilde{g}, B)$ be the two hesitant fuzzy soft groups over G.Then their intersection $(\tilde{f}, A) \bigcap (\tilde{g}, B)$ is a hesitant fuzzy soft subgroup over G.

Proof: Let

$$(\tilde{f}, A) \bigcap (\tilde{g}, B) = (\tilde{h}, C), where\ C = A \bigcap B\ and\ \tilde{h}_c(x) = \tilde{f}_c(x) \wedge \tilde{g}_c(x)\ \ c \in C\ and\ x \in G.$$
$$(\tilde{f}_c \cap \tilde{g}_c)(x.y) = \tilde{f}_c(x.y) \cap \tilde{g}_c(x.y) \geq \min(\min(\tilde{f}_c(x), \tilde{f}_c(y)), \min(\tilde{g}_c(x), \tilde{g}_c(y)))$$
$$\geq \min(\min(\tilde{f}_c(x), \tilde{g}_c(x)), \min(\tilde{f}_c(y), \tilde{g}_c(y)))$$
$$\geq \min((\tilde{f}_c \cap \tilde{g}_c)(x), (\tilde{f}_c \cap \tilde{g}_c)(y))$$

$$and\ also\ (\tilde{f}_c \cap \tilde{g}_c)(x^{-1}) \geq (\tilde{f}_c \cap \tilde{g}_c)(x).$$

Theorem 4.5: Let $(\tilde{f}, A)\ and\ (\tilde{g}, B)$ be the two hesitant fuzzy soft groups over G. $If\ \ A \cap B = \phi, then\ (\tilde{f}, A) \cup (\tilde{g}, B)\ is$ a hesitant fuzzy soft subgroup over G.

Proof: Let $(\tilde{f}, A) \cup (\tilde{g}, B) = (\tilde{h}, C)$.

Since $A \cap B = \phi$,if follows that either $c \in A - B\ or\ c \in B - A\ \ for\ all\, c \in C.$

If $c \in A - B, then\ \tilde{h}_c = \tilde{f}_c$ is a hesitant fuzzy subgroup of G and

If $c \in B - A, then\ \tilde{h}_c = \tilde{g}_c$ is a hesitant fuzzy subgroup of G.Thus, $(\tilde{f}, A) \cup (\tilde{g}, B)\ is$ a hesitant fuzzy soft subgroup over G.

Cartesian Product of Hesitantfuzzy soft groups

Definition 4.6: Let $(\tilde{f}, A)\ and\ (\tilde{g}, B)$ be the two hesitant fuzzy soft groups over G. The Cartesian product of $(\tilde{f}, A)\ and\ (\tilde{g}, B)$ is given by

$$\Re = \bigcup_{e_i \in A \times B} H_{\tilde{f}}(e_i) \times H_{\tilde{g}}(e_i)$$
$$= \bigcup_{e_i \in A \times B} \{Median(H_{\tilde{f}}(e_i)^{(j)}, H_{\tilde{g}}(e_i)^{(k)}\ )\}$$

$$Where\ \ i = \{1,2,...,n\}\ , j = \{1,2,....,l_{\tilde{f}}(e_i)\}\ \ , k = \{1,2,.....,l_{\tilde{g}}(e_i)\}$$

Hence each paired hesitant fuzzy soft element comprises all pairs of elements such that the first element of the pair is from $H_{\tilde{f}}(e_i)$ and the second element of the pair is from $H_{\tilde{g}}(e_i)$ .The total number of pair of values in $\Re\ is\ |\Re| = \sum_{i=1}^{n}(l_{\tilde{f}}(e_i) \times l_{\tilde{g}}(e_i))$ where $l_{\tilde{f}}(e_i)$ and $l_{\tilde{g}}(e_i)$ is the length of $(\tilde{f}, A)\ and\ (\tilde{g}, B)$.

Theorem 4.7: The direct product of hesitant fuzzy soft groups is a hesitant fuzzy soft group.

Proof: Let $\tilde{f} : [0,1]^n \rightarrow [0,1]\ , and\ H_i$ be hesitant soft groups of group G_i ,i=1, 2,3,...,n.Then ,the extension of $\tilde{f}\ \ on\ \ H_i$ is defined by

$$\tilde{f}_a(x_i) = (H_1 \times H_2 \times H_3 \times \dots \times H_n)(x_1, x_2, x_3, \dots, x_n) = \bigcup_{x_i \in H_i} \{\text{Median of } (H_1(x_1), H_2(x_2) \dots H_n(x_n)\}$$

Let $x_i = (x_1, x_2, x_3, \dots, x_n), y_i = (y_1, y_2, y_3, \dots, y_n)$

$$\{(x_1, x_2, x_3, \dots, x_n), (y_1, y_2, y_3, \dots, y_n)\} \in H_1 \times H_2 \times H_3 \times \dots \times H_n$$

HFSG 1:

$$\begin{aligned}
\tilde{f}_a(x_i y_i) &= \tilde{f}[(x_1, x_2, x_3, \dots, x_n), (y_1, y_2, y_3, \dots, y_n)] \\
&= (H_1 \times H_2 \times \dots \times H_n)(x_1 y_1, x_2 y_2, \dots, x_n y_n) \\
&= \bigcup_{x_i y_i \in H_i} \{\text{Median}(H_1(x_1 y_1), H_2(x_2 y_2) \dots H_n(x_n y_n))\} \\
&\geq \min[\bigcup_{x_i y_i \in H_i} \{\text{Median}((H_1(x_1), H_1(y_1)), (H_2(x_2), H_2(y_2)), \dots, (H_n(x_n), H_n(y_n))\}] \\
&\geq \min[\bigcup_{x_i y_i \in H_i} \{\text{Median}(H_1(x_1), H_1(y_1), H_2(x_2), H_2(y_2), \dots, H_n(x_n), H_n(y_n)\}] \\
&\geq \min[\bigcup_{x_i y_i \in H_i} \{\text{Median}((H_1(x_1),), H_2(x_2), \dots, H_n(x_n)), (H_1(y_1), H_2(y_2). , \dots, H_n(y_n))\}] \\
&\geq \min[\bigcup_{x_i \in H_i} \{\text{Median}((H_1(x_1),), H_2(x_2), \dots, H_n(x_n)), \bigcup_{y_i \in H_i} \{\text{Median of}(H_1(y_1), H_2(y_2)). , \dots, H_n(y_n))\}] \\
&\geq \min[(H_1 \times H_2 \times \dots \times H_n)(x_1, x_2, x_3, \dots, x_n), (H_1 \times H_2 \times \dots \times H_n)(y_1, y_2, y_3, \dots, y_n)]
\end{aligned}$$

$$\tilde{f}_a(x_i y_i) \geq \min[\tilde{f}_a(x_i), \tilde{f}_a(y_i)].$$

HFSG 2 :

$$\begin{aligned}
\tilde{f}_a(x^{-1}) &= (H_1 \times H_2 \times H_3 \times \dots \times H_n)(x_1^{-1}, x_2^{-1}, x_3^{-1}, \dots, x_n^{-1}) \\
&= \bigcup_{x_i \in H_i} \{\text{Median}(H_1(x_1^{-1}), H_2(x_2^{-1}), \dots, H_n(x_n^{-1})\} \\
&= \bigcup_{x_i y_i \in H_i} \{\text{Median}(H_1(x_1), H_2(x_2), \dots, H_n(x_n))\} \\
&= (H_1 \times H_2 \times \dots \times H_n)(x_1, x_2, x_3, \dots, x_n) \\
&= \tilde{f}_a(x).
\end{aligned}$$

Homomorphism of Hesitant fuzzy soft groups:

Inthis section,we first define hesitant fuzzy soft function,and then define the image and pre-image of a hesitant fuzzy soft set under hesitant fuzzy soft function. Furthermore we define hesitant fuzzy soft homomorphism and show that the homomorphic image and pre-image of a hesitant fuzzy soft group are also hesitant fuzzy soft group.

Definition 4.8: Let $(\tilde{f}, A)$ *and* $(\tilde{g}, B)$ be two hesitant fuzzy soft sets of G *and* G^1 over subsets of [0, 1], respectively and let (φ, ψ) be the hesitant fuzzy soft function from G *to* G^1.

(1) The image of $(\tilde{f}, A)$ under the soft function (φ, ψ),denoted by (φ, ψ) $(\tilde{f}, A)$,is a hesitant fuzzy soft set over G^1 defined by $(\phi, \psi)(\tilde{f}, A) = (\phi(\tilde{f}), \psi(A))$ where

$$\phi(\tilde{f})_k(y) = \left\{ \bigvee_{\phi(x)=y} \bigvee_{\psi(a)=k} \tilde{f}_a(x) \quad if \quad x \in \phi^{-1}(y) \right. \quad , \forall k \in \psi(A), \forall y \in G^1.$$

(2) The pre-image of $(\tilde{g}, B)$ under the hesitant soft function (φ, ψ), denoted by

$(\varphi,\psi)\,(\tilde{g},B)$, is a hesitant fuzzy soft set over B defined by $(\phi,\psi)^{-1}(\tilde{g},B)=(\phi^{-1}(\tilde{g}),\psi^{-1}(A)),\ where\ \phi^{-1}(\tilde{g})_a(x)=\tilde{g}_{\psi(a)}(\phi(x)),\ \ a\in\psi^{-1}(A),\ \forall\ x\in G.$

Definition 4.9: Let (φ,ψ) be the hesitant fuzzy soft function from $G\ to\ G^1$. Then the pair (φ,ψ) is said to be hesitant fuzzy soft homomorphism if

(1) ϕ is a isomorphism from $G\ to\ G^1$.

(2) ψ is a one-to-one mapping from A onto B.

Then (φ,ψ) is said to be hesitant fuzzy soft isomorphism.

Theorem 4.10: Let $(\tilde{f},A)$ be a hesitant fuzzy soft group over G, (φ,ψ) be a hesitant fuzzy soft homomorphism from $G\ to\ G^1$. then $(\varphi,\psi)\,(\tilde{f},A)$ is a fuzzy soft group over G^1.

Proof. Let $k\in\psi(A)\ and\ y_1,y_2\in G^1$.

$If\ \ \varphi^{-1}(y_1)=\phi\ or\ \varphi^{-1}(y_2)=\phi$ the proof is straight forward.

Let assume that there exist $x_1,x_2\in G\ such\ that\ \varphi(x_1)=y_1\ and\ \varphi(x_2)=y_2$.

$$\varphi(\tilde{f})_k(y_1.y_2^{-1})=\underset{\varphi(t)=y_1.y_2^{-1}}{\vee}\ \underset{\psi(a)=k}{\vee}\tilde{f}_a(t)$$

$$\geq\underset{\psi(a)=k}{\vee}\tilde{f}_a(x_1.x_2^{-1})$$

$$\geq\underset{\psi(a)=k}{\vee}\min(\tilde{f}_a(x_1),\tilde{f}_a(x_2))$$

$$\geq\min(\underset{\psi(a)=k}{\vee}\tilde{f}_a(x_1),\underset{\psi(a)=k}{\vee}\tilde{f}_a(x_2))$$ $.This\ inequality\ is\ satisfied\ \ for\ each\ x_1,x_2\in G$

$which\ satisfy\ \varphi(x_1)=y_1\ and\ \varphi(x_2)=y_2.$

$$\varphi(\tilde{f})_k(y_1.y_2^{-1})\geq\min(\underset{\psi(t_1)=y_1}{\vee}\ \underset{\psi(a)=k}{\vee}\tilde{f}_a(t_1),\underset{\psi(t_2)=y_2}{\vee}\ \underset{\psi(a)=k}{\vee}\tilde{f}_a(t_2))$$

$$=T(\varphi(\tilde{f})_k(y_1),\varphi(\tilde{f})_k(y_2).$$

$This\ completes\ the\ proof.$

Theorem 4.11: Let $(\tilde{g},B)$ be a fuzzy soft group over G^1 and (φ,ψ) be a fuzzy soft homomorphism from $G\ to\ G^1$. Then $(\phi,\psi)^{-1}(\tilde{g},B)$ is a fuzzy soft group over G.

Proof:

$Let\ a\in\psi^{-1}(B)\ and\ x_1,x_2\in G.$

$$\varphi^{-1}(\tilde{g})_a(x_1.x_2^{-1})=\tilde{g}_{\psi(a)}(\varphi(x_1.x_2^{-1}))$$

$$=\tilde{g}_{\psi(a)}(\varphi(x_1).\varphi(x_2)^{-1})$$

$$\geq\min((\tilde{g}_{\psi(a)}(\varphi(x_1)),(\tilde{g}_{\psi(a)}(\varphi(x_2)))$$

$$=\min(\varphi^{-1}(\tilde{g})_a(x_1)\ ,\varphi^{-1}(\tilde{g})_a(x_2))$$

$This\ completes\ the\ proof.$

Section 5 - Normal hesitant fuzzy soft groups

In this section ,we define the normal hesitant fuzzy soft groups and study some of their properties.Further more we show that the homomorphic image and pre-image of a normal hesitant fuzzy soft group are also normal hesitant fuzzy soft group.

Definition 5.1: Let $(\tilde{f}, A)$ be a hesitant fuzzy soft group over G. Then $(\tilde{f}, A)$

Is said to be normal hesitant fuzzy soft group if it satisfies

(1) $\tilde{f}(y.x.y^{-1}) = \tilde{f}(x), \ \forall x, y \in G.$

(2) $\tilde{f}(x.y) \geq \tilde{f}(y.x), \ \forall x, y \in G.$

Theorem 5.2: Let $(\tilde{f}, A)$ *and* $(\tilde{g}, B)$ be two hesitant fuzzy soft sets of G over subsets of [0, 1] and $A \cap B \neq \phi$. Then their intersection $(\tilde{f}, A) \cap (\tilde{g}, B)$ is a normal hesitant fuzzy soft subgroup over G.

Proof: Let $(\tilde{f}, A)$ *and* $(\tilde{g}, B)$ be two hesitant fuzzy soft sets over G .

Now for all $x, y \ in \ G,$

$$(\tilde{f}, A) \cap (\tilde{g}, B)(y.x.y^{-1}) = \min(\tilde{f}(y.x.y^{-1}), \tilde{g}(y.x.y^{-1})) \qquad \text{[According to theorem 3.2]}$$
$$= \min(\tilde{f}(x), \tilde{g}(x))$$
$$= (\tilde{f} \cap \tilde{g})(x).$$

This completes the proof.

Theorem 5.3: Let $(\tilde{f}, A)$ be a normal fuzzy soft group over $G, (\varphi, \psi)$ be a surjective fuzzy soft homomorphism from $G \ to \ G^1$. Then $(\varphi, \psi)(\tilde{f}, A)$ is a normal fuzzy soft group over G^1.

Proof: Let assume that there exist $x_1, x_2 \in G$ *such that* $\varphi(x_1) = y_1$ *and* $\varphi(x_2) = y_2$.

$$\varphi(\tilde{f})_k(y_1.y_2.y_1^{-1}) = \bigvee_{\varphi(t)=y_1.y_2.y_1^{-1}} \ \bigvee_{\psi(a)=k} \tilde{f}_a(t)$$
$$\geq \bigvee_{\psi(a)=k} \tilde{f}_a(x_1.x_2.x_1^{-1})$$
$$\geq \bigvee_{\psi(a)=k} \tilde{f}_a(x_1^{-1}.x_1.x_2)$$
$$\geq \bigvee_{\psi(a)=k} \tilde{f}_a(x_2)$$

.This inequality is satisfied for each $x_1, x_2 \in G$ *which satisfy* $\varphi(x_1) = y_1$ *and* $\varphi(x_2) = y_2$.

Then we have

$$\varphi(\tilde{f})_k(y_1.y_2.y_1^{-1}) \geq \bigvee_{\psi(t)=y_2} \ \bigvee_{\psi(a)=k} \tilde{f}_a(t))$$
$$= \varphi(\tilde{f})_k(y_2).$$

This completes the proof.

Theorem 5.4: Let $(\tilde{g}, B)$ be a normal fuzzy soft group over G^1 and (φ, ψ) be a fuzzy soft homomorphism from $G \ to \ G^1$. Then $(\phi, \psi)^{-1}(\tilde{g}, B)$ is a normal fuzzy soft group over G.

Proof:

$Let\ a \in \psi^{-1}(B)\ and\ x_1, x_2 \in G.$

$$\begin{aligned}
\varphi^{-1}(\tilde{g})_a(x_1.x_2) &= \tilde{g}_{\psi(a)}(\varphi(x_1.x_2)) \\
&= \tilde{g}_{\psi(a)}(\varphi(x_1).\varphi(x_2)) \\
&= \tilde{g}_{\psi(a)}\ (\varphi(x_2).\varphi(x_1)) \\
&= \tilde{g}_{\psi(a)}\ (\varphi(x_2.x_1) \\
&= \varphi^{-1}(\tilde{g})_a(x_2\ .\ x_1)
\end{aligned}$$

$This\ completes\ the\ proof.$

Hesitant fuzzy soft set in decision making problems

In this section,we have investigated the application of hesitant soft set in group decision making problems.Let U be the universal set consisting of set of alternatives.We can represent a group decision making problem using hesitant fuzzy soft approach using the following way.

Definition 5.6: For a hesitant fuzzy element $h(x)$, $s(h) = \bigcup_{\gamma \in h(x)} Median\{\gamma\}$ is called the score function of $h(x)$.

Definition 5.7: let (F, A) denotes hesitant soft set.Then the fuzzy soft set (F_s, A) in which each entries in the fuzzy element $F_s(e)$ is the score function of the respective entries in the hesitant fuzzy set $F(e)$ is called the score matrix.

Definition 5.8: The table obtained by calculating the median of $F_s(e)$ for each x_j is called the decision table. This table determines the optimal outcome for the decision making problem.Now we will propose an algorithm which is shown by considering score matrix, a hesitant fuzzy soft based decision making problem can be reduced into simpler fuzzy soft set.

Algorithm 5.9:

Step 1. Input the hesitant fuzzy soft set (F, A).

Step 2. Obtain the score matrix (F_s, A) corresponds to (F, A) by

Sub step 1. Arrange the hesitant fuzzy element $h_i(x), i = 1,2,...,n.$ is in ascending order.

Sub step 2.

$$s(h) = \bigcup_{\gamma \in h(x)} Median\{\gamma\}$$

$$(i)\ \ If\ \ n\ is\ odd, Median(\gamma) = value\ at\ (\frac{n+1}{2})^{th}\ \ position.$$

$$(ii)\ If\ \ n\ is\ even, Median(\gamma) = value\ at\ (\frac{n}{2})^{th}\ \ position\ and\ (\frac{n+1}{2})^{th}\ \ position.$$

Step 3. Calculate the median of $F_s(e_i)$ for each x_j and it be denoted a_j.

Step 4. Select the optimal alternative $u_k\ \ if\ \ a_k = \max_j\ a_j$.

Step 5. If k has more than one value then any one of u_k may be chosen.

Example 5.10: Consider a retailer planning to open a new store in the city. There are five sites, $x_j(j = 1,2,3,4,5)$ to be selected. Five attributes are considered:market (e_1), traffic (e_2), rentprice (e_3), competition (e_4).Suppose that the retailer evaluates the optional five sites under various attributes with hesitant fuzzy element; then, hesitant fuzzy soft set (F, A) can describe the characteristics of the sites under hesitant fuzzy information shown in table 1.

	x_1	x_2	x_3	x_4	x_5
e_1	{0.8,0.5,0.77,0.87,0.2}	{0.2,0.3,0.45,0.5}	{0.3,0.95}	{0.4,0.1,0.5, 0.64}	{0.5,0.7,0.65}
e_2	{0.7,0.8,0.58}	{0.6,0.7,0.65}	{0.8,0.7,0.65}	{0.1,0.4,0.8 0.43}	{0.6,0.45,0.5,0.78}
e_3	{0.7,0.82,0.65,0.7}	{0.1,0.4,0.56,0.35}	{0.3,0.21}	{0.45,0.22, 0.6,0.9,0.6}	{0.6,0.88,0.67,0.11,0.56}
e_4	{0.8,0.82,0.88,0.9,0.3}	{0.74,0.68,0.52}	{0.5,0.67,0.4}	{0.4,0.6,0.7, 0.34,0.56}	{0.7}

Table 1

Then the score matrix (F_s, A) corresponds to (F, A) given in the table 1 is as follows:

	x_1	x_2	x_3	x_4	x_5
e_1	{0.77}	{0.45,0.3}	{0.625}	{0.5,0.4}	{0.65}
e_2	{0.7}	{0.65}	{0.65}	{0.43,0.4}	{0.5,0.6}
e_3	{0.7,0.7}	{0.4,0.35}	{0.25}	{0.6}	{0.6}
e_4	{0.82}	{0.68}	{0.67}	{0.56}	{0.7}

Table 2

The decision table for each site x_j obtained as follows.

a_j	Values
a_1	0.7
a_2	0.45,0.4
a_3	0.65,0.625
a_4	0.5,0.43
a_5	0.6

Table 3

From table 3,it is clear that the optimal alternate is the site x_1.

Conclusion: Hesitant fuzzy set theory is a newly emerging mathematical tool to deal with uncertain problems. This paper proposed a hesitant fuzzy soft group in combination of hesitant soft set and fuzzy groups. We define some operations on hesitant fuzzy soft groups, direct product of hesitant fuzzy groups and decision making problem under hesitant soft set information.

In this chapter, the concept of hesitant fuzzy soft group is introduced and some of the properties are studied. Furthermore, definitions fuzzy soft function and hesitant fuzzy soft homomorphism are defined and the theorems of homomorphic image and pre-image are given. The definition of hesitant fuzzy soft group is given and some of the basic properties are studied.

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