Common Fixed Point Theorems For Six Self Mappings Complying Common (E.A.) Property in Fuzzy Metric Space

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Abstract

In this paper, we prove common fixed point theorems for six self mappings in a fuzzy metric spaces complying common (E. A.) property. In order to prove the results, we utilize implicit relations. We also give an example to validate main results.

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1 Introduction

The idea of fuzzy mathematics is initiated by Zadeh [22] in 1965. Last few decades were very dynamic for fuzzy mathematics and the recent researches witnessed the fuzzification in almost every area of mathematics e.g. arithmetic, topology, graph theory, logic, differential equations etc. Fuzzy set theory has practical applications in applied sciences such as image processing, medical sciences, control theory, neural work theory, mathematical modeling. Fuzzy metric space is introduced by Kramosil and Michalek [20] by generalizing the probabilistic metric space to fuzzy background. George and Veeramani [17] vaguely modified the above concept to get a Hausedorff topology on this space.

On the other side, Fixed Point Theory is one of most valuable research branches in Nonlinear Analysis. It can be utilized to various different abstract metric spaces. In last two decades fixed point theorems have been largely explored in the settings of fuzzy metric spaces.

Jungck [19] launched the notion of compatible maps for a pair of self mappings and Aamri et al [8] generalized the concept of non-compatibility by introducing the notion of property (E.A.) for self mappings which contained the class of non-compatible mappings in metric spaces, many results showed contraction maps satisfying property(E.A.) in settings of fuzzy metric spaces for instance Kumar et al in [4], Sedghi et al in [3] and Imdad et al in [6]. Popa and Turkoglu [12] proved some fixed point theorems for hybrid mappings satisfying implicit relations. Popa [9] utilized the family of implicit real functions to prove the existence of fixed points.

In this paper we prove common fixed point theorems for six self mappings complying common (E.A.) property on fuzzy metric space employing implicit relation. Our result extends the several existing results in literature.

2 Preliminaries

Definition 2.1.([20]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:
(i) $*$ is commutative and associative,
(ii)$*$ is continuous,
(iii) a $*1=$ a for every a $\in$ [0,1],
(iv) a $*$ b $\leq$ c $*$ d whenever a $\leq$ b and c $\leq$ d for all a, b, c, d $\in$ [0,1].

Definition 2.2.([13]) The 3-tuple (X,M,$*$) is called a fuzzy metric space(FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ satisfying, for every $x, y, z \in X$ and $t, s > 0$, the following conditions:
(FM-1)$M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4)$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5)$M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Remark 1 ([18]). Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, \cdot)$ is nondecreasing on $(0, \infty)$ for all $x, y \in X$.

Remark 2 ([13]) Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, \cdot)$ is continuous function on $X^2 \times (0, \infty)$ for all $x, y \in X$.

Definition 2.3. ([21]) A pair of self mappings (A,B) of a fuzzy metric space (X,M,$*$) is said to be commuting if M(ABx, BAx, t) =1 for all x $\in$ X $\ and\ t > 0$.

Definition 2.4. ([10]) A pair of self mappings (A, B) of a fuzzy metric space (X, M , $*$) is said to be weakly commuting if $M(ABx, BAx, t) \geq M(Ax, Bx, t)$ for all $x \in X\ and\ t > 0$,

Remark 3 ([16]) Let $(X, M, *)$ be a fuzzy metric space. If there exists $r \in (0, 1)$ such that $M(x, y, rt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$

Definition 2.5. ([19]) A pair of self mappings (A, B) of a fuzzy metric space (X, M, $*$) is said to be compatible (or asymptotically commuting) if for all $t > 0$

$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$,

Definition 2.6. ([14]) A pair of self mappings (A, B) of a fuzzy metric space $(X, M, *)$ is said to be weakly compatible if they commute at the coincidence points i.e., if $Au = Bu$ for some $u = X$, then $ABu = BAu$.

Definition 2.7. ([5]) A pair of self mappings (A, B) of a fuzzy metric space $(X, M, *)$ is said to have the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that $lim_{n\longrightarrow\infty} Ax_n =$

$lim_{n\longrightarrow\infty}Bx_n = y$ for some $y \in X$.

We can see that compatible as well as noncompatible pairs satisfy the property (E.A.).

Definition 2.8. ([5]) Two pairs of self mappings (A, P) and (B, Q) defined on fuzzy metric space $(X, M, *)$ are said to share common property (E.A.) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that,
$\lim_{n\to\infty} Ax_n = lim_{n\to\infty}Px_n = lim_{n\to\infty}By_n = lim_{n\to\infty}Qy_n = z$ for some $z \in X$.

Definitions 2.9. Let A and B be self mappings of fuzzy metric space $(X, M, *)$, then a point $x \in X$ is said to be a
(i) coincidence point of A and B if $Bx = Ax$,
(ii) fixed point of B if $Bx = x$.

Implicit Relation

We utilize implicit relations to prove common fixed point results. Let $\mathcal{M}$ be the set of all continuous functions $\psi : [0, 1]^4 \longrightarrow R$ non-decreasing 4 coordinate variables in first argument and satisfying the following conditions:

(a) $\psi(v, 1, v, 1) \geq 0 \Rightarrow v \geq 1$,

(b) $\psi(v, 1, 1, v) \geq 0 \Rightarrow v \geq 1$,

(c) $\psi(v, v, 1, 1) \geq 0 \Rightarrow v \geq 1$.

Example 2.1: Define $\psi : [0, 1] \longrightarrow R\ as\ \ \psi(t_1, t_2, t_3, t_4) = 20t_1 - 13t_2 + 3t_3 - 10t_4$.

We can see clearly ψ satisfies all conditions (a), (b) and (c). Thus $\psi \in \mathcal{M}$.

3 Main Results

Theorem 3.1. Let A, B, P, Q, S and T be self mappings of a fuzzy metric space $(X, M, *)$ satisfying the following conditions:

(i) The pairs (AP, S) and (BQ, T) share the common E.A. property,

(ii) For any $x, y \in X$ and $t > 0$, ψ in $\mathcal{M}$ such that,

$\psi\Big(M(APx, BQy, t), M(Sx, Ty, t), M(Sx, APx, t), M(Ty, BQy, t)\Big) \geq 0$,

(iii) $AP = PA$ and either $AS = SA$ or $PS = SA$,

(iv) $BQ = QB$ and either $BT = TB$ or $QT = TQ$,

If the range of one of S and T is closed subspace of X then $A, B, P, Q, S\ and\ T$ have unique common fixed point.

Proof: The pairs (AP, S) and (BQ, T) share the common (E.A.) property i.e. there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$lim_{n\longrightarrow\infty}APx_n = lim_{n\longrightarrow\infty}Sx_n = lim_{n\longrightarrow\infty}BQy_n = lim_{n\longrightarrow\infty}Ty_n = Sv = BQv = z(say)\ for\ z \in X.$

Suppose $S(X)$ is closed space of X, thus there exists a point $v \in X$ such that,

$z = Sv = BQv$

Now, we claim that $APv = z$, take $x = v\ and\ y = y_n$, by (ii),

$\psi\Big(M(APv, BQy_n, t), M(Sv, Ty_n, t), M(Sv, APv, t), M(Ty_n, BQy_n, t)\Big) \geq 0$

as n $\longrightarrow \infty$

$\psi\Big(M(APv, z, t), M(z, z, t), M(z, APv, t), M(z, z, t)\Big) \geq 0,$

$\psi\Big(M(APv, z, t), 1, M(z, APv, t)1\Big) \geq 0,$

as ψ is non decreasing in the first argument,

$\psi\Big(M(APv, z, t), 1, M(APv, z, t), 1\Big) \geq 0$

using condition (a) of implicit relations

$M(APv, z, t) \geq 1$, hence $APv = z = Sv = BQv$

Implies the pair (AP, S) has a coincident point v

similarly by taking $y = v$ and $x = x_n$ in (ii), we get $BQv = Tv = z$

Thus $APv = Sv = BQv = Tv = z,\ Now,\ APSv = SAPv\ and\ BQTv = TBQv$

i.e. $APz = Sz\ and\ BQz = Tz$

we now show that $APz = z$, Taking $x = z$ and $y = v$ in (ii) we get,

$\psi\Big(M(APz, BQv, t), M(Sz, Tv, t), M(Sz, APz, t), M(Tv, BQv, t)\Big) \geq 0$

$\psi\Big(M(APz, z, t), M(APz, z, t), M(APz, APz, t), M(BQv, BQv, t)\Big) \geq 0$

$\psi\Big(M(APz, z, t), M(APz, z, t), 1, 1) \geq 0$

From the condition (c) of implicit relations,

$M(APz, z, t) \geq 1$ hence we get, $APz = z$ Thus $APz = z = Sz$

Now since $AP = PA,\ hence\ z = APz = PAz\ finally\ PAz = APz = Sz = z$

Suppose A commutes with S so $AS = SA$ thus $SAz = ASz = z$

Since $AP = PA$, we have $APAz = A(PAz) = Az$

Now we show that $Az = z$, $Taking\ x = Az\ and\ y = v$ in (ii) we get,

$\psi\Big(M(APAz, BQv, t), M(SAz, Tv, t), M(SAz, APAz, t), M(Tv, BQv, t)\Big) \geq 0$

$\psi\Big(M(Az, z, t), M(z, z, t), M(z, Az, t), M(z, z, t)\Big) \geq 0$

as ψ is non decreasing in the first argument,

$\Big(M(Az, z, t), 1, M(Az, z, t), 1\Big) \geq 0$

from the condition (a) of implicit relation $M(Az, z, t) \geq 1$

Hence, $M(Az, z, t) = 1$ thus we have $Az = z$ i.e. $Az = Sz = z$

Similarly if P commutes with S then by taking, $x = Pz\ and\ y = v$

we get, $Pz = Az = Sz = z$

we now show that $BQz = z$ by taking $x = v$ and $y = z$

we get $BQz = z$, since B commutes with Q

$z = BQz = QBz$ i.e. $z = QBz = BQz = Tz$

Now suppose B commutes with T i.e. $BT = TB$ thus we have $TBz = BTz = Bz$

And if $BQ = QB$, we have $BQBz = B(BQz) = Bz$

Taking $x = v$ and $y = Bz$ in (ii),

we get, $Bz = z$ Since $BQz = z$ i.e. $BQz = QBz = Qz = z$ Thus, $Bz = Qz = z$

Similarly if Q commutes with T, then by taking $x = v$ and $y = Tz$, we get $Tz = z$

Therefore, we have proven that, $Az = Bz = Pz = Qz = Sz = Tz = z$

Hence z is a common fixed point of six self mappings $A, B, P, Q, S\ and\ T\ in\ X$.

Uniqueness: Now we'll prove uniqueness of the fixed point,

If l is also a common fixed point of $A, B, P, Q, S\ and\ T$ then by taking $x = z\ and\ y = l$ in (ii)

$\psi\Big(M(APz, BQl, t), M(Sz, Tl, t), M(Sz, APz, t), M(Tl, BQl, t)\Big) \geq 0$

Since $z\ and\ l$ are fixed points of mappings $A, B, P, Q, S\ andT$

we have $Az = Bz = Pz = Qz = Sz = Tz = z$ and $Al = Bl = Pl = Ql = Sl = Tl = l$

$\psi\Big(M(z, l, t), M(z, l, t), M(z, z, t), M(l, l, t)\Big) \geq 0$

$\psi\Big(M(z,l,t), M(z,l,t), 1, 1\Big) \geq 0$

From the condition (c) of implicit relations,

$M(z,l,t) \geq 1$

$z = l.$

Theorem 3.2. Let A, B, P and Q be self mappings of a fuzzy metric space $(X, M, *)$ satisfying the following:

(v) The pair (AP, BQ) satisfies the common E.A. property,
(vi) With another pair of self mapping S and T, AB and BQ satisfies the condition(ii).

If the range of one of AP and BQ is closed subspace of X then AB and PQ have common fixed point.

Proof: All the conditions of the Theorem 3.1 are satisfied ensuring the results. Now one needs to prove that AB and PQ have common fixed point, the pair (AP, BQ) satisfies E.A. property,

Hence, $lim_{n\longrightarrow\infty}APx_n = lim_{n\longrightarrow\infty}BQx_n$

Now take $x = z,\ y = x_n$ and$S = AP, T = BQ$ in the condition (ii) we get,

$\psi\Big(M(APz, BQx_n, t), M(APz, BQx_n, t), M(APz, APz, t), M(BQx_n, BQx_n, t) \geq 0$

as $n \longrightarrow \infty$

$\psi\Big(M(APz, z, t), M(APz, z, t), 1, 1\Big) \geq 0$

From the condition (c) of implicit relations

$M(APz, z, t) = 1 \Rightarrow APz = z$

Now we claim that $BQz = z,\ take\ x = x_n\ and\ y = z\ in\ (ii),\ we\ get,$

$\psi\Big(M(APx_n, BQz, t), M(APx_n, BQz, t), M(APx_n, APx_n, t), M(BQz, BQz, t) \geq 0$

$\psi\Big(M(z, BQz, t), M(z, BQz, t), 1, 1)\Big) \geq 0$

From the condition (c) of implicit relation, we get,

$z = BQz \Rightarrow APz = BQz = z$,

thus AP and BQ have common fixed point z.

Example 3.1. Let (X, M, $*$) is a fuzzy metric space where $X = [0, 20]$ and $M(x, y, t) = \dfrac{t}{t+ \mid x - y \mid},\ for\ t > 0,\ define\ \psi : [0, 1]^4 \longrightarrow R,$

$\psi(t_1, t_2, t_3, t_4) = 20t_1 - 13t_2 + 3t_3 - 10t_4 \ldots (A)$

Clearly ψ satisfies all the conditions (a), (b) and (c) of implicit relations.

Now define, $Ax = 2, Bx = 2$

$Px = x, Qx = x$

$$Sx = \begin{cases} 2 & x \leq 10 \\ \frac{1}{4} & x > 10 \end{cases}$$

$$Tx = \begin{cases} 2 & x \leq 10 \\ \frac{1}{2} & x > 10 \end{cases}$$

At the coincidence point 2, the pairs (AP, S) and (BQ, T) share the common E.A. property,

Condition (1): x , y ≤ 10

LHS of inequality (ii)

$\psi\Big((M(2,2,t), M(2,2,t), M(2,2,t), M(2,2,t)\Big)$

$= \psi\Big(1,1,1,1)$, now from (A)

$= 0$ which satisfies (ii)

Condition (2): x > 10, y ≤ 10

From the LHS of (ii)

$\psi\Big(M(2,2,t), M(\frac{1}{4},2,t), M(\frac{1}{4},2,t), M(2,2,t)\Big)$

$= \psi\Big(1, \dfrac{t}{t+\frac{7}{4}}, \dfrac{t}{t+\frac{7}{4}}, 1\Big)$

Now from (A)

$= 20 - 13\dfrac{t}{t+\frac{7}{4}} + 3\dfrac{t}{t+\frac{7}{4}} - 10$

$= 10 - 10\dfrac{t}{t+\frac{7}{4}}$

≥ 0 which satisfies (ii)

Condition (3): $x \leq 10, y > 10,$

From the LHS of (ii)

$= \psi\Big(M(2,2,t), M(2,\frac{1}{2},t), M(2,2,t), M(\frac{1}{2},2,t)\Big)$

$= \psi\Big(1, M(2,\frac{1}{2},t), 1, M(\frac{1}{2},2,t)\Big)$

Now from (A)

$$= 20 - 13\frac{t}{t+\frac{3}{2}} + 3 - 10\frac{t}{t+\frac{3}{2}}$$
$$= 23 - 23\frac{t}{t+\frac{3}{2}}$$
≥ 0 which satisfies (ii)

Condition (4): $x > 10, y > 10$

Now defined mappings and LHS of (ii) give us,

$$= \psi\Big(M(2,2,t), M(\tfrac{1}{4},\tfrac{1}{2},t), M(\tfrac{1}{4},2,t), M(\tfrac{1}{2},2,t)\Big)$$
$$= \psi\Big(1, \frac{t}{t+\frac{1}{2}}, \frac{t}{t+\frac{7}{4}}, \frac{t}{t+\frac{3}{2}}\Big)$$
≥ 0 (For $t > 0$)

Hence all four conditions of variables satisfy (ii).

Clearly, A, B , P, Q, S and T satisfy the hypothesis of Theorem 3.1 and have a unique common fixed point 2 in X.

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