A Non-Linear Stability Analysis of Rayleigh Bènard Magnetoconvection of a Couple Stress Fluid in the Presence of Rotational Modulation

Ramya Rajagopal #1, Shelin Elizabeth Jacob#2, Sangeetha George K #3
P.G. Student, Department of Mathematics, Christ (Deemed to be University), Bengaluru, India #1, 2
Department of Mathematics, Christ (Deemed to be University), Bengaluru, India #3

Abstract— *The non-autonomous Ginzburg-Landau equation with time-periodic coefficients is derived for Rayleigh Bènard convection of couple stress fluid in the presence of rotational modulation and magnetic field. Nusselt number, which is obtained as function of the slow time scale, is used to quantify heat transport. The effects of Chandrasekhar number, Prandtl number, Magnetic Prandtl number, Taylor number and Couple stress parameter are studied in detail.*

Keywords—*Couple stress fluid, Rayleigh Bènard convection, Magnetic field, Ginzburg- Landau equation, Rotational modulation*

NOMENCLATURE

$\vec{q} = (u, v, w)$	Velocity of the fluid	$\vec{H} = (H_x, 0, H_z)$	Magnetic field
$\vec{\Omega}$	Rotational speed vector	μ	Dynamic viscosity
μ'	Couple stress viscosity	ρ	Density of the fluid
μ_m	Magnetic permeability	P_{rm}	Pressure
K_C	Critical wave number	χ_t	Thermal conductivity
T	Temperature	α_t	Coefficient of thermal expansion
γ_m	Magnetic viscosity	τ	Time scale
ε	Perturbation	δ	Amplitude of rotation
ω	Frequency of modulation	$'$	Perturbed value
∇^2	Laplacian operator	R	Thermal Rayleigh number
Q	Chandrasekhar number	Pr	Prandtl number
Pm	Magnetic Prandtl number	Ta	Taylor number
Nu	Nusselt number	ψ	Velocity stream function
ϕ	Magnetic potential function	C	Couple stress parameter

I. INTRODUCTION

The classical Rayleigh Bènardconvection studies the convectionin fluids that arise due to variation in density as a result of temperature differences in the fluid layers.Couple stress fluids are used as a continuum model in many fluid applications that involvesuspended particles. Couple stress fluids have distinct features, such as polar effects. These fluids find their uses in a numberof processes that occur in industry, such as the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath, exotic lubrication and colloidal and suspension solutions. The constitutive equationsfor couple stress fluids were given by Stokes [1]. Since then, many authors have done some extensive studies on Rayleigh Bènard convection of couple stress fluids. Some of the notable studies performed in this area are by Sharma and Thakur [3], R.C Sharma and Shivani Sharma [4], Vivek Kumar and Sudhir Kumar [5]. Various practical applications of couple stress fluid in industries led to the studies on Rayleigh Bènard convection in couple stress fluids in the presence of rotation and magnetic field. The works of A.S. Banyal [6],R.C. Sharma and Monika Sharma [7], A. K. Agarwal and Suman Makhija [8], Pradeep Kumar [9] are some of the important works in this area. All these works strongly established the stabilising effect of magnetic field and rotation on the onset of convection in a couple stress fluid. Thereafter, extensive research is being conducted on the effects of modulation on the onset of convection in a couple stress fluid. The early works of N. Rudraiah and M.S. Malashetty [10], B. S. Bhadauria [11], B. S. Bhadauria and Lokenath Debnath [12], P.G. Siddheshwar, et.al [13], [17] , B. S. Bhadauria et.al [14], J. K. Bhattacharjee [15], [16] have all established the effects of temperature and gravity modulation on the convection in a couple stress fluid. Premila Kollur and G. Sudhaamsh Mohan Reddy [18] recently studied the effect of rotational modulation on the onset of Rayleigh Bènard convection in a couple stress fluid.

Although rotation speed modulation led to the ideas of temperature and gravity modulation, the research work in this fieldis scarce. Therefore in the present work, we intend to investigate the onset of Rayleigh Bènard convection in a couple stress fluid in the presence of magnetic field and rotational modulation using weak nonlinear stability analysis with the help of Ginzburg- Landau model. The objective of this paper isto study how the onset criteria for convection is affected by Prandtl number, Magnetic Prandtl number, Taylornumber, Chandrasekhar number and Couple stress parameter.

II. MATHEMATICAL FORMULATION

Consider a layer of couple stress liquid confined between two infinite horizontal surfaces separated by a distance d. A Cartesian system is taken withorigin in the lower boundary and z-axis vertically upward. Lower surfaceis maintained at higher temperature $T_0 + \Delta T$ and upper surface is maintained at temperature T_0. A uniform magnetic field and variable rotational speed $\vec{\Omega}(t) = \Omega_0(1 + \varepsilon^2 \delta \cos \omega t)$ isapplied along the positive z axis (See Fig.1)

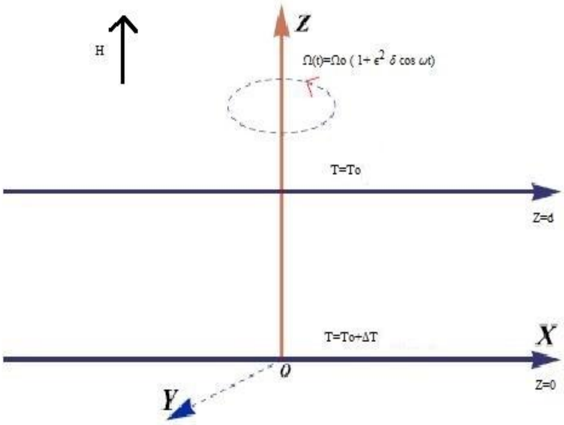

Fig.1.

The basic governing equations are:

$$\nabla . \vec{q} = 0 \tag{1}$$

$$\rho_0 \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} + 2\vec{\Omega}(t) \times \vec{q} \right] = -\nabla P_{rm} + \rho \vec{g} + \mu \nabla^2 \vec{q} - \mu' \nabla^4 \vec{q} + \mu_m (\vec{H}.\nabla)\vec{H} \tag{2}$$

$$\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = \chi_t \nabla^2 T \tag{3}$$

$$\rho = \rho_0 (1 - \alpha_t (T - T_0)) \tag{4}$$

$$\frac{\partial \vec{H}}{\partial t} + (\vec{q}.\nabla)\vec{H} = (\vec{H}.\nabla)\vec{q} + \gamma_m \nabla^2 \vec{H} \tag{5}$$

$$\nabla . \vec{H} = 0 \tag{6}$$

The basic state is assumed to be quiescent, i.e,

$$\vec{q}_b = (0,0,0), \vec{\Omega} = \Omega_0 \hat{k}, T = T_b(z), P_{rm.b} = P_{rm.b}(z), \rho = \rho_b(z), \vec{H}_b = H_o \hat{k} \tag{7}$$

where the subscript b denotes the basic state.

Now we introduce infinitesimally small perturbations to the basic state to get

$$\vec{q} = \vec{q}\,', \rho = \rho_b + \rho', P_{rm} = P_{rm.b} + P_{rm}', T = T_b + T', \vec{H} = H_0 \hat{k} + \vec{H}' \tag{8}$$

Substituting (8) in (1) to (6), eliminating pressure by cross differentiation, introducing the stream functions $u' = \frac{\partial \psi}{\partial z}, w' = -\frac{\partial \psi}{\partial x}, H_x = \frac{\partial \phi}{\partial z}, H_z = -\frac{\partial \phi}{\partial x}$, we get,

$$\rho_0 \left[\frac{\partial}{\partial t}(\nabla^2 \psi) + J(\nabla^2 \psi, \psi) \right] = 2\rho_0 \vec{\Omega}(t) \frac{\partial v'}{\partial z} + \mu \nabla^4 \psi - \mu' \nabla^6 \psi + \mu_m H_0 \frac{\partial}{\partial z} \nabla^2 \phi + \mu_m J(\nabla^2 \phi, \phi) - \rho_0 \alpha_t \frac{\partial T'}{\partial x} g \tag{9}$$

$$\rho_0 \left[\frac{\partial v'}{\partial t} + J(v', \psi) + 2\vec{\Omega}(t) \frac{\partial \psi}{\partial z} \right] = \mu \nabla^2 v' - \mu' \nabla^4 v' \tag{10}$$

$$\frac{\partial T'}{\partial t} + J(T', \psi) = \chi_t \nabla^2 T' - \frac{\Delta T}{d} \frac{\partial \psi}{\partial x} \tag{11}$$

$$\frac{\partial \phi}{\partial t} - J(\psi, \phi) = H_0 \frac{\partial \psi}{dt} + \gamma_m \nabla^2 \phi \tag{12}$$

Equations (9) to (12) are rendered dimensionless using the following transformations.

$$(x^*, z^*) = \left(\frac{x}{d}, \frac{z}{d} \right), \psi^* = \frac{\psi}{\chi_t}, \nabla^* = \frac{\nabla}{(1/d)}, t^* = \frac{t}{(d^2/\chi_t)}, T^* = \frac{T}{\Delta T}, u^* = \frac{u}{(\chi_t/d)}, w^* = \frac{w}{(\chi_t/d)}, H^* = \frac{H}{H_0}$$

The dimensionless equations are given by

$$\begin{bmatrix} -\nabla^4 + C\nabla^6 & R_0 \frac{\partial}{\partial x} & -QPm \frac{\partial}{\partial z}\nabla^2 & -Ta^{1/2}(1+\varepsilon^2 \delta \cos \omega t)\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & -\nabla^2 & 0 & 0 \\ -\frac{\partial}{\partial z} & 0 & -Pm\nabla^2 & 0 \\ Ta^{1/2}(1+\varepsilon^2 \delta \cos \omega t)\frac{\partial}{\partial z} & 0 & 0 & -\nabla^2 + C\nabla^4 \end{bmatrix} \begin{bmatrix} \psi \\ T \\ \phi \\ v \end{bmatrix} = \begin{bmatrix} -\frac{1}{\Pr}\left[\frac{\partial}{\partial t}\nabla^2 + J(\nabla^2 \psi, \psi) \right] + QPmJ(\nabla^2 \phi, \phi) \\ -\frac{\partial T}{\partial t} + J(\psi, T) \\ -\frac{\partial \phi}{\partial t} + J(\psi, \phi) \\ -\frac{1}{\Pr}\left[\frac{\partial v}{\partial t} - J(\psi, v) \right] \end{bmatrix} \tag{13}$$

where

$$\Pr = \frac{\mu}{\rho_0 \chi_t}, Ta = \left(\frac{2\rho_0 d^2 \Omega_0}{\mu} \right)^2, Pm = \frac{\gamma_m}{\chi_t}, Q = \frac{\mu_m H_0^2 d^2}{\mu \gamma_m}, C = \frac{\mu'}{\mu d^2} \text{ and } R = \frac{\alpha_t g d^3 \Delta T \rho_0}{\mu \chi_t}.$$

The system of equations (13) are solved for free- free isothermal boundary conditions $\psi = \nabla^2 \psi = T = \phi = D\phi = 0$ at $z = 0, 1$.

Finite Amplitude Equation, Heat and Mass Transfer

We now introduce the following asymptotic expansions in (13).

$$R = R_0 + \varepsilon^2 R_2 + \varepsilon^4 R_4 + \ldots$$
$$\psi = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \ldots$$
$$v = \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \ldots$$
$$T = \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + \ldots$$
$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots$$

At the lowest mode, we have

$$\begin{bmatrix} -\nabla^4 + C\nabla^6 & R_0\frac{\partial}{\partial x} & -QPm\frac{\partial}{\partial z}\nabla^2 & -Ta^{1/2}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & -\nabla^2 & 0 & 0 \\ -\frac{\partial}{\partial z} & 0 & -Pm\nabla^2 & 0 \\ Ta^{1/2}\frac{\partial}{\partial z} & 0 & 0 & -\nabla^2 + C\nabla^4 \end{bmatrix}\begin{bmatrix}\psi_1 \\ T_1 \\ \phi_1 \\ v_1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad (14)$$

The solution to the lowest order system is given as

$$\psi_1 = A(\tau)\sin(K_c x)\sin(\pi z)$$
$$T_1 = -\frac{K_c}{\delta^2}A(\tau)\cos(K_c x)\sin(\pi z)$$
$$\phi_1 = \frac{\pi}{Pm\,\delta^2}A(\tau)\sin(K_c x)\cos(\pi z)$$
$$v_1 = -\frac{\pi Ta^{1/2}}{\delta^2(1 + C\delta^2)}A(\tau)\sin(K_c x)\cos(\pi z)$$

where $\delta^2 = K_c^2 + \pi^2$.

The critical Rayleigh number is given by $R_0 = \frac{Ta\pi^2}{K_c^2(1 + C\delta^2)} + \frac{Q\pi^2\delta^2}{K_c^2} + \frac{\delta^6(1 + C\delta^2)}{K_c^2}$

At the second order, we have,

$$\begin{bmatrix} -\nabla^4 + C\nabla^6 & R_0\frac{\partial}{\partial x} & -QPm\frac{\partial}{\partial z}\nabla^2 & -Ta^{1/2}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & -\nabla^2 & 0 & 0 \\ -\frac{\partial}{\partial z} & 0 & -Pm\nabla^2 & 0 \\ Ta^{1/2}\frac{\partial}{\partial z} & 0 & 0 & -\nabla^2 + C\nabla^4 \end{bmatrix}\begin{bmatrix}\psi_2 \\ T_2 \\ \phi_2 \\ v_2\end{bmatrix} = \begin{bmatrix}R_{21}\\R_{22}\\R_{23}\\R_{24}\end{bmatrix}$$

where

$$R_{21} = 0$$
$$R_{22} = -\frac{A(\tau)^2 K_c^2\pi}{2\delta^2}\sin(2\pi z)$$

$$R_{23} = -\frac{A(\tau)^2 K_c^2 \pi^2}{2Pm\delta^2}\sin(2K_c x)$$

$$R_{24} = \frac{A(\tau)^2 K_c \pi^2 Ta^{1/2}}{2\Pr\delta^2(1+C\delta^2)}\sin(2K_c x)$$

The second order solutions are as follows.

$$\psi_2 = 0$$

$$T_2 = -\frac{K_c^2}{8\pi\delta^2}A(\tau)^2\sin(2\pi z)$$

$$\phi_2 = -\frac{\pi^2}{8K_c Pm^2\delta^2}A(\tau)^2\sin(2K_c x)$$

$$v_2 = \frac{\pi^2 Ta^{1/2}}{8K_c\Pr\delta^2(1+C\delta^2)(1+4K_c^2)}A(\tau)^2\sin(2K_c x)$$

The horizontally averaged Nusselt number is given by $Nu(\tau) = 1 + \frac{K_c^2}{8\pi\delta^2}A(\tau)^2$ (15)

At the third order, we have

$$\begin{bmatrix} -\nabla^4 + C\nabla^6 & R_0\frac{\partial}{\partial x} & -QPm\frac{\partial}{\partial z}\nabla^2 & -Ta^{1/2}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & -\nabla^2 & 0 & 0 \\ -\frac{\partial}{\partial z} & 0 & -Pm\nabla^2 & 0 \\ Ta^{1/2}\frac{\partial}{\partial z} & 0 & 0 & -\nabla^2 + C\nabla^4 \end{bmatrix}\begin{bmatrix}\psi_3 \\ T_3 \\ \phi_3 \\ v_3\end{bmatrix} = \begin{bmatrix}R_{31} \\ R_{32} \\ R_{33} \\ R_{34}\end{bmatrix} \quad (16)$$

where

$$R_{31} = -\frac{1}{\Pr}\frac{\partial}{\partial\tau}\nabla^2\psi_1 + \frac{1}{\Pr}\left[J(\psi_1,\nabla^2\psi_2) + J(\psi_2,\nabla^2\psi_1)\right] + QPm\left[J(\nabla^2\phi_2,\phi_1) + J(\nabla^2\phi_1,\phi_2)\right] - R_2\frac{\partial T_1}{\partial x} + Ta^{1/2}\delta\cos\omega t\frac{\partial v_1}{\partial z}$$

$$R_{32} = J(\psi_1,T_2) + J(\psi_2,T_1) - \frac{\partial T_1}{\partial\tau}$$

$$R_{33} = J(\psi_1,\phi_2) + J(\psi_2,\phi_1) - \frac{\partial\phi_1}{\partial\tau}$$

$$R_{34} = -\frac{1}{\Pr}\frac{\partial v_1}{\partial\tau} - Ta^{1/2}\delta\cos\omega t\frac{\partial\psi_1}{\partial z}$$

Now we apply the solvability condition for the criterion of the third order solution of the system (16) to arrive at the non-autonomous Ginzburg Landau equation for stationary convection with time periodic coefficients in theform

$$A_1 A'(\tau) - A_2 A(\tau) + A_3 A(\tau)^3 = 0 \quad (17)$$

where $A_1 = \frac{\delta^2}{\Pr} + \frac{R_0 K_c^2}{\delta^4} - \frac{Q\pi^2}{\delta^2 Pm} - \frac{Ta\pi^2}{\delta^4(1+C\delta^2)^2\Pr}$, $A_2 = \frac{K_c^2 R_2}{\delta^2} - \frac{2Ta\pi^2\delta\cos\omega t}{\delta^2(1+C\delta^2)}$

$$A_3 = \frac{Q\pi^4 K_c^2}{2Pm^2\delta^4} - \frac{Q\pi^4}{4Pm^2\delta^2} + \frac{K_c^4 R_0}{8\delta^4}$$

The Ginzburg-Landau equation given in (17) is Bernoulli equation and hence, obtaining its analytical solution is difficult as it is autonomous in nature. Inview of this, it has been solved numerically using Mathematica, subject tothe initial condition A(0) = a0 , where a0 is the chosen initial amplitude ofconvection.

III. Results and discussions

In this paper, a nonlinear analysis is performed to study the effect of magnetic field and rotational modulation on Rayleigh Bènard convection ofa couple stress fluid. Autonomous Ginzburg-Landau equation is derived to study the effects of parameters of theproblem on heat and mass transfer.

Fig (2) shows the variation of *Nu* with τ for different values of couplestress parameter *C*. It is observed that as *C* increases *Nu* decreases. Thepresence of couple stress increases the viscosity of the fluid and hence as *C*increases, more heating is required to make the system unstable, indicating the stabilising effect of *C*. Fig (3) shows the variation of *Nu* with τ for different values of*Pr*. It is found that *Nu* increasesas *Pr* increases, indicating the destabilising effect. Fig (4) to (6) show the variation of *Nu* with τ for different values of *Ta, Pm* and *Q* respectively. It is found that*Nu* decreases with increase in *Ta, Pm* and *Q*, showing the delayed onset ofconvection.

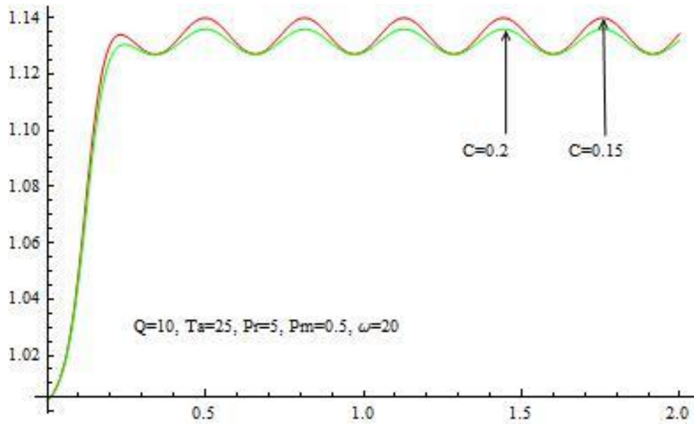

Fig. 2. Variation of *Nu* with τ for different values of *C*

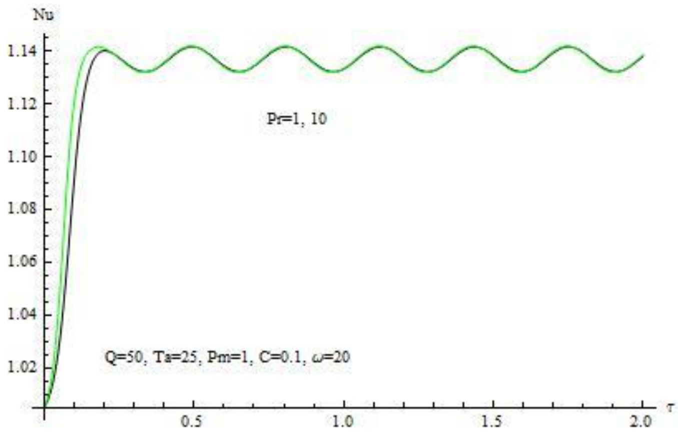

Fig. 3. Variation of *Nu* with τ for different values of *Pr*

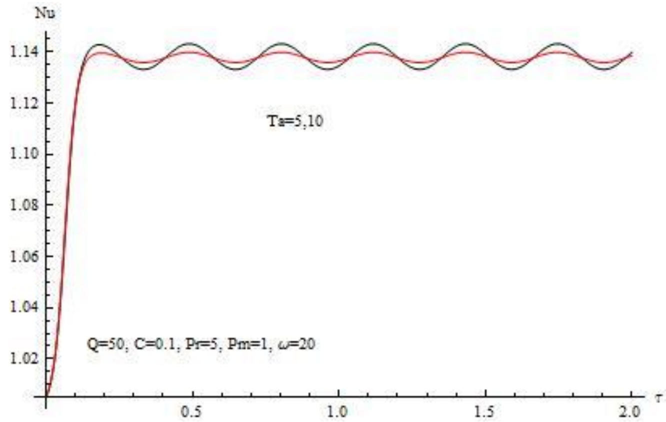

Fig. 4. Variation of *Nu* with τ for different values of *Ta*

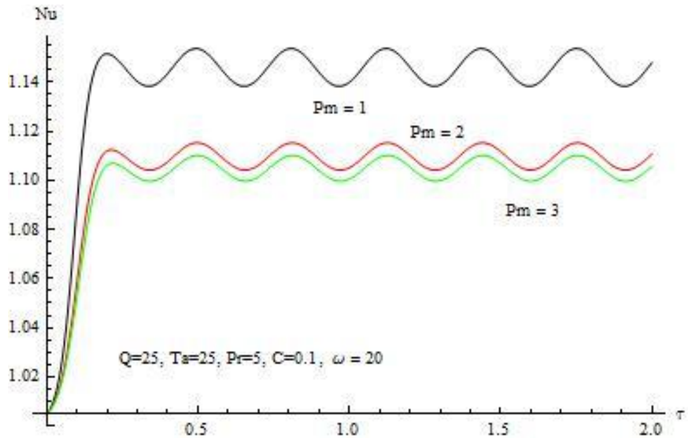

Fig. 5. Variation of *Nu* with τ for different values of *Pm*

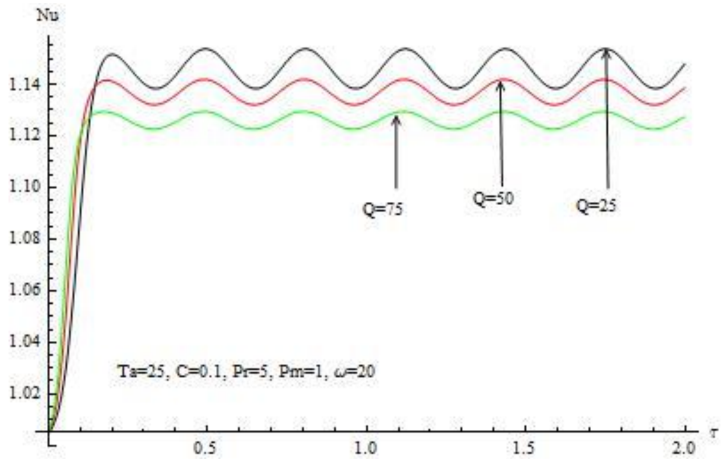

Fig. 6. Variation of *Nu* with τ for different values of *Q*

IV. Conclusions

The combined effect of magnetic field and rotational modulation on Rayleigh Bènard convection of a couple stress fluid has been studied by employing nonlinear analysis using Ginzburg- Landau model. The results have been obtained in terms of Nusselt number, and the effects ofvarious parameters have been obtained graphically. The following are theobservations.

1. The effect of increasing Couple stress parameter is to decrease the Nusselt number, thus indicating stabilising effect.
2. The effect of increasing Prandtl number is to advance the onset of convection, and thus heat and mass transfer.

3. Increasing Taylor number has a stabilising effect by delaying the onset of convection, hence reducing heat and mass transfer.
4. The effect of increasing Magnetic Prandtl number is to decrease Nusselt number, thus indicating stabilising effect.
5. The effect of increasing Chandrasekhar number is to decrease Nusselt number, hence delaying the onset of convection.

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