Maximal Decomposition of the Turaev-Viro TQFT

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Abstract —*In[1]we have built aHQFT from the universal graduation of a spherical category. In the present paper, we show that every graduation (G, p) of a spherical category C defines a Turaev-Viro HQFT. Furthermore we show that the Turaev-Viro TQFT will be split into blocks coming from this HQFT. We show that this decomposition is maximal for the universal graduation of the category, which means that for every graduation (G, p) we define a HQFT which will be split into blocks coming from the HQFT obtained from the universal graduation.*

Keywords —*Quantum invariants, Turaev-Viro invariant, TQFTs, HQFTs*

I. Introduction

The Turaev-Viro invariant[2]is a quantum invariant of 3-manifold with boundary. In the original construction, Turaev and Viro used the quantum group $\boldsymbol{U_q(\mathfrak{sl}_2)}$ to build this invariant. In [3]and[4]the authors generalize this construction to spherical categories with invertible dimension. A spherical category is a semisimple sovereign category over a commutative ring **k** such that the left and right traces coincide. The dimension of a spherical category is the sum of squares of dimensions of simple objects. The Turaev-Viro invariant of a closed 3-manifold **M** is a state-sum indexed by the colorings of a triangulation of **M**. The colorings of a triangulation **T** are maps from the set of oriented 1-simplices to the set of scalar objects (up to isomorphism) of a spherical category **C**. The set of colorings of a triangulation **T** is denoted **Col(T)**. The Turaev-Viro invariant is:

$$TV_C(\mathrm{M}) = \Delta_C^{-n_0(T)} \sum_{c \in Col(T)} w_c W_c \in \boldsymbol{k},$$

where Δ_C is the dimension of the category, $n_0(T)$ is the number of 0-simplices of **T**, $w_c$is a scalar obtained from the coloring of the 1-simplices and the trace of the category and W_c is a scalar obtained from the 6j-symbols of the category.

The Turaev-Viro invariant extends to a *Topological Quantum Field Theory* (TQFT) [5]called *Turaev-Viro TQFT*. In dimension 2+1, a TQFT assigns to every closed surface a finite dimensional vector space and to every cobordism a linear map. In [1], we show that the Turaev-Viro TQFT can decomposed as a sum of HQFT [6]. A *Homotopy Quantum Field Theory* (HQFT)[6] is a TQFT for surfaces and cobordims endowed with homotopy classes of continuous map to *target space***X.**To obtain this decomposition, we use the universal graduation $(\Gamma_C, |?|)$of the spherical category**C**in order to build a homotopy invariant and then obtain a HQFT. A graduation of a semisimple tensor category is a pair ***(G, p)*** where**G** is a group and **p** is a map from **G** to the set of isomorphism classes of scalar objects such that **p**(Z)=**p**(X)**p**(Y) if Z is a scalar subobject of $X \otimes Y$. Using the group $\boldsymbol{\Gamma_C}$we define ahomotopy invariant HTV_C called*the homotopyTuraev-Viro invariant*. The Turaev-Viro invariant will be a sum of this invariant. More precisely, we observe that for every coloring **c** of a triangulation T of a closed 3-manifold M leads to ahomotopyclass$x_c \in [M, B\Gamma]$, where $\mathbf{B}\boldsymbol{\Gamma_C}$is the classifying space of the group $\boldsymbol{\Gamma_C}$and[M,$\mathbf{B}\boldsymbol{\Gamma_C}$**]** is the set of homotopy classes of continuous map from M to $\mathbf{B}\boldsymbol{\Gamma_C}$. These remarks lead to the following homotopy invariant of closed 3-manifolds:

$$HTV_C^{(\Gamma_C,|?|)}(M, x) = \Delta_C^{-n_0(T)} \sum_{\substack{c \in Col(T) \\ x_c = x}} w_c W_c$$

with$x \in [M, \mathbf{B}\boldsymbol{\Gamma_C}]$. In [1], we show that the homotopyTuraev-Viro extends to an HQFT with target space $\mathbf{B}\boldsymbol{\Gamma_C}$ denoted $\mathcal{H}_C^{(\Gamma_C,|?|)}$ and we obtain the following decomposition of the Turaev-Viro TQFT$\mathcal{V}_C$:

$$\mathcal{V}_C(\Sigma) = \sum_{x \in [\Sigma, B\Gamma]} \mathcal{H}_C^{(\Gamma_C,|?|)}(\Sigma, x)$$

for every closed and oriented surfaceΣ.

The motivation for this paper is to study other decomposition of the Turaev-Viro TQFT and compare them. To fulfill this objective, for every graduation **(G, p)** of Cwe build ahomotopyTuraev-Viro invariant and we express the Turaev-Viro invariant with this invariant:

Theorem5.3

*Let **C**be a spherical category with an invertible dimension, **M**be 3-manifold, Σ be the boundary of **M**and T_0 be a triangulation ofΣ. For every coloring $c_0 \in Col(T_0)$and for every homotopyclass$x \in [M,BG]_{\Sigma,x_0}$, where $x_{c_0} \in [M,BG]$ is obtained fromc_0, the vector:*

$$HTV_C^{(G,p)}(M,c_0,x) = \Delta_C^{-n_0(T)+\frac{n_0(T_0)}{2}} \sum_{c \in Col_{c_0,x}(T)} w_c W_c \in V_C(\Sigma,T_0,c_0)$$

is an invariant of the triple(M,c_0,x). We have the following equality:

$$TV_C(M,c_0) = \sum_{x \in [M,BG]_{\Sigma,x_{c_0}}} HTV_C^{(G,p)}(M,c_0,c)$$

Using the universal property of the universal graduation, we can compare the decompositions of the Turaev-Viro invariant obtained from a graduation **(G,p)**and from the universal graduation. The universal property of the universal graduation induces a map $F : [M, \boldsymbol{B\Gamma_C}] \to [M, \boldsymbol{BG}]$, using this map we show that for every graduation **(G,p)** the homotopy Turaev-Viro invariant $HTV_C^{(G,p)}$comes from the homotopy Turaev-Viro invariant $HTV_C^{(\Gamma,|?|)}$:

Corollary6.3

*Let **C**be a spherical category with an invertible dimension, **M**be a 3-manifold, Σ be the boundary of **M**and T_0 be a triangulation of Σ. For every graduation **(G,p)** of**C**, one gets:*

$$TV_C(M,c_0) = \sum_{x \in [M,BG]_{\Sigma,x_{c_0}}} HTV_C^{(G,p)}(M,c_0,c) \in V_C(\Sigma,T_0,c_0),$$

with $c_0 \in Col(T_0)$, and

$$HTV_C^{(G,p)}(M,c_0,x) = \sum_{y \in F^{-1}(x)} HTV_C^{(\Gamma_C,|?|)}(M,c_0,y),$$

*where**F** is the map induced by the universal graduation $(\Gamma_C, |?|)$.*

For every graduation **(G,p)**of **C** we prove that the homotopy invariant $HTV_C^{(G,p)}$ extends to an HQFT $\mathcal{H}_C^{(G,p)}$with target space **BG**such:

$$\mathcal{V}_C(\Sigma) = \sum_{x \in [\Sigma,BG]} \mathcal{H}_C^{(G,p)}(\Sigma,x),$$

for every closed and oriented surface Σ. Using **Theorem 5.3**and**Corollary6.3**, we show that the decomposition of the Turaev-Viro TQFT is given by the universal graduation is maximal:

Theorem 8.1

*Let **C**be a spherical category**, (G,p)** be a graduation of **C**. The Turaev-Viro HQFT obtained from the graduation **(G,p)** is decomposed in the following way:*

$$\mathcal{H}_C^{(G,p)}(M,x) = \sum_{y \in F^{-1}(x)} \mathcal{H}_C^{(\Gamma_C,|?|)}(M,y),$$

*for every closed surface Σ, for every $x \in [M, \boldsymbol{BG}]$ and with $F : [M, \boldsymbol{B\Gamma_C}] \to [M, \boldsymbol{BG}]$ the map obtained from the universal graduation (**Lemma 6.1**).*

The rest of the paper is organized as follows. In Section III, we review several facts aboutmonoidal categories and we define the universal graduation of semisimple tensor categories. In Section IV, we recall the construction of the Turaev-Viro invariant. In Section V, we build ahomotopy Turaev-Viro invariant for every graduation **(G,p)** of a spherical category C. Furthermore we show that the Turaev-Viro is obtained from the homotopyTuraev-Viro invariant $HTV_C^{(G,p)}$ (**Theorem 4.3**). In Section VI, we compare the different splitting of the Turaev-Viro invariant. We show that The Turaev-Viro invariant and the invariant$HTV_C^{(G,p)}$ are obtained from the homotopy Turaev-Viro invariant $HTV_C^{(\Gamma_C,|?|)}$ (**Corollary 6.3**). In Section VII, we use the homotopy invariant to build an HQFT. The target of this HQFT will be the classifying space of the graduation. In Section VIII, we prove **Theorem 7.1**, it follows that the Turaev-Viro TQFT and the Turaev-Viro HQFT obtained from any graduation of **C**are decomposed into blocks which come from the Turaev-Viro HQFT obtained from the universal graduation.

II. Notations and conventions

Throughout this paper, **k**will be a commutative, algebraically closed and characteristic zero field. Unless otherwise specified, categories are assumed to be small and monoidal categories are assumed to be strict and spherical categories are assumed to be strict.

Throughout this paper, we use the following notation. For an oriented manifold **M**, we denote by $\overline{\mathbf{M}}$ thesame manifold with the opposite orientation.

III.GRADUATIONS OF TENSOR CATEGORIES

In the present section, we review a few general facts about categories with structure, which we use intensively throughout this text.

Let **C**be a monoidal category. A *duality*of **C**is a data (X,Y,e_h), where X and Y are objects of C and $e : X \otimes Y \hookrightarrow I$ (*evaluation*) and $h : I \hookrightarrow Y \otimes X$(*coevaluation*) are morphisms of **C**, satisfying:

$$(e \otimes id_X)(id_X \otimes h) = id_X \text{ and } (id_Y \otimes e)(h \otimes id_Y) = id_Y$$

If (X,Y,e_h)is a duality, we say that (Y,e_h) is *a right dual of X*, and (X,e_h) is *a left dual of Y*. If a right or left dual of an object exists, it is unique up to unique isomorphism.

A *right autonomous* (resp. *left autonomous*, resp. *autonomous*) category is a monoidal category for which every object admits a right dual (resp. a left dual, resp. both a left and a right dual).

If C has right duals, we may pick a right dual ($X^\vee$,e_X,h_X) for each object X. This defines a monoidal functor$?^{\vee\vee} : \boldsymbol{C} \to \boldsymbol{C}$defined by $X \mapsto X^{\vee\vee}$and $f \mapsto f^{\vee\vee}$, called the *double right dual functor*.

A. *Sovereign categories*

A *sovereign structure* on a right autonomous category **C**consists in the choice of a right dual for each object of **C**together with a monoidal isomorphism $\phi : 1_C \to ?^{\vee\vee}$, where 1_C is the identity functor of **C**. Two sovereign structures are *equivalent* if the corresponding monoidal isomorphism coincides via the canonical identification of the double dual functor.

A *sovereign category* is a right autonomous category endowed with an equivalence class of sovereign structures.

Let C be a sovereign category, with chosen right duals ($X^\vee$,e_X,h_X) and sovereign isomorphism$\phi_X: X \mapsto X^{\vee\vee}$. For each object X of **C**, we set :

$$\epsilon_X = e_{X^\vee}(id_{X^\vee} \otimes \phi_X) \text{and} \eta_X = (\phi_X^{-1} \otimes id_{X^\vee})h_{X^\vee}$$

Then ($X^\vee$,ε_X,η_X) is a left dual of X. Therefore **C**is autonomous. Moreover the right left functor $^\vee ?$ defined by this choice of left duals coincides with $?^\vee$ as a monoidal functor. From now on, for each sovereigncategory **C**we will make this choice of duals.

The sovereign categories are an appropriate categorical setting for a good notion of trace. Let **C**be a sovereign category and X be an object of**C**. For each endomorphism$f \in Hom_C(X,X)$, we have :

- $tr_l(f) = \epsilon_X(id_{X^\wedge} \otimes f)h_X \in Hom_C(I,I) = \boldsymbol{k}$is the *left trace* of f
- $tr_r(f) = e_X(f \otimes id_{X^\wedge})\eta_X \in Hom_C(I,I) = \boldsymbol{k}$is the *right trace* of f.

We denote by $\dim_r(X)=tr_r(id_X)$ (resp. $\dim_l(X)=tr_l(id_X)$) the *right dimension* (resp.*left dimension*) of X.

B. *Tensor categories*

By a **k**-linear category, we shall mean a category for which the set of morphisms are **k**-spaces, the composition is **k**-bilinear there exists a null object and for every objects X, Y the direct sum $X \oplus Y$ exists in **C**.

A **k**-linear category is *abelian* if it admits finite direct sums, every morphism has a kernel and a cokernel, every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel, and every morphism is expressible as the composite of an epimorphism followed by a monomorphism.

An object X of an abelian **k**-category C is *scalar* if $Hom_C(X,X) \cong \boldsymbol{k}$.

A *tensor category over **k*** is an autonomous category endowed with a structure of **k**-linear abelian category such that the tensor product is **k**-bilinear and the unit object is a scalar object.

A **k**-linear category is *semisimple* if :

- every object of **C** is a finite direct sum of scalar objects,
- for every scalar objects X and Y, we have : $X \cong Y$ or $Hom_C(X,Y) = \mathbf{0}$.

C. *Graduations*

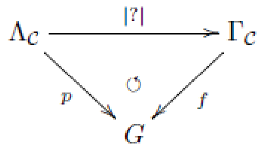
Let **C**be semisimple tensor **k**-category and **G** be a group. A ***G**-graduation of **C***is a map $: G \to \Lambda_C$:

- p(Z)=p(X)p(Y), for every scalar objects X,Y,Z such that Z is a subobject of$X \otimes Y$.

A *graduation of C* is a pair **(G,p)**, where **G** is group and p is a **G**-graduation of **C**.By induction, the multiplicity property of a graduation can be extended to **n**-terms.In [1], we prove that every semisimple tensor **k**-category admits a universal graduation:

Proposition 3.1

Let **C**be a semisimple tensor **k-**category. There exists a graduation $(\Gamma_C, |?|)$ of **C**satisfying the following universal property: for every graduation **(G,p)** of **C**, there exists unique group morphism $f : \Gamma_C \to G$ such that the diagram:



commutes.

Let **C**be a semisimple tensor **k-**category, the group $\Gamma_C$which defines the universal graduation $(\Gamma_C, |?|)$ is called *the graduator of C*. The graduator can be used to describe the sovereign (resp. spherical) structures of a sovereign (resp. spherical) category [1].

Examples : The graduator of the category of representations of finite dimension of$U_q(\mathfrak{sl}_n)$ is $\mathbb{Z}_n$.

D. ***Spherical categories***

A *spherical category* is a sovereign, finitely semisimple tensor **k-**category satisfying:

- for every object X of **C** and for every morphism $f : X \to X$ $tr_r(f)=tr_l(f)$.

A *spherical structure* on **C**is a sovereign structure on **C**such that **C**is a spherical category.

From now on, for every spherical category the left and right trace (resp. dimension) will be denoted by tr (resp. dim).The dimension of a spherical category is the scalar: $\Delta_C = \sum_{X \in \Lambda_C} \dim(X)^2 \in \boldsymbol{k}$. From now on, unless otherwise specified, spherical categories are assumed to have an invertible dimension.

IV. The Turaev-Viro invariant

In this Section, we recall the construction of the Turaev-Viro invariant. For further reading on the Turaev-Viro invariant, we refer the reader to [2] (the original construction), [3] (the construction using a spherical category), [4]and [5]. Throughout this Section, **C**will be a spherical category.

An *orientation* of a **n-**simplex **F** is a map$o : Num(F) \to \{\pm 1\}$, where Num(F) is the set of numberings of **F**, invariant under the action of the alternated group$\mathfrak{U}_{N+1} \subset \mathfrak{S}_{N+1}$.

Let**T**be an oriented simplicial complex, we denote the set of oriented p-simplices by T_o^p. A coloring of **T**is a map $c : T_0^1 \to \Lambda_C$satisfying:

- $c(x1x2) = c(x2x1)^\vee$, for every oriented 1-simplex (x_1x_2),
- the unit object **I**is a subobject of $c(x_1x_2) \otimes c(x_2x_3) \otimes c(x_3x_1)$ for every oriented 2-simplex $(x_1x_2x_3)$.

We denote by Col(T)the set of colorings of **T**.

Let f be an oriented 2-simplex, c be a coloring of **T**and $v=(x_1x_2x_3)$ be a numbering of f compatible with the orientation of f. Set :

$$V_C(f,c)_v = Hom_C\big(\boldsymbol{I}, c(x_1x_2) \otimes c(x_2x_3) \otimes c(x_3x_1)\big)$$

The vector space $V_C(f,c)_v$ does not depend on the choice of the numbering compatible with the orientation (e.g. [3], [4], [5]). From now on, the vector space$V_C(f,c)_v$, with $v=(x_1x_2x_3)$ will be denoted by$V_C(x_1x_2x_3, c)$. If there is no ambiguity on the choice of the coloring c, then $V_C(x_1x_2x_3, c)$ will be denoted by $V_C(x_1x_2x_3)$.

Let us recall some properties of the vector space defined above. For every scalar objects X, Y andZ, we set:

$$\omega_C\colon\ Hom_c(I, X \otimes Y \otimes Z) \otimes_{\boldsymbol{k}} Hom_c(I, Z^\wedge \otimes Y^\wedge \otimes X^\wedge) \to \boldsymbol{k}^*$$
$$f \otimes g \ \mapsto tr(f^\vee g)$$

For every spherical category **C**, the bilinear formω_C is non degenerate (e.g. [3], [4], [5]). Let **f** be an oriented 2-simplex, we denote by $\bar{\boldsymbol{f}}$the 2-simplex **f** endowed with the opposite orientation. Let c be a coloring of**f**, the bilinear form ω_C induces: $V_C(\boldsymbol{f}, c)^* \cong V(\bar{\boldsymbol{f}}, c)$.

In the construction of the Turaev-Viro invariant, we assign to every oriented 3-simplex of a colored 3-manifold M, a vector which lies in the vector space defined by the faces of the 3-simplex. The vector assigned to each 3-simplex is obtained by the 6j-symbols of the category. A contraction of these vectors along the 2-simplices contained inside the 3-manifold M leads to a scalar if the manifold M is without boundary or to a vector in

$$\bigotimes_{f \in T^2_{\partial M}} V_C(f,c)$$

if the manifold M has a boundary ∂M. We denote this vector (or scalar) by W_c, for every coloring c.

We introduce some notations. Let Σ be an oriented closed surface endowed with a triangulation $\mathbf{T_0}$. For every coloring c_0 of$\mathbf{T_0}$, we set :

$$V_C(\Sigma, T_0, c_0) = \bigotimes_{f \in T^2_{\partial M}} V_C(f, c_0)$$

$$V_C(\Sigma, T_0) = \bigoplus_{c \in Col(T_0)} V_C(\Sigma, T_0, c)$$

Let Mbe 3-manifold with boundary Σ and T be a triangulation of M such that its restriction to Σ is T_0. For every coloring $c_0 \in Col(T_0)$, we denote by $Col_{c_0}(T)$ the set of colorings of T such that the restriction to T_0 is c_0. With this notation, for every coloring $c \in Col_{c_0}(T)$, we have: $W_c \in V_C(\Sigma, T_0, c_0)$. Furthermore we choose a square root$\Delta_C^{1/2}$of Δ_C.

For every scalar object X of C, we set $\dim(X)^{1/2}$ a square root of dim(X). The equalities $\dim(X)^{1/2} = \dim(X^\vee)^{1/2}$and dim(X) = $\dim(X^\vee)$ ensure for every coloring c the independence for dim(c(e)) and $\dim(c(e))^{1/2}$ of the choice of the orientation of e.

Theorem 4.1[Turaev-Viroinvariant [3], [4], [5], [2]]

Let **C***be a spherical category with an invertible dimension, M be a compact oriented 3-manifold and ∂M be the boundary of M endowed with a triangulation T_0. For every coloring $c_0 \in Col(T_0)$, we set :*

$$TV_C(M, c_0) = \Delta_C^{-n_0(T)+n_0(T_0)/2} \sum_{c \in Col_{c_0}(T)} \prod_{e \in T_0^1} dim\big(c_0(e)\big)^{1/2} \prod_{e \in T^1 \setminus T_0^1} dim\big(c(e)\big) W_c \in V(\partial M, c_0, T_0)$$

where $n_0(T)$ (resp. $n_0(T_0)$) is the number of 0-simplices of T (resp. T_0) and $T^1 \setminus T^1_0$ is the set of 1-simplices ofM\∂M.

For every coloring $c \in Col_{c_0}(T_0)$, the vector $TV_C(M, c_0)$is independent on the choice of the triangulation of M which extends T_0. The Turaev-Viro invariant is the vector:

$$TV_{C(M)} = \sum_{c_0 \in Col(T_0)} TV_c(M, c_0) \in V_C(\partial M, T_0) = \bigoplus V(\partial M, T_0, c_0)_{c \in Col(T_0)}$$

From now on, for every coloring $c \in Col_{c_0}(T)$ we denote by w_c the scalar:

$$\prod_{e \in T_0^1} dim\big(c_0(e)\big)^{1/2} \prod_{e \in T^1 \setminus T_0^1} dim\big(c(e)\big).$$

V. THE HOMOTOPY TURAEV-VIRO INVARIANT

In this section, we will extend the construction of the homotopy turaev-viro invariant defined in [1]. Thus we will obtain ahomotopyTuraev-Viro invariant for every graduation of a spherical category with an invertible dimension.

1. *G-colorings*

Throughout this paragraph **C** will be a finitely semisimple tensor **k**-category and **G** will be a group. Let **T** be a simplicial complex. A **G**-coloringc of t is a map :

$$c : T_0^1 \to G$$
$$e \longmapsto c(e)$$

Satisfying:

- for every oriented 1-simplex (x_1x_2) of **T**: $c(x_1x_2)=c(x_2x_1)^{-1}$
- for every oriented 2-simplex $(x_1x_2x_3)$ of**T**: $c(x_1x_2)c(x_2x_3)c(x_3x_1)=1$,

We denote by $Col_G(T)$ the set of **G**-colorings of **T**.

In [1], we define an action on the set of **G**-colorings of **T** using the gauge group of **T**. *A gauge of **T** with value in **G*** is a map $\delta : T^0 \to G$ and we denote $\mathcal{g}_T^G$ the gauge group of **T** with value in **G**. The action of the gauge group on **T**is defined in the following way:

$$\mathcal{g}_T^G \times Col_G(T) \to Col_G(T)$$
$$(\delta, c) \mapsto c^\delta$$

Wherec^δ is the **G**-coloring:$c^\delta(x,y) = \delta(x)c(xy)\delta(y)^{-1}$, for every oriented 1-simplex (xy). We denote by $Col_G(T)/\mathcal{g}_T^G$ the quotient set of $Col_G(T)$ by the action of the gauge group $\mathcal{g}_T^G$. We have the following topological interpretation of $Col_G(T)/\mathcal{g}_T^G$:

Proposition 5.1

*Let **T** be a simplicial complex, c be a semisimple tensor k-category and **G** be a group and $\mathcal{G}$the associated groupoid. Themap :*

$$Col_G(T) \to Fun(\pi_1(T), \mathcal{G})$$
$$c \mapsto F_c$$

Where F_c is the functor which sends every 0-simplex of $\boldsymbol{T}$ to the unique object of the groupoid $\mathcal{G}$ and sends every oriented 1-simplex (xy) to c(xy), induces the following isomorphism :

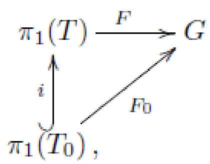
$$Col_G(T)/\mathcal{g}_T^G \cong Fun(\pi_1(T), G)/(iso) \cong [|T|, BG]$$

Where $[|T|, BG]$ is the set of homotopy classes of continuous maps from the topological space $|T|$, to the classifying space BG.

Let us recall the topological interpretation of the **G**-colorings, in the case of manifolds with boundary.

Let M be a 3-manifold, Σ be the boundary of M and T_0 be a triangulation of Σ. We set $Col_{G,c_0}(T)$ the set of **G**-colorings of T such that the restriction to $\mathbf{T_0}$ is c_0. In this case we consider the gauge action which does not change the **G**-coloring on the boundary, i.e the restriction of c^δ to $\mathbf{T_0}$ is c_0.

From now we denote by **G** the groupoid build from the group **G**. For every functor $F_0: \pi_1(T_0) \to \boldsymbol{G}$, $Fun(\pi_1(T), \boldsymbol{G})_{F_0}$ is the set of functors F from $\pi_1(T)$ to the groupoid **G** such that the diagram :



commutes, with i is the inclusion functor. We denote by $Fun(\pi_1(T), G)_{F_0}/(iso)$ the set of isomorphisms classes of functors such that the restriction of the natural isomorphisms to $\pi_1(T)$ is id_{F_0}.

Proposition 5.2[1]

Let $\boldsymbol{C}$ be a semisimple tensor $\boldsymbol{k}$-category, $\boldsymbol{T}$ be a simplicial complex and T_0 be a subcomplex of $\boldsymbol{T}$. For every coloring $c_0 \in Col(T_0)$, the map:

$$Col_{G,c_0}(T) \to Fun(\pi_1(T), G)_{F_{c_0}}$$
$$c \mapsto F_c$$

Where the functor F_c sends every 0-simplex of $\boldsymbol{T}$ to the unique object of the groupoid **G** *and every oriented 1-simplex (xy) to c(xy), induces the following isomorphism :*

$$Col_{G,c_0}(T)/\mathcal{g}_T^G \cong Fun(\pi_1(T), G)_{F_{c_0}}/(iso) \ \mathbf{(1)}$$

From now on, **C** is a spherical category and (**G**,p) is a graduation on **C**.

Let us introduce some notations. Let M be a 3-manifold and T be a triangulation of M. By definition of the graduation, for every coloring $c \in Col(T)$, p_c is a **G**-coloring of **T**. Then for every $x \in [M, BG]$, we denote by $Col_{(G,p),x}(T)$ the set of colorings c of **T** such that the equivalence class $[p_c]$ in $Col_G(T)/\mathcal{g}_T^G$ corresponds to x (bijection **(1)**). We obtain a partition of the set $Col(T) = \coprod_{x \in [M,BG]} Col_{(G,p),x}(T)$. If $c \in Col(T)$, we denote by $x_c \in [M, BG]$ the homotopy class associated to p_c.

Let **M** be a 3-manifold, Σ be the boundary of M and $\mathbf{T_0}$ be a triangulation of Σ. For every homotopy class $x_0 \in [\Sigma, BG]$, we denote by $[M, BG]_{\Sigma, x_0}$ the set of homotopy classes of maps from M to the classifying space BG such that the homotopy class of the restriction to Σ is x_0. Thus for every coloring $c_0 \in Col(T_0)$ and for every triangulation **T** of M such that its restriction to Σ is T_0, we have:

$$Col_{G,c_0}(T)/\mathcal{g}_T^G \cong Fun(\pi_1(T), G)_{F_{c_0}}/(iso) \cong [M, BG]_{\Sigma, x_{C_0}} \mathbf{(2)}$$

For every coloring $c_0 \in Col(T_0)$ and for every homotopy class $y \in [M, BG]_{\Sigma, x_{C_0}}$, we denote by $Col_{(G,p),c_0,y}(T)$ the set of colorings $c \in Col(T)$ satisfying:

- $c_{T_0} = c_0$
- the equivalent class $p_c \in Col_{G,c_0}/\mathcal{g}_T^G$corresponds to$y \in [M, BG]_{\Sigma, x_{C_0}}$ by the bijections **(2)**

Let us define the homotopy turaev-viro invariant obtained from the graduation (**G**,p).

Let **M** be a 3-manifold, Σ be the boundary of m, T_0be a triangulation of Σ and $c_0 \in Col(T_0)$. We can break up the Turaev-Viro state sum in the following way:

$$TV_C(M, c_0) = \Delta_c^{-n_0(T)+n_0(T_0)/2} \sum_{c \in Col_{c_0}(T)} w_c W_c$$

$$= \Delta_c^{-n_0(T)+n_0(T_0)/2} \sum_{x \in [M,BG]_{(\Sigma, x_{c_0})}} \sum_{c \in Col_{(G,p),c_0,x}(T)} w_c W_c$$

We set: $HTV_C^{(G,p)}(M, x, c_0) = \Delta_c^{-n_0(T)+n_0(T_0)/2} \sum_{c \in Col_{(G,p),c_0,x}(T)} w_c W_c$.The vector $HTV_C^{(G,p)}(M, x, c_0)$is an invariant for the triple (M,x,c_0). The proof of the invariance is similar to the proof given in [1] (theorem 4.6).

Theorem 5.3

*Let **C** be a spherical category with an invertible dimension, **M** be 3-manifold, Σ be the boundary of **M** and T_0 be a triangulation of Σ. For every coloring $c_0 \in Col(T_0)$ and for every homotopy class$x \in [M, BG]_{(\Sigma, x_{c_0})}$, where $x_{c_0} \in [M, BG]$is obtained from c_0, the vector :*

$$HTV_C^{(G,p)}(M, x, c_0) = \Delta_c^{-n_0(T)+n_0(T_0)/2} \sum_{c \in Col_{(G,p),c_0,x}(T)} w_c W_c \in V_C(\Sigma, T_0, c_0)$$

is an invariant of the triple (M,x,c_0). We have the following equality:

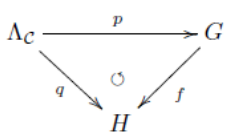
$$TV_C(M, c_0) = \sum_{x \in [M,BG]_{\Sigma, x_{c_0}}} HTV_C^{(G,p)}(M, x, c_0)$$

The vector $HTV_C^{(G,p)}$ is the (**G**,p)-homotopy Turaev-Viro invariant. The homotopy invariant defined in [1] is the $(\Gamma_C, |?|)$-homotopy turaev-viro invariant.

VI. Maximal decomposition of the Turaev-Viro invariant

Every graduation of a spherical category defines an homotopy Turaev-Viro invariant and a splitting of the Turaev-Viro invariant. We will compare these homotopy invariants. Throughout this Section, **C**will be a spherical category.

Let (**G**,p) and (**H**, q) be two graduations of **C**. A morphism of graduation f from (**G**,p) to (H, q)} is group morphism $f : \boldsymbol{G} \rightarrow \boldsymbol{H}$ such that the diagram:



commutes. Notice that in the category of graduations of **C**, where objects are the graduations of **C**and morphisms are the morphisms of graduation, the universal graduation is the unique initial object (up to isomorphism)

Lemma6.1

*Let **T**be a simplicial complex, **C**be a finitely semisimple tensor category, (**G**,p) and (**H**, q) be two graduations of **C**and $f : \boldsymbol{G} \rightarrow \boldsymbol{H}$ be a morphism of graduation. The morphism of graduation f induces the following map:*

$$\bar{F} : Col_G(T)/\mathcal{g}_T^G \rightarrow Col_H(T)/\mathcal{g}_T^H \, \mathbf{(3)}$$

$$[c] \mapsto [f \circ c]$$

Proof:

Let us show that the map **(3)**is well defined. First since $f : \boldsymbol{G} \to \boldsymbol{H}$is group morphism then for every **G**-coloring c, f_c is a **H**-coloring. Let us show that $\bar{F}$ does not depend on the choice of the representative. Let $c \in Col_G(T)$ and $\delta \in \mathcal{G}_T^G$, for every oriented 1-simplex (xy), one gets:

$$f(c^\delta)(xy) = f(\delta(x)c(xy)\delta(y)^{-1})$$
$$= f\delta(x)fc(xy)\big(f\delta(y)\big)^{-1}$$
$$= (fc)^{f\delta}(xy)$$

Thus the map$\bar{F}$ is well defined.□

The **lemma 6.1** asserts that if there is a groupmorphism between two graduation then we can relate the set of colorings (up to gauge actions) and since the homotopy Turaev-Viro invariants are state-sum invariants indexed by the set of colorings, we can relate those invariants.

Theorem 6.2

*Let **C** be a spherical category with an invertible dimension, M be a 3-manifold, Σ be the boundary of M and T_0 be a triangulation of Σ. For every graduation(**G**,p) and (**H**,q) of **C**such that there exists a morphism of graduation $f : (\boldsymbol{G}, p) \to (\boldsymbol{H}, q)$, we have:*

$$HTV_C^{(H,q)}(M, x, c_0) = \sum_{y \in \bar{F}^{-1}(x)} HTV_C^{(G,p)}(M, y, c_0)$$

*where$F : [M, BG] \to [M, BH]$ is the map induced by f (**Lemma 6.1**).*

Proof

Let us recall that for every coloring $c_0 \in Col(T_0)$and for every homotopy class $x \in [M, BH]_{\Sigma, x_0}$where $x_0 \in [\Sigma, BH]$ is the homotopy class obtained from c_0 the vector $HTV_C^{(H,q)}(M, x, c_0)$is the state sum:

$$HTV_C^{(H,q)}(M, x, c_0) = \Delta_c^{-n_0(T)+n_0(T_0)/2} \sum_{c \in Col_{(H,q),c_0,x}(T)} w_c W_c$$

□

Using **Lemma 6.1**, we have the map:

$$\bar{F} : Col_G(T)/\mathcal{G}_T^G \to Col_H(T)/\mathcal{G}_T^H$$
$$[c] \mapsto [f \circ c]$$

the map $\bar{F}$induces a map $F : [M : BG] \to [M, BH]$ (**Proposition 5.1**). It follows that for every $c \in Col_{(H,q),x,c_0}(T)$, we have : $c \in Col_{c_0}(T)$ and $\bar{F}([pc]) = [fpc] = [qc]$thus the homotopy class $y \in [M, BG]$ defined by [pc] belongs to the set $F^{-1}(x)$. We have shown that :

$$Col_{(H,q),x,c_0}(T) \subset \coprod_{y \in F^{-1}(x)} Col_{(G,p),y,c_0}(T)$$

Let us show that for every $y \in F^{-1}(x)$ and for every $c_0 \in Col(T_0)$, we have : $Col_{(G,p),y,c_0}(T) \subset Col_{(H,q),x,c_0}(T)$. Let $c \in Col_{(G,p),y,c_0}(T)$, it follows that$c \in Col_{c_0}(T)$ and $[qc] = [fpc] = \bar{F}([pc])$, since $y \in F^{-1}(x)$ one gets that the homotopy classes defined from the class [qc] is X. It follows:

$$HTV_C^{(H,q)}(M, x, c_0) = \Delta_c^{-n_0(T)+n_0(T_0)/2} \sum_{c \in Col_{(H,q),c_0,x}(T)} w_c W_c$$

$$= \Delta_c^{-n_0(T)+\frac{n_0(T_0)}{2}} \sum_{y\in F^{-1}(x)} \sum_{c\in Col_{(G,p),c_0,p}} w_c W_c$$
$$= \sum_{y\in F^{-1}(x)} HTV_C^{(G,p)}(M,y,c_0)$$

Notice that if we consider the trivial graduation we obtain the Turaev-Viro invariant.

By definition of the universal graduation and using Theorem 5.2, we can conclude that the splitting given by $HTV_C^{(\Gamma_C,|?|)}$ is maximal.

Corollary6.3

Let **C***be a spherical category with an invertible dimension, M be a 3-manifold, Σ be the boundary of M and T_0 be a triangulation of Σ. For every graduation* (**G**,p) *of***C***, one gets:*

$$TV_C(M,c_0) = \sum_{x\in[M,BG]} HTV_C^{(G,p)}(M,x,c_0) \in V_C(\Sigma,T_0,c_0)$$

with $c_0 \in Col(T_0)$, *and*

$$HTV_C^{(G,p)}(M,x,c_0) = \sum_{y\in F^{-1}(x)} HTV_C^{(\Gamma_C,|?|)}(M,y,c_0)$$

whereF is the map induced by the universal graduation $(\Gamma_C,|?|)$.

Example. Lens spaces L(p,q), with 0<q<p and (p,q)=1, are oriented compact 3-manifolds, which result from identifying on the sphere$S^3 = \{(x,y)\in\mathbb{C}^2 \mid |x|^2+|y|^2=1\}$ the points which belong to the same orbit under the action of the cyclic group $\mathbb{Z}_p$ defined by $(x,y)\mapsto(wx,w^q y)$ with$w=\exp(2i\pi/p)$.

A singular triangulation of L(p,q) is obtained by gluing together p tetrahedra (a_i,b_i,c_i,d_i), i=0,...,p-1 according to the following identification of faces (i+1 and i+q are understood modulo p):

$$(a_i b_i, c_i) = (a_{i+1} b_{i+1}, c_{i+1})\mathbf{(4)}$$
$$(a_i b_i, c_i) = \left(b_{i+q}, c_{i+q}, d_{i+q}\right)\mathbf{(5)}$$

The identification of **(4)** can be realized by embedding the p tetrahedra in Euclidean three-space, leading to a prismatic solid with p+2 0-simplices a,b,c_i, 2p external faces,3p external edges and one internal axis (a,b). Then formula **(5)** is interpreted as the identification of the surface triangles (a,c_i,c_{i+1}) and (b,c_{i+q},c_{i+1+q}). A coloring of L(p,q) is determined by the colors of the edges : (ab), $(c_i c_{i+1})$ and (bc_i) such that the triple is admissible. From now on, a coloring c of L(p,q) will be denoted by $(c(ab),c(c_i c_{i+1}),c(bc_i))$.

In [1], we have shown that for the category of representation of $U_q(\mathfrak{sl}_2)$ with q root of unity, there are two homotopy classes in $[L(p,q),B\mathbb{Z}_2]$ and we have:

$$TV_{U_q(sl_2)}\big(L(p,q)\big) = \Delta_{U_q(sl_2)}^{-2} \sum_{x=(X,Z,Y_i)} w_c W_c$$
$$= \Delta_{U_q(sl_2)}^{-2}\left(\sum_{\substack{x=(X,Z,Y_i)\\ |X|=1}} w_c W_c + \sum_{\substack{x=(X,Z,Y_i)\\ |X|=-1}} w_c W_c \right)$$
$$= HTV_0\big(L(p,q)\big) + HTV_1\big(L(p,q)\big)$$

where $HTV_0(L(p,q))$ (resp. $HTV_1(L(p,q))$) is the state sum $\Delta_{U_q(\boldsymbol{sl_2})}^{-2}\sum_{\substack{x=(X,Z,Y_i)\\|X|=1}} w_c W_c$ (resp. $\Delta_{U_q(\boldsymbol{sl_2})}^{-2}\sum_{\substack{x=(X,Z,Y_i)\\|X|=-1}} w_c W_c$) . The state sum HTV_0 is the homotopy Turaev-Viro invariant for the trivial homotopy class, and HTV_1 is the homotopy Turaev-Viro obtained for the other homotopy class.

Let us describe the decomposition of the homotopy Turaev-Viro invariant defined for a graduation (**G**,p) of $U_q(\mathfrak{sl}_2)$. Using the universal property of the graduator, one gets a morphism of graduation: $f:(G,p)\to(\mathbb{Z}_2,|?|)$. This morphism induces a map: $F:[L(p,q),B\mathbb{Z}_2]\to[L(p,q),BG]$ and **Corollary 6.3** gives the following equality:

$$HTV_{U_q(\boldsymbol{sl_2})}^{(G,p)}(M,x,c_0)=\sum_{y\in F^{-1}(x)} HTV_{U_q(\boldsymbol{sl_2})}^{(\Gamma_C,|?|)}(M,y,c_0)$$

VII. THE TURAEV-VIROHQFT

In the present Section, we recall the construction of the Turaev-Viro TQFT and we will show that for every graduation of a spherical category, we can obtain a Turaev-Viro HQFT which splits the Turaev-Viro TQFT. Furthermore we will show that the splitting obtained using the universal graduation is maximal. Throughout this Section **C**will be a spherical category.

1. The Turaev-Viro TQFT

a. Cobordisms category

Let Σ and Σ′ be two oriented closed surfaces, a cobordism from Σ to Σ' is a 3-manifold whose boundary is the disjoint union : $\overline{\Sigma}\coprod\Sigma$. Let M and M' be two cobordisms from Σ to Σ', M and M' are equivalents if there exists an isomorphism between M and M' such that it preserves the orientation and its restriction to the boundary is the identity.

The cobordism category is the category where objects are closed and oriented surfaces and morphisms are equivalent classes of cobordisms. The cobordism category is denoted by Cob_{1+2}. The disjoint union and the empty manifold ∅define a strict monoidal structure on Cob_{1+2}.

b. TQFT

A TQFT is a monoidal functor from the cobordism category to the category of finite dimensional vector spaces.

Let us recall the construction of the Turaev-Viro TQFT. Let Σ be an oriented closed surface and T be a triangulation of Σ. We associate to the pair (Σ,T) a vector space :

$$V_C(\Sigma,T)=\bigoplus_{c\in Col(T)}\bigotimes_{f\in T_0^2} V(f,c)$$

Where $V(f,c)=Hom_C(\boldsymbol{I},c(01)\otimes c(12)\otimes c(20))$ for every f=(012). The vector space V(f,c) does not depend on the choice of a numbering which respects the orientation. Since the category **C**is the semi-simple, the vector space $V_C(\Sigma,T)$ is dual to $V_C(\overline{\Sigma},T)$,the duality is induced by the trace of the category ([5], [4]and [1]).

Let Σ (resp. Σ') be an oriented surface endowed with a triangulation T (resp.T') and M be a cobordism from Σto Σ', for every colorings $c\in Col(T)$and $c'\in Col(T')$ we have the following vector :

$$TV_C(M,c,c')\in V_C(\overline{\Sigma},T,c)\otimes V_C(\Sigma',T',c')\cong V_C(\Sigma,T,c)^*\otimes V_C(\Sigma',T',c')$$

The vector spaces$V_C(\Sigma,T,c)$ and $V_C(\Sigma',T',c')$ are finite dimensional vector spaces, thus we can build the following linearmap: $\overline{TV_C}(M)_{c,c'}:V_C(\Sigma,T,c)\to V_C(\Sigma',T',c')$. It follows that the matrix $\left(\overline{TV_C}(M)_{c,c'}\right)_{c\in Col(T),c'\in Col(T')}$ defines the following linear map :

$$[M]=\left(\overline{TV_C}(M)_{c,c'}\right)_{c\in Col(T),c'\in Col(T')}:V_C(\Sigma,T)\to V_C(\Sigma',T')$$

By construction of the Turaev-Viro invariant (Theorem 1.8 [5]), we have the following relation : $[M' \cup_{\Sigma'} M] = [M'] \circ [M]$ and the map $[\Sigma \times I]: V_C(\Sigma, T) \to V_C(\Sigma, T)$is an idempotent denoted by $p_{\Sigma,T}$. We set $\mathcal{V}_C(\Sigma, T) = im(p_{\Sigma,T})$ and for every cobordism $M : \Sigma \to \Sigma'$ we denote by $\mathcal{V}_c(M) = [M]_{im(p_{\Sigma,T})}$the restriction of [M] to$im(p_{\Sigma,T})$. It follows that the vector space $\mathcal{V}_C(\Sigma, T)$is independent on the choice of the triangulation T.

From now on, we will denote by $\mathcal{V}_C$ the Turaev-Viro TQFT.

c. The Turaev-Viro HQFT

i. B-manifolds

Let B be a d-dimensional manifold, a d-dimensionalB-manifold is a pair (X,g) where X is closed d-manifold and $g : X \to B$ is a continuous map called characteristic map.

A B-cobordism from (X,g) to (Y,h) is a pair (W,F) where W is a cobordism from X to Y and f is a relative homotopy class of a map from W to B such that the restriction to X (resp. Y) is g (resp. h). From now on, we make no notational distinction between a homotopy class and any of its representatives.

Let $(W, F): (M, g) \to (N, h)$ and $(W', F'): (N', h') \to (P, k)$ be two B-cobordisms and $\Psi : N \to N'$ be a diffeomorphism such that $h'\Psi = h$. The composition of B-cobordisms is defined in the following way:$(W', F') \circ (W, F) = (W' \cup W, F.F')$, where F.F' is the following homotopy class :

$$F.F'(x) = \begin{cases} F(x) x \in W \\ F'(x) x' \in W' \end{cases}$$

Since $h'\Psi = h$ the map F.F' is well defined.

The identity of (X,g) is the B-cobordism $(X \times I, 1_g)$, with $1_g$the homotopy class of the map:

$$X \times I \to B$$
$$(x, t) \mapsto g(x)$$

The disjoint union of B-cobordisms is defined in the same way that disjoint union of cobordisms is.

The category of d+1 B-cobordisms isthe category whose objects are d-dimensional B-manifolds and morphisms are isomorphism classes of B-cobordisms. The category of d+1B-cobordism is denoted by Hcob(B,d+1), this is a strict monoidal category.

ii. HQFTs

A d+1 dimensional HQFT with target space B is a monoidal functor from the category Hcob(d+1,B) to the category of finite dimensional vector spaces.

The vector space obtained from a B-manifold only depends (up to isomorphism) on the manifold and the homotopy class of the characteristic map ([1]).

d. The construction of the Turaev-Viro HQFT

In [1], we have built the Turaev-Viro HQFT using the universal graduation. To build this HQFT we use the homotopy Turaev-Viro invariant $HTV_C^{(\Gamma_C,|?)}$. Since we have built ahomotopy Turaev-Viro invariant for every graduation (G,p) of a spherical category **C**, we will obtain in the same way a Turaev-Viro HQFT. In this case the target space will be the classifying space of the group of the graduation. Throughout this Section, (G, p) will be a graduation **C**.

From now on, for every homotopy classes $[x \in [\Sigma, \mathrm{BG}]$ and $x' \in [\Sigma', \mathrm{BG}]$ we denote by $[M, BG]_{(\Sigma,x),(\Sigma',x')}$the set of homotopy classes of [M,BG] such that the homotopy class of the restriction to Σ (resp. Σ‘) is x (resp. x').

For every oriented surface Σ endowed with a triangulation T, we have the following decomposition:

$$V_C(\Sigma, T) = \bigoplus_{c \in Col_{(G,p),x}(T)} \bigoplus_{x \in [\Sigma, BG]} V_C(\Sigma, T, c) = \bigoplus_{x \in [\Sigma, BG]} V_C(\Sigma, T, x)$$

With

$$V_C(\Sigma,T,x)=\bigoplus_{c\in Col_{(G,p),x}(T)} V_C(\Sigma,T,c)$$

Let M be a cobordism from (Σ,T) to (Σ',T'), c be a coloring of T and c' be a coloring of T'. For every homotopy class $y\in[M,BG]_{(\Sigma,x_c),(\Sigma',x_{c'})}$, the vector $HTV_C^{(G,p)}(M,y,c,c')\in V_C(\Sigma,T,c)^*\otimes V_C(\Sigma',T',c')$ induces the following linear map :

$$\overline{HTV_C^{(G,p)}}(M,y,c,c')\in V_C(\Sigma,T,c)\to V_C(\Sigma',T',c')$$

Let $x\in[\Sigma,\mathrm{BG}]$ and $x'\in[\Sigma,\mathrm{BG}]$, for every $y\in[M,BG]_{(\Sigma,x),(\Sigma',x')}$ the matrix $\left(\overline{HTV_C^{((G,p)}}(M,y,c,c')\right)_{c\in Col_x(T),c'\in Col_x(T')}$ defines a map from $V_C(\Sigma,T,x)$ to $V_C(\Sigma',T',x')$:

This map is denoted by $\widetilde{HTV_C^{(G,p)}}(M,y)_{x,x'}$.

Let Σ be a closed and oriented surface, the inclusion $\Sigma\hookrightarrow\Sigma\times I$ is a deformation retract, thus there exists a unique homotopy class $y\in[\Sigma\times I,BG]$ such that the homotopy class of the restriction to $\Sigma\times\{0\}$ is X. More precisely, y is the homotopy class of the following map :

$$\Sigma\times\mathrm{I}\to\mathrm{BG}$$
$$(\mathrm{z},\mathrm{t})\mapsto x(z)$$

and we have: $[\Sigma\times I]_{(\Sigma,x),(\Sigma',x')}=\begin{cases}1_x\,if\ x=x'\\ \emptyset\ otherwise\end{cases}$

We denoted by $p_{\Sigma,T,x}^{(G,p)}$ the idempotent $\widetilde{HTV_C^{(G,p)}}(\Sigma\times I,1_x)_{x,x'}$.

For every closed surface Σ endowed with a triangulation T, we set $\mathcal{W}_C^{(G,p)}(\Sigma,T,x)=im\left(p_{\Sigma,T,x}^{(G,p)}\right)$. Let M be a cobordism from (Σ,T) to (Σ',T'), for every $x\in[\Sigma,BG]$, $x'\in[\Sigma',BG]$ and $y\in[M,BG]_{(\Sigma,x),(\Sigma',x')}$, we denote $\mathcal{W}_C^{(G,p)}(M,y)_{x,x'}$ the restriction of $\widetilde{HTV_C^{(G,p)}}(M,y)_{x,x'}$ to the vector spaces $\mathcal{W}_C^{(G,p)}(\Sigma,T,x)$ and $\mathcal{W}_C^{(G,p)}(\Sigma',T',x')$. For every closed surface Σ and for every triangulation T and T'of Σ, the linear map $\mathcal{W}_C^{(G,p)}(\Sigma\times\mathrm{I},1_x)$: $\mathcal{W}_C^{(G,p)}(\Sigma,T,x)\to\mathcal{W}_C^{(G,p)}(\Sigma,T',x)$ is an isomorphism. Thus the space $\mathcal{W}_C^{(G,p)}(\Sigma,T,x)$ does not depend on the choice of the triangulation. Similarly to [1], where the HQFT is obtained from $HTV_C^{(\Gamma_C,|?|)}$, we have the following HQFT :

Theorem 7.1

*Let **C** be a spherical category and (**G**,p) be a graduation of **C**. Weset :*

$$\mathcal{H}_C^{(G,p)}: Hcob(BG,2+1)\to Vect_{\boldsymbol{k}}$$
$$(\Sigma,g)\mapsto\mathcal{W}_C^{(G,p)}(\Sigma,g)$$
$$(M,F)\mapsto\mathcal{W}_C^{(G,p)}(M,F)$$

Where the vector space $\mathcal{W}_C^{(G,p)}(\Sigma,g)$ is defined for the homotopy class of g. The functor $\mathcal{H}_C^{(G,p)}$ is a 2+1 dimensional HQFT with target space the classifying space BG

To obtain the decomposition of the Turaev-Viro TQFT, we will use the decomposition of the idempotent which defines the Turaev-Viro TQFT.

Lemma7.2

Let **C** be a spherical category, (G,p) be a graduation of **C.** For every surface Σ endowed with a triangulation T, we have :

$$p_{\Sigma,T} = \sum_{x\in[\Sigma,BG]} p_{\Sigma,T,x}^{(G,p)}$$

Proof

For every 3-manifold with boundary Σ, for every triangulation T of Σ and for every coloring $c \in Col(T)$ we have :

$$TV_C(M,c) = \sum_{x\in[M,BG]_{\Sigma,x_c}} HTV_C^{(G,p)}(M,x,c)$$

If $M = \Sigma \times I$, then we have $[\Sigma \times I]_{(\Sigma,x),(\Sigma',x')} = \begin{cases} 1_x if\ x = x' \\ \emptyset\ otherwise \end{cases}$. It follows that $c, c' \in Col_x^{(G,p)}(T)$ then $TV_C(\Sigma \times I, c, c') = HTV_C^{(G,p)}(\Sigma \times I, 1_x, c, c')$ and if $c \in Col_x^{(G,p)}(T)$ and $c' \in Col_{x'}^{(G,p)}(T)$ with $x \neq x'$ then $TV_C(\Sigma \times I, c, c') = 0$. One gets

$$p_{\Sigma,T} = \sum_{x\in[\Sigma,BG]} p_{\Sigma,T,x}^{(G,p)}$$

□

Using Lemma 7.2 and Theorem 7.1, one gets that every graduation of a spherical category gives a decomposition of the Turaev-Viro TQFT in terms of HQFT, whose target space is given by the classifying space ofthe graduation.

Theorem 7.3

*Let **C**be a spherical category with an invertible dimension and (G,p) be a graduation of **C**. The Turaev-Viro TQFT $\mathcal{V}_C$ is obtained from the HQFT $\mathcal{H}_C^{(G,p)}$:*

$$\mathcal{V}_C(\Sigma) = \bigoplus \mathcal{H}_C^{(G,p)}(\Sigma, x)_{x\in[\Sigma,BG]}$$

For every cobordism$M$: $\Sigma_0 \to \Sigma_1$ and for every $x_0 \in [\Sigma_0, BG]$, $x_1 \in [\Sigma_1, BG]$, we denote by $\mathcal{V}_C(M)_{x_0,x_1}$ the following restriction of the map $\mathcal{V}_C(M)$:

$$\begin{array}{ccc} \mathcal{V}_C(\Sigma_0) & \xrightarrow{\mathcal{V}_C(M)} & \mathcal{V}_C(\Sigma_1) \\ \uparrow & & \uparrow \\ \mathcal{V}_C(\Sigma_0, x_0) & \xrightarrow[\mathcal{V}_C(M)_{x_0,x_1}]{} & \mathcal{V}_C(\Sigma_1, x_1) \end{array}.$$

We have the following splitting:

$$\mathcal{V}_C(M)_{x_0,x_1} = \sum_{y\in[M,BG]_{(\Sigma_0,x_0),(\Sigma_1,x_1)}} \mathcal{H}_C(M,y)_{x_0,x_1}$$

VIII. Maximal decomposition of the Turaev-Viro TQFT

In this Section we will compare the different decompositions of the Turaev-Viro TQFT. The decomposition obtained from the universal graduation will be the maximal decomposition.

Let **C**be a spherical category, (G,p) be a graduation on **C**, $f : \Gamma_C \to G$ the group morphism obtained form the universal property of the graduator (**Proposition 3.1**) and Σ be a closed and oriented surface endowed with a triangulation **T**. For every homotopy class $x \in [M, BG]$ the vector space $\mathcal{V}_C^{(G,p)}(\Sigma)$ is the image of the idempotent $p_{\Sigma,T,x}^{(G,p)}$ and this idempotent is obtained from the vector $HTV_C^{(G,p)}(\Sigma \times I, 1_x)$. Using **Theorem 6.2**, we have the following decomposition of thevector $HTV_C^{(G,p)}(\Sigma \times I, 1_x)$, for every $x \in [\Sigma, BG]$ and for every $c_0 \in Col_x^{(G,p)}(T_0)$we have:

$$HTV_C^{(G,p)}(\Sigma \times I, c_0, 1_x) = \sum_{y \in F^{-1}(x)} HTV_C^{(\Gamma_C, |?|)}(\Sigma \times I, c_0, y)$$

where$F : [\Sigma \times I, B\Gamma_C] \to [\Sigma \times I, BG]$ is the map induced by f (**Lemma 6.1**). We have shown that

$$Col_x^{(G,p)}(T_0) = \coprod_{y \in F^{-1}(x)} Col_y^{\Gamma_C}(T_0)$$

, furthermore we have: $[M, B\Gamma_C]_{(\Sigma,x),(\Sigma,x')} = \begin{cases} \emptyset\ x \neq x' \\ 1_x x = x' \end{cases}$

It follows that :

$$HTV_C^{(G,p)}(\Sigma \times I, c_{0,1_x}) = \sum_{y \in F_\Sigma^{-1}(x)} HTV_C^{\Gamma_C}(\Sigma, c_{0,1_y}).$$

where F_Σ is the restriction of F to Σ. Let us take the image of the induced idempotent, one gets:

$$\mathcal{V}_C(\Sigma, x) = \sum_{y \in F_\Sigma^{-1}(x)} \mathcal{V}_C^{(\Gamma_C, |?|)}(\Sigma, y)$$

We obtain in the same way a decomposition of linear map defined by the HQFT. It follows:

Theorem8.1

*Let **C**be a spherical category, (**G**,p) be a graduation of **C**. The Turaev-Viro HQFT obtained from the graduation (**G**,p) is decomposed in the following way:*

$$\mathcal{V}_C(\Sigma, x) = \sum_{y \in F_\Sigma^{-1}(x)} \mathcal{V}_C^{(\Gamma_C, |?|)}(\Sigma, y)$$

*for every closed surface Σ, and for every $[x \in [\Sigma, BG]$, the map $F : [\Sigma, B\Gamma_C] \to [\Sigma, BG]$is the map obtained from the universal graduation (**Lemma 6.1**).*

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