N-fractional calculus and multivariable Aleph function and generalized multivariable polynomials

$F.Y.\ AYANT^1$

1 Teacher in High School , France

Abstract
In this paper, we obtain Nishimoto's N-fractional differintegral of the multivariable Aleph-function and class of multivariable polynomials whose arguments involves the product of two power functions $(z-a)^{-\lambda}$ and $(z-b)^{\mu}$, $(\lambda,\mu>0)$. On account of the general nature of our result, N-fractional differintegral of a large variety of special functions of one and several variables having general arguments follows as special cases of our findings. At the end, we shall see several corollaries.

Keywords: General class of multivariable polynomials, fractional calculus, Mellin-barnes integrals contour, multivariable Aleph-function, Aleph-function.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Introduction and Preliminaries.

Recently, Saxena and Nishimoto [16], Mishra and Purohit [10], Garg and Mishra [3], Gupta et al. [6], Jaimini and Nishimoto [7] have studied the N-fractional calculus and multivariable H-function or generalized Lauricella function with general arguments. In our paper, we evaluate the N-fractional calculus concerning a class of multivariable polynomials defined by Srivastava [22] and the multivariable Aleph-function with general arguments.

Following Nishimoto [11], we define the N-fractional diffeintegral of a function of one variable in the following form :

Let $D=\left\{\underset{-}{D},\underset{+}{D}\right\}, C=\left\{\underset{-}{C},\underset{+}{C}\right\}$

$\underset{-}{C}$ be a curve along the cut joining two points z and $-\infty+\omega Im(z)$,

$\underset{+}{C}$ be a curve along the cut joining two points z and $\infty+\omega Im(z)$,

$\underset{-}{D}$ be a domain surrounded by $\underset{-}{C}$,

$\underset{+}{D}$ be a domain surrounded by $\underset{+}{C}$.

Further, let $f=f(z)$ be an analytic function of one variable in a domain D where D is surrounded by C then the fractional differintegral of an arbitrary order υ for $z(\upsilon\in\mathbb{R}_*^+, z\in\mathbb{C})$ of the function $f(z)$, if$|(f)_\upsilon|$ exists, is defined by

$$f_\upsilon=f_\upsilon(z)={}_C(f)_\upsilon=\frac{\Gamma(\upsilon+1)}{2\pi\omega}\int_C\frac{f(\zeta)}{(\zeta-z)^{\upsilon+1}}\mathrm{d}\zeta \tag{1.1}$$

$$(f)_{-m}=\lim_{\upsilon\to -m}f_\upsilon(m\in\mathbb{Z}^+)$$

where

$$-\pi\leqslant arg(\zeta-z)\leqslant\pi \text{ for } C=\underset{-}{C}\quad 0\leqslant arg(\zeta-z)\leqslant 2\pi \text{ for } C=\underset{+}{C}, \zeta\neq z.$$

The generalized multivariable polynomials defined by Srivastava [22], is given in the following manner :

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v]=\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \tag{1.2}$$

where $\mathfrak{M}_1, \cdots, \mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots ; N_v, K_v]$ are arbitrary constants real or complex. On suitably specializing the coefficients $A[N_1, K_1; \cdots ; N_v, K_v]$, $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1, \cdots , y_v]$ yields a number of known polynomials, the Laguerre polynomials, the Jacobi polynomials, and several other ([25], page. 158-161]. We shall note.

$$a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1, K_1; \cdots ; N_v, K_v] \tag{1.3}$$

The Aleph-function of several variables is an extension the multivariable I-function defined by Sharma and Ahmad [19] , itself is a generalisation of G and H-functions of several variables defined by Srivastava et Panda [23,24]. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function of $r-$ variables throughout our present study and will be defined and represented as follows (see Ayant [1]).

We have : $\aleph(z_1, \cdots , z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} [(a_j; \alpha_j^{(1)}, \cdots , \alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot \\ \cdots\cdots\cdots\cdots , \end{array} \right.$

$$[\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots , \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] : [(c_j^{(1)}), \gamma_j^{(1)})_{1,n_1}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}]; \cdots ;$$
$$[\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots , \beta_{ji}^{(r)})_{1,q_i}] : [(d_j^{(1)}), \delta_j^{(1)})_{1,m_1}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}]; \cdots ;$$

$$\left. \begin{array}{c} [(c_j^{(r)}), \gamma_j^{(r)})_{1,n_r}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \cdot \\ [(d_j^{(r)}), \delta_j^{(r)})_{1,m_r}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array} \right) = \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots , s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.4}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, \cdots , s_r) = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} [\tau_i \prod_{j=\mathfrak{n}+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k)]} \tag{1.5}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]} \tag{1.6}$$

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0,$$

with $k = 1, \cdots , r, i = 1, \cdots , R \; , i^{(k)} = 1, \cdots , R^{(k)}$ (1.7)

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots , z_r) = 0(|z_1|^{\alpha_1}, \cdots , |z_r|^{\alpha_r}) , max(|z_1|, \cdots , |z_r|) \rightarrow 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\to\infty$$

where $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k$$

For convenience, we shall use the following notations in this paper.

$$V=m_1,n_1;\cdots;m_r,n_r \tag{1.8}$$

$$W=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.9}$$

$$A=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}\}:\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},$$

$$\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$B=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\},\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}\;;\cdots;$$

$$\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

3. Main result.

We have the following result.

Theorem.

$$\left[(z-a)^{\rho}(z-b)^{\sigma}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(y_1(z-a)^{-a_1}(z-b)^{b_1},\cdots,y_v(z-a)^{-a_v}(z-b)^{b_v}\right)\aleph\left(z_1(z-a)^{-\lambda_1}(z-b)^{\mu_1},\cdots,z_r(z-a)^{-\lambda_r}(z-b)^{\mu_r}\right)\right]_{\upsilon}$$

$$=e^{-\omega\pi\upsilon}(z-b)^{\rho+\sigma-\upsilon}\sum_{k=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\prod_{j=1}^{v}y_j^{K_j}\left(\frac{a-b}{z-b}\right)^k\frac{1}{k!}(z-b)^{\sum_{i=1}^{v}K_i(b_i-a_i)}$$

$$\aleph_{p_i+2,q_i+2,\tau_i;R:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c}\mathrm{z}_1(z-b)^{\mu_1-\lambda_1} & \mathrm{A}_1,A_2,A\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r(z-b)^{\mu_r-\lambda_r} & \mathrm{B}_1,B_2,B\end{array}\right) \tag{2.1}$$

where

$$A_1=(1-\upsilon+\rho+\sigma-k-\sum_{i=1}^{v}K_i(a_i-b_i);\lambda_1-\mu_1,\cdots,\lambda_r-\mu_r) \tag{2.2}$$

$$A_2=(1+\rho-k-\sum_{i=1}^{v}K_ia_i;\lambda_1,\cdots,\lambda_r) \tag{2.3}$$

$$B_1=(1+\rho+\sigma-k-\sum_{i=1}^{v}K_i(a_i-b_i);\lambda_1-\mu_1,\cdots,\lambda_r-\mu_r) \tag{2.4}$$

$$B_2=(1+\rho-\sum_{i=1}^{v}K_ia_i;\lambda_1,\cdots,\lambda_r) \tag{2.5}$$

Provided that

$z \neq a, b \left|\frac{a-b}{z-b}\right| < 1, a_i, b_i > 0$ for $i = 1, \cdots, v$; $\lambda_j, \mu_j > 0, \mu_j \leqslant \lambda_j$ for $j = 1, \cdots, r$.

$$Re\left(\sigma + \eta - \sum_{i=1}^{v} K_i(a_i - b_i)\right) - \sum_{j=1}^{r}(\lambda_j - \mu_j) \max_{1 \leqslant l \leqslant n_j} Re\left(\frac{c_l^{(j)} - 1}{\gamma_l^{(j)}}\right) < v < 0.$$

$\left|z_i(z-a)^{-\lambda_i}(z-b)^{\mu_i}\right| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.7) and the serie in the left-hand side of (3.1) is absolutely and uniformely convergent.

Proof

To establish (2.1), we first apply the definition of differintegral given by (1.1) in the left-hand side of equation (2.1), express the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ in finite series form with the help of (1.3) and the multivariable Aleph-function in terms of its equivalent multiple Mellin-Barnes integrals contour with the help of (1.4), interchanging the order of summations and $(s_1, \cdots, s_r)$-integrals (which is permissible under the stated conditions), we obtain

$$\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v \prod_{j=1}^{v} y_j^{K_j} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \left[(z-a)^{\rho - \sum_{i=1}^{v} K_i a_i - \sum_{j=1}^{r} s_j \lambda_j}\right.$$

$$\left.(z-b)^{\sigma + \sum_{i=1}^{v} K_i b_i + \sum_{j=1}^{r} s_j \mu_j}\right]_v \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{2.6}$$

Next we collect the terms involving the powers of $(z-a)$ and $(z-b)$ and write them in terms of the powers of $(z-b)$ as follows, we obtain

$$(z-a)^{\rho - \sum_{i=1}^{v} K_i a_i - \sum_{j=1}^{r} s_j \lambda_j}(z-b)^{\sigma + \sum_{i=1}^{v} K_i b_i + \sum_{j=1}^{r} s_j \mu_j} = \sum_{k=0}^{\infty} \frac{\left(-\rho + \sum_{i=1}^{v} K_i a_i + \sum_{j=1}^{r} s_j \lambda_j\right)_k (a-b)^k}{k!}$$

$$(z-b)^{\sigma + \rho + \sum_{i=1}^{v} K_i(b_i - a_i) + \sum_{j=1}^{r} s_j(\mu_j - \lambda_j) - k} \tag{2.7}$$

Provided that : $z \neq a, b \left|\frac{a-b}{z-b}\right| < 1.$

We interchange $(s_1, \cdots, s_r)$ and ζ-integrals and we evaluate the ζ-integral, which is N-fractional differintegral of order v of power function $(z-b)^{\sigma + \rho + \sum_{i=1}^{v} K_i(b_i - a_i) + \sum_{j=1}^{r} s_j(\mu_j - \lambda_j) - k}$, we follows the method given by Nishimoto [11, vol. (5), p. 1, chap 1] and we obtain the left hand side of equation (2.1) as follows.

$$\frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \sum_{k=0}^{\infty} \frac{\left(-\rho + \sum_{i=1}^{v} K_i a_i + \sum_{j=1}^{r} s_j \lambda_j\right)_k (a-b)^k}{k!} e^{-\omega\pi v}$$

$$\frac{\Gamma\left(v - \rho - \sigma + \sum_{i=1}^{v}(a_i - b_i)K_i + \sum_{j=1}^{r}(\lambda_j - \mu_j)s_j + k\right)}{\Gamma\left(-\rho - \sigma + \sum_{i=1}^{v}(a_i - b_i)K_i + \sum_{j=1}^{r}(\lambda_j - \mu_j)s_j + k\right)} (z-b)^{\sigma + \rho + \sum_{i=1}^{v} K_i(b_i - a_i) + \sum_{j=1}^{r} s_j(\mu_j - \lambda_j) - v - k}$$

$$\mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{2.8}$$

Provided

$\lambda_j, \mu_j > 0, \mu_j \leqslant \lambda_j$ for $j = 1, \cdots, r$.

$$Re\left(\sigma + \eta - \sum_{i=1}^{v} K_i(a_i - b_i)\right) - \sum_{j=1}^{r}(\lambda_j - \mu_j) \max_{1 \leqslant l \leqslant n_j} Re\left(\frac{c_l^{(j)} - 1}{\gamma_l^{(j)}}\right) < v < 0.$$

Interchanging the k-serie and $(s_1,\cdots,s_r)$-integrals (which is permissible under the stated conditions), now interpreting the multiple Mellin-Barnes integrals contour in multivariable Aleph-function, we obtain the desired result (2.1) after algebric manipulations.

Particular case.

If in (2.1), we take $a=b,\sigma=0,b_1=\cdots=b_v=\mu_1=\cdots=\mu_r=0$, we get the following result after simplifications.

$$\left[(z-a)^{\rho}S_{N_1,\cdots,N_v}^{\mathfrak{M}_2,\cdots,\mathfrak{M}_v}\left(y_1(z-a)^{-a_1},\cdots,y_v(z-a)^{-a_v}\right)\aleph\left(z_1(z-a)^{-\lambda_1},\cdots,z_r(z-a)^{-\lambda_r}\right)\right]_{\upsilon}$$

$$=e^{-\omega\pi\upsilon}(z-a)^{\rho-\upsilon}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\prod_{j=1}^{v}y_j^{K_j}\,(z-a)^{-\sum_{i=1}^{v}K_ia_i}$$

$$\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}+1:V}\left(\begin{array}{c|c}\mathrm{z}_1(z-b)^{-\lambda_1} & (1\text{-}\upsilon+\rho-\sum_{i=1}^{v}K_ia_i;\lambda_1,\cdots,\lambda_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r(z-b)^{-\lambda_r} & (1+\rho-\sum_{i=1}^{v}K_ia_i;\lambda_1,\cdots,\lambda_r),B\end{array}\right) \tag{2.9}$$

under the same conditions that (2.1) with $a=b,\sigma=0,b_1=\cdots=b_v=\mu_1=\cdots=\mu_r=0.$

3. Corollaries.

If the multivariable I-function and class of multivariable polynomials reduce respectively to Aleph-function defined by Sudland [26,27] and class of polynomials of one variable defined by Srivastava [21], we obtain

Corollary 1.

$$\left[(z-a)^{\rho}(z-b)^{\sigma}S_N^M\left(y_1(z-a)^{-a_1}(z-b)^{b_1}\right)\aleph\left(z_1(z-a)^{-\lambda_1}(z-b)^{\mu_1}\right)\right]_{\upsilon}=e^{-\omega\pi\upsilon}(z-b)^{\rho+\sigma-\upsilon}$$

$$\sum_{k=0}^{\infty}\sum_{K=0}^{[N/M]}A_{N,K}y_1^K\left(\frac{a-b}{z-b}\right)^k\frac{(z-b)^{b_1-a_1}}{k!}\aleph_{p_{i^{(1)}}+2,q_{i^{(1)}}+2,\tau_{i^{(1)}};R}^{m_1,n_1+2}\left(\begin{array}{c|c}\mathrm{z}_1(z-b)^{\mu_1-\lambda_1} & \begin{array}{c}\mathrm{A}_1',A_2',A'\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{B}_1',B_2',B'\end{array}\end{array}\right) \tag{3.1}$$

where

$$A_1'=(1-\upsilon+\rho+\sigma-k-K(a_1-b_1);\lambda_1-\mu_1)\ ;\ A_2'=(1+\rho-k-Ka_1;\lambda_1) \tag{3.2}$$

$$B_1'=(1+\rho+\sigma-k-K(a_1-b_1);\lambda_1-\mu_1)\ ;B_2'=(1+\rho-Ka_1;\lambda_1) \tag{3.3}$$

$$A'=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}\ ;\ B'=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\} \tag{3.4}$$

$$A_{N,K}=\frac{(-N)_{MK}}{K!}A[N,K] \tag{3.5}$$

Provided that

$$z\neq a,b\left|\frac{a-b}{z-b}\right|<1,a_1,b_1>0\,;\lambda_1,\mu_1>0,\mu_1\leqslant\lambda_1.$$

$$Re\left(\sigma+\eta-K(a_1-b_1)\right)-(\lambda_1-\mu_1)\max_{1\leqslant l\leqslant n_1} Re\left(\frac{c_l^{(1)}-1}{\gamma_l^{(1)}}\right)<\upsilon<0.$$

$$\left|z_1(z-a)^{-\lambda_1}(z-b)^{\mu_1}\right|<\frac{1}{2}A_1^{(k)}\pi,\ \text{ where } A_1^{(k)}=\sum_{j=1}^{n_1}\gamma_j^{(1)}-\tau_{i^{(1)}}\sum_{j=n_1+1}^{p_{i^{(1)}}}\gamma_{ji^{(1)}}^{(1)}+\sum_{j=1}^{m_1}\delta_j^{(1)}-\tau_{i^{(1)}}\sum_{j=m_1+1}^{q_{i^{(1)}}}\delta_{ji^{(1)}}^{(1)}>0$$

and the serie in the left-hand side of (4.1) is absolutely and uniformely convergent.

Consider the above corollary, the Aleph-function reduces to I-function defined by Saxena [17], we get

Corollary 2.

$$\left[(z-a)^{\rho}(z-b)^{\sigma}S_N^M\left(y_1(z-a)^{-a_1}(z-b)^{b_1}\right)I\left(z_1(z-a)^{-\lambda_1}(z-b)^{\mu_1}\right)\right]_{\upsilon}=e^{-\omega\pi\upsilon}(z-b)^{\rho+\sigma-\upsilon}$$

$$\sum_{k=0}^{\infty}\sum_{K=0}^{[N/M]}A_{N,K}y_1^K\left(\frac{a-b}{z-b}\right)^k\frac{(z-b)^{b_1-a_1}}{k!}I^{m_1,n_1+2}_{p_{i^{(1)}}+2,q_{i^{(1)}}+2;R}\left(z_1(z-b)^{\mu_1-\lambda_1}\ \middle|\ \begin{matrix}\mathrm{A}_1',A_2',A''\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{B}_1',B_2',B''\end{matrix}\right) \tag{3.6}$$

where

$$A_1'=(1-\upsilon+\rho+\sigma-k-K(a_1-b_1);\lambda_1-\mu_1)\ ;\ A_2'=(1+\rho-k-Ka_1;\lambda_1) \tag{3.7}$$

$$B_1'=(1+\rho+\sigma-k-K(a_1-b_1);\lambda_1-\mu_1)\ ;B_2'=(1+\rho-Ka_1;\lambda_1) \tag{3.8}$$

$$A''=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\{(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}\ ;\ B''=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\} \tag{3.9}$$

under the same conditions that (3.1) and

$$\left|z_1(z-a)^{-\lambda_1}(z-b)^{\mu_1}\right|<\frac{1}{2}B_1^{(k)}\pi,\ \text{ where } B_1^{(k)}=\sum_{j=1}^{n_1}\gamma_j^{(1)}-\sum_{j=n_1+1}^{p_{i^{(1)}}}\gamma_{ji^{(1)}}^{(1)}+\sum_{j=1}^{m_1}\delta_j^{(1)}-\sum_{j=m_1+1}^{q_{i^{(1)}}}\delta_{ji^{(1)}}^{(1)}>0$$

Consider the above corollary, the I-function reduces to H-function defined by Fox ([2],[9]), we have :

Corollary 3.

$$\left[(z-a)^{\rho}(z-b)^{\sigma}S_N^M\left(y_1(z-a)^{-a_1}(z-b)^{b_1}\right)H\left(z_1(z-a)^{-\lambda_1}(z-b)^{\mu_1}\right)\right]_{\upsilon}=e^{-\omega\pi\upsilon}(z-b)^{\rho+\sigma-\upsilon}$$

$$\sum_{k=0}^{\infty}\sum_{K=0}^{[N/M]}A_{N,K}y_1^K\left(\frac{a-b}{z-b}\right)^k\frac{(z-b)^{b_1-a_1}}{k!}H^{m_1,n_1+2}_{p_1+2,q_1+2}\left(z_1(z-b)^{\mu_1-\lambda_1}\ \middle|\ \begin{matrix}\mathrm{A}_1',A_2',C\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{B}_1',B_2',D\end{matrix}\right) \tag{3.10}$$

$$A_1'=(1-\upsilon+\rho+\sigma-k-K(a_1-b_1);\lambda_1-\mu_1)\ ;\ A_2'=(1+\rho-k-Ka_1;\lambda_1) \tag{3.11}$$

$$B_1'=(1+\rho+\sigma-k-K(a_1-b_1);\lambda_1-\mu_1)\ ;B_2'=(1+\rho-Ka_1;\lambda_1) \tag{3.12}$$

$$C=(c_j;\gamma_j)_{1,p_1}\ ;\ D=(d_j;\delta_j)_{1,q_1} \tag{3.13}$$

under the same conditions that (3.1) and

$$\left|z_1(z-a)^{-\lambda_1}(z-b)^{\mu_1}\right| < B_1^{(k)} = \sum_{j=1}^{n_1}\gamma_j - \sum_{j=n_1+1}^{p_1}\gamma_j + \sum_{j=1}^{m_1}\delta_j - \sum_{j=m_1+1}^{q_1}\delta_j > 0$$

Consider the above corollary , by applying our result given in (4.4) to the case the Laguerre polynomials ([28], page 101, eq.(15.1.6)) and ([25], page 159) and by setting

$$S_N^1(x) \rightarrow L_N^\alpha(x)$$

In which case $M = 1, A_{N,K} = \binom{N+\alpha}{N}\frac{1}{(\alpha+1)_K}$ we have the following interesting consequencies of the main result.

Corollary 4.

$$\left[(z-a)^\rho(z-b)^\sigma L_N^\alpha\left(y_1(z-a)^{-a_1}(z-b)^{b_1}\right)H\left(z_1(z-a)^{-\lambda_1}(z-b)^{\mu_1}\right)\right]_\upsilon = e^{-\omega\pi\upsilon}(z-b)^{\rho+\sigma-\upsilon}$$

$$\sum_{k=0}^\infty\sum_{K=0}^N A_{N,K}y_1^K\left(\frac{a-b}{z-b}\right)^k\frac{(z-b)^{b_1-a_1}}{k!}y_1^K\left(\frac{a-b}{z-b}\right)^k\frac{1}{k!}H_{p_1+2,q_1+2}^{m_1,n_1+2}\left(\begin{array}{c|c} z_1(z-b)^{\mu_1-\lambda_1} & \begin{array}{c}\mathrm{A}_1', A_2', C\\ \cdot\\ \cdot\\ \mathrm{B}_1', B_2', D\end{array}\end{array}\right) \tag{3.14}$$

under the same conditions that (3.10).

If we take $a = b, \sigma = 0, b_1 = \mu_1 = 0$ in (3.14), we have the following result.

Particular case.

$$\left[(z-a)^\rho L_N^\alpha\left(y_1(z-a)^{-a_1}\right)H\left(z_1(z-a)^{-\lambda_1}\right)\right]_\upsilon = e^{-\omega\pi\upsilon}(z-a)^{\rho-\upsilon}$$

$$\sum_{k=0}^\infty\sum_{K=0}^N \frac{(-N)_K}{K!}\binom{N+\alpha}{N}\frac{1}{(\alpha+1)_K}y_1^K\left(\frac{a-b}{z-b}\right)^k\frac{(z-b)^{-a_1}}{k!}H_{p_1+2,q_1+2}^{m_1,n_1+2}\left(\begin{array}{c|c} z_1(z-b)^{-\lambda_1} & \begin{array}{c}\mathrm{A}_1'', A_2'', C\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{B}_1'', B_2'', D\end{array}\end{array}\right) \tag{3.15}$$

where

$$A_1'' = (1-\upsilon+\rho+\sigma-k-Ka_1;\lambda_1)\ ;\ A_2'' = (1+\rho-k-Ka_1;\lambda_1) \tag{3.16}$$

$$B_1'' = (1+\rho+\sigma-k-Ka_1;\lambda_1)\ ;\ B_2'' = (1+\rho-Ka_1;\lambda_1) \tag{3.17}$$

under the same conditions that (3.10) with $a = b, \sigma = 0, b_1 = \mu_1 = 0$.

Remarks :
By the similar methods, we obtain the analog relations with the I-function of several variables ([12],[14],[19]), Aleph-function of two variables [18], the I-function of two variables [8,20], the I-function of one variable [15], the multivariable A-function [5], the A-function [4] and the modified multivariable H-function [13].

4. Conclusion.

Finally, it is interesting to observe that due to fairly general character of the multivariable Aleph-function and class of multivariable polynomials, numerous interesting special cases of the main result (2.1) associated with potentially useful a variety special functions of one and several variables, orthogonal polynomials, multivariable H-function, H-function, G-function and Generalized Lauricella functions etc.

References

[1] F.Y. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT). 31 (3) (2016),142-154.

[2] C. Fox, The G and H-functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98 (1961), 395-429.

[3] M. Garg and R. Mishra, On r-dimensional N-fractional differintegral of multivariable H-function, J. Indian Acad. Math, 28 (2006), 189-198.

[4] B.P. Gautam and A.S. Asgar, The A-function. Revista Mathematica. Tucuman (1980).

[5] B.P. Gautam and A.S. Asgar, On the multivariable A-function. Vijnana Parishas Anusandhan Patrika,29(4) (1986), 67-81.

[6] K.C. Gupta, S.P. Goyal and R.Garg, N-fractional differintegral of multivariable H-function, Ganita Sandesh, 16 (2002), 5-12.

[7] B.B. Jaimini and K. Nishimoto, N-fractional calculus of generalized Lauricella function of several variables, J. Frac. Calc. 29 (2006), 65-74.

[8] K.S. Kumari, T.M. Vasudevan Nambisan and A.K. Rathie, A study of I-function of two variables, Le Matematiche, 69(1) (2014), 285-305.

[9] A.M. Mathai and R.K. Saxena, The H-function with applications in statistics and other disciplines, John Wiley and Sonc Inc, New York, 1978.

[10] R. Mishra and M. Purohit, N-fractional calculus of H-function of several variables, J. Rajasthan Acad. Phy. Sci. 3(3)
(2004), 191-196.

[11] K. Nishimoto, Fractional calculus, Descartes Press, Koriyama, Japan (1984), Vol I ; (1987), Vol II ; (1989), Vol III ; (1991), Vol IV and (1996), Vol V.

[12] Y.N. Prasad, Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , 231-237.

[13] Y.N. Prasad and A.K.Singh, Basic properties of the transform involving and H-function of r-variables as kernel, Indian Acad Math, (2) (1982), 109-115.

[14] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journalof Engineering Mathematics Vol (2014), 1-12.

[15] A.K. Rathie, A new generalization of generalized hypergeometric functions, LeMatematiche,52(2) (1997), 297 - 310.

[16] R.K. Saxena and K. Nishimoto, *N*-fractional calculus of the multivariable *H*-functions, J. Frac. Calc, 31 (2007), 43-52.

[17] V.P. Saxena, Formal solution of certain new pair of dual integral equations involving H-function, Proc. Nat. Acad.Sci. IndiaSect., (2001), A51, 366–375.

[18] K. Sharma, On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences 3(1) (2014), 1-13.

[19] C.K. Sharma and S.S. Ahmad, On the multivariable I-function. Acta ciencia Indica Math , 20(2) (1994), 113-116.

[20] C.K. Sharma and P.L. Mishra, On the I-function of two variables and its properties. Acta Ciencia Indica Math,17 (1991), 667-672.

[21] H.M. Srivastava, A contour integral involving Fox's H-function. Indian. J. Math, (14) (1972), 1-6.

[22] H.M. Srivastava, A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by

Laguerre polynomial, Pacific. J. Math. 177(1985), 183-191.

[23] H.M.Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[24] H.M.Srivastava and R.Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.

[25] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials. Rend. Circ. Mat. Palermo. Vol 32 (No 2) (1983), 157-187.

[26] N. Südland, B. Baumann and T.F. Nonnenmacher, Open problem : who knows about the Aleph-functions? Fract. Calc. Appl. Anal., 1(4) (1998), 401-402.

[27] N.Sudland, B. Baumann and T.F. Nannenmacher, Fractional drift-less Fokker-Planck equation with power law diffusion coefficients, in V.G. Gangha, E.W. Mayr, W.G. Vorozhtsov (Eds.), Computer Algebra in Scientific Computing (CASC Konstanz 2001), Springer, Berlin, 2001, 513–525.

[28] C. Szego, (1975), Orthogonal polynomials. Amer. Math. Soc. Colloq. Publ. 23 fourth edition. Amer. Math. Soc. Providence. Rhodes Island, 1975.