Soft Sets and Sigma Algebras

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Abstract-*This paper studies the properties of inverse image of a real valued function defined on the set of all soft sets S(U). It is proved that if A is a sigma algebra of subsets of R, then its inverse image* $f^{-1}(A)$ *is also sigma algebra and smallest sigma algebra generated by a set of soft sets is defined.*

Key Words- *Soft set, algebra, sigma algebra*

I. INTRODUCTION

Soft set is a parameterized family of subsets of a universal set U.Because of its simple structure, it has been used by many researchers to make decisions in problems with uncertainty. Soft set was first introduced by Modolstov [1] and he applied this concept in economics. Basing on the definition of soft set given by Modolstov, P.K Maji [4] defined many set theoretic operations between soft sets which established a pavement for the development of soft set theory.

Algebras of sets are closed under finite unions. Sigma algebras of sets are closed under countable unions. Researchers in finance, economics and statistics use sigma algebras to model the problems arising in their fields. In view of this, we intend to introduce the sigma algebra of soft sets. John H Halton [3] discussed about sigma algebras whose elements are inverse images of a real valued function. The method introduced by John H Halton motivated us to consider sigma algebras of classes of soft sets. This is obtained by defining a mapping from the set of all soft sets to the real number set R. By considering subsets $A_1, A_2, A_3,\ldots\ldots A_n,\ldots\ldots$ of R, we analyze the properties of the set $f^{-1}(A_n)$, $n \in N$ which are obviously set of soft sets. Next we study the properties of the class $\{ f^{-1}(A_n)\}$, showing that it is closed under arbitrary unions and proved that it is a sigma algebra whenever $\{A_n\}$ is a sigma algebra.Finally we find sigma algebra of class of soft sets generated by a set of soft sets.

This paper is structured as follows. In section 2 we review the basic literature on soft set and sigma algebras. Section 3 presents the definition and properties of sigma algebra of soft sets.

II. PRELIMINARIES

Definition 2.1: Let U be the initial universal set and E be the set of all parameters defined over U i.e. elements of E are characteristics or properties of elements of U.

A pair (F, E) is called a soft set if F: $E \rightarrow P(U)$ is a mapping.

Definition 2.2: Let (F,A) and (G,B) be two soft sets defined over the same universal set U. Then (F,A) is said to be soft subset of (G,B) if

(i) $A \subseteq B$,

(ii) For all $e \in A$, $F(e) \subseteq G(e)$

Definition 2.3: Let (F,A) and (G,B) be two soft sets over the common Universe U. The union of two soft sets (F,A) and (G,B) is a soft set (H,C) where $C = A \cup B$ and H is defined as

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cap G(e) & if\ e \in A \cap B \end{cases}$$

Definition 2.4: Let (F,A) and (G,B) be two soft sets over the common Universe U such that $A \cap B \neq \emptyset$. The intersection of (F,A) and (G,B) is the soft set (H,C) where $C = A \cap B$ and

$H(e) = F(e) \cap G(e), \forall e \in C$.

Definition 2.5: Let (F,A) be a soft set over U. Then the compliment of (F,A) is defined by $F^c = U - F(e), \forall e \in A$ and is denoted by (F^c , A)

Definition 2.6: A collection $\aleph$ of subsets of X is called an algebra of sets if (i) $A \cup B$ is in $\aleph$ whenever A and B are (ii) A^c is in $\aleph$ whenever A is.

Definition 2.7: A collection $\Im$ of subsets of X is called a sigma algebra of sets if (i) X is in $\Im$ (ii) A^c is in $\Im$ whenever A is (iii) countable union of subsets in $\Im$ is also in $\Im$.

II.SIGMA ALGEBRAS OF SOFT SETS

Let S(U) be the set of all soft sets defined over U with the parameter set E. Let $f : S(U)\rightarrow R$ be any real valued function on S(U).Let $A\subseteq R$ then the inverse image of A under f is defined as

$f^{-1}(A) = \{(F_i, E) \in S(U): f(F_i, E) \in A\}\subseteq S(U)$.

If A = {a} then $f^{-1}(A)$ will be $\{(F_i, E) / f(F_i, E) = a\} = f^{-1}(a)$.

Here we are assigning real value to every soft set defined over U.

If A contains more than one element then $f^{-1}(A)$ will be a class of soft sets in S(U) satisfying $f(F_i, E) \in A$.

Observe that the range f, $f(S(U)) = \{a \in R/ f(F_i, E) = a\}$.

Without loss of generality, we define $f^{-1}(r) = \phi$ for real number $r \in R$ and r is not in the image set of f(S(U)). We may assume that $[f(S(U))]^c = \{r \in R/ f^{-1}(r) = \phi\}$.

Now $f^{-1}([f(S(U))]^c) = \phi$.

Lemma 3.1:Given $f : S(U)\rightarrow R$ is a function. Let A be any arbitrary subset of R and let A^c denote the complement of A in R then

(i) $f^{-1}(R) = S(U)$

(ii) $f^{-1}(A^c) = (f^{-1}(A))^c$

(iii) $f^{-1}(\phi) = \phi$

Proof: (i) By definition $f^{-1}(A) = \{(F_i, E) \in S(U): f((F_i, E))\in A \subseteq R\}$

$f : S(U)\rightarrow R$ means that for all $(F_i, E) \in S(U)$, $f((F_i, E))\in R$ implies that $f^{-1}(R) = S(U)$.

(ii) $A \subseteq R$ implies $A^c\subseteq R$

By definition $f^{-1}(A^c) = \{(F_i, E) \in S(U):f(F_i, E) \in A^c\}$

Since $A \cap A^c = \phi$, $f^{-1}(A^c) = \{(F_i, E) \in S(U):f((F_i, E))\notin A\}$

$= \{(F_i, E) \in S(U):f((F_i, E))\in A\}^c = (f^{-1}(A))^c$.

(iii) $\phi^c = R$ by (ii) $[f^{-1}(\phi)]^c = f^{-1}(\phi^c) = f^{-1}(R) = S(U)$ implies that $f^{-1}(\phi) = \phi$

Lemma 3.2: Given $f : S(U)\rightarrow R$ is a function. Let $A_1, A_2 \ldots\ldots A_n$ be arbitrary subsets of R, then

(i) $f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$

(ii) $f^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(A_n)$

Proof: (i) $f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \{(F_i, E) :f((F_i, E)) \in \bigcup_{n=1}^{\infty} A_n\} = \{(F_i, E):f((F_i, E))\in A_j\}$ for atleast one j

$= \bigcup_{n=1}^{\infty} \{(F_i, E) :f((F_i, E))\in A_j\} = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$

(ii) we have $\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c$

Therefore $f^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = f^{-1}\left[\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c\right] = \left[f^{-1}\left(\bigcup_{n=1}^{\infty} A_n^c\right)\right]^c = \left[\bigcup_{n=1}^{\infty} f^{-1}(A_n^c)\right]^c = \left[\bigcup_{n=1}^{\infty} (f^{-1}(A_n))^c\right]^c =$

$\bigcap_{n=1}^{\infty} f^{-1}(A_n)$

From the above lemma it is clear that for any two subsets A and B of R, $A \cap B = \phi \Rightarrow f^{-1}(A) \cap f^{-1}(B) = \phi$*Lemma 3.3*: For a function $f: S(U) \rightarrow R$, and subsets A and B of R such that $A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$

Proof: $f^{-1}(A) = \{(F_i, E) \in S(U): f((F_i, E)) \in A\}$, $f^{-1}(B) = \{(F_i, E) \in S(U): f((F_i, E)) \in B\}$

If $A \subseteq B$ then for all $(F_i, E) \in S(U)$, $f(F_i, E) \in A \subseteq B$ implies $f(F_i, E) \in B$ and hence $f^{-1}(A) \subseteq f^{-1}(B)$.

If$\boldsymbol{A}$is any class of subsets of R, and $\boldsymbol{A}=\{A_t\}$, write $f^{-1}(\boldsymbol{A}) = \{(F_i, E) = f^{-1}(A_t) / A_t \in \boldsymbol{A}\}$ as class of soft sets (F_i, E) of S(U) which are inverse images of subsets A_t of R. Observe that all these subsets A_t of R are in the class $\boldsymbol{A}$.

Lemma3.4: If $\boldsymbol{A}$,$\boldsymbol{B}$are any two classes of subsets of R such that $\boldsymbol{A} \subseteq \boldsymbol{B}$then $f^{-1}(\boldsymbol{A}) \subseteq f^{-1}(\boldsymbol{B})$.

Proof. Let $A \in \boldsymbol{A}$impliesA $\in \boldsymbol{B}$

$A \in \boldsymbol{A}$implies $f^{-1}(A) \in f^{-1}(\boldsymbol{A})$and $A \in \boldsymbol{B}$implies $f^{-1}(A) \in f^{-1}(\boldsymbol{B})$. Hence $f^{-1}(\boldsymbol{A}) \subseteq f^{-1}(\boldsymbol{B})$.

Theorem 3.5: Let $\boldsymbol{A}$ be a sigma algebra of subsets of R. For the function $f: S(U) \rightarrow R$, let $f^{-1}(\boldsymbol{A}) = \{(F_i, E) : f(F_i, E) \in A_t, A_t \in \boldsymbol{A}\}$. Then $f^{-1}(\boldsymbol{A})$ is also a sigma algebra of subsets of (F_i, E) of S(U).

Proof: Because $\boldsymbol{A}$ is a sigma algebra of subsets of R, $\phi \in \boldsymbol{A}$

By (iii) of lemma (3.1) $\phi = f^{-1}(\phi) \in f^{-1}(\boldsymbol{A})$ implies $\phi \in f^{-1}(\boldsymbol{A})$

Also by (ii) of lemma (3.1) $S(U) \in f^{-1}(\boldsymbol{A})$

$\bigcup_{n=1}^{\infty} A_n \in \boldsymbol{A}$implies that $f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) \in f^{-1}(\boldsymbol{A})$

By (i) of lemma (3.2) $\bigcup_{n=1}^{\infty} f^{-1}(A_n) \in f^{-1}(\boldsymbol{A})$

Thus for all $n \in N$, $f^{-1}(A_n) \in f^{-1}(\boldsymbol{A})$implies that the countable union of $f^{-1}(A_n)$ is also in $f^{-1}(\boldsymbol{A})$.

For all $A_t \in \boldsymbol{A}$, $f^{-1}(A_t) \in f^{-1}(\boldsymbol{A})$

For $A_t^c \in \boldsymbol{A}$ $f^{-1}(A_t^c) \in f^{-1}(\boldsymbol{A})$ implies that $(f^{-1}(A_t))^c \in f^{-1}(\boldsymbol{A})$ and thus $f^{-1}(\boldsymbol{A})$ is a sigma algebra.

Theorem 3.6: If $f^{-1}(\boldsymbol{A})$ and $f^{-1}(\boldsymbol{B})$ are sigma algebras of classes of soft sets, their intersection is also a sigma algebra.

Proof: Since each of the sigma algebras $f^{-1}(\boldsymbol{A})$ and $f^{-1}(\boldsymbol{B})$contains ϕ and $f^{-1}(R)$, the intersection $f^{-1}(\boldsymbol{A}) \cap f^{-1}(\boldsymbol{B})$also contains ϕ and $f^{-1}(R)$. If $f^{-1}(C)$ is in $f^{-1}(\boldsymbol{A}) \cap f^{-1}(\boldsymbol{B})$then $f^{-1}(C)$ is in both $f^{-1}(\boldsymbol{A})$ and $f^{-1}(\boldsymbol{B})$and therefore $(f^{-1}(C))^c$ is also a member of $f^{-1}(\boldsymbol{A}) \cap f^{-1}(\boldsymbol{B})$. And finally the intersection of sigma algebras, $f^{-1}(\boldsymbol{A}) \cap f^{-1}(\boldsymbol{B})$ is closed under countable unions for the same reason as above.

Corollary 3.7: The intersection of finite number sigma algebras, $\bigcap_{i=1}^{n} f^{-1}(\boldsymbol{A}_i)$ is also sigma algebra.

Proof: The proof is the immediate consequence of theorem 3.6

Theorem 3.8:For a set, $f^{-1}(\boldsymbol{X}) = \{(F_i, E) : f((F_i, E)) \in X_t$,for all $X_t \in \boldsymbol{X}\}$, there exists a unique minimal sigma algebra containing $f^{-1}(\boldsymbol{X})$. This minimal sigma algebra is the sigma algebra generated by $f^{-1}(\boldsymbol{X})$.

Proof: Since $f^{-1}(R)$ is a sigma algebra and $f^{-1}(\boldsymbol{X}) \subseteq f^{-1}(R)$ implies that there exists a sigma algebra $f^{-1}(R)$ containing $f^{-1}(\boldsymbol{X})$. Let $f^{-1}(\boldsymbol{A})$ be the intersection of all sigma algebras containing $f^{-1}(\boldsymbol{X})$. Then by corollary 3.7, $f^{-1}(\boldsymbol{A})$ is also sigma algebra and is the smallest sigma algebra containing $f^{-1}(\boldsymbol{X})$.

Definition 3.9:For $f^{-1}(\boldsymbol{A}) = \{(F_i, E) : f((F_i, E)) \in A_t$, for$A_t \in \boldsymbol{A}\}$and $f^{-1}(B) = \{(F_j, E) / f(F_i, E) = b\}$, $B = \{b\}$, We define $f^{-1}(\boldsymbol{A}) \cap f^{-1}(B) = \{f^{-1}(A) \cap f^{-1}(B) / A \in \boldsymbol{A}\}$.

Theorem 3.10: Let $f^{-1}(\boldsymbol{A})$ be a sigma algebra over $f^{-1}(R)$ and $f^{-1}(B) \subseteq f^{-1}(R)$ then $f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$ is sigma algebra over $f^{-1}(R)$.

Proof: It is clear that $f^{-1}(B) = f^{-1}(R) \cap f^{-1}(B)$

$f^{-1}(R) \in f^{-1}(\boldsymbol{A})$, $f^{-1}(R) \cap f^{-1}(B) \in f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$ implies that $f^{-1}(B) \in f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$.

Since $\phi \in f^{-1}(\boldsymbol{A})$, $\phi \in f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$

Let $f^{-1}(D) \in f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$ then clearly $f^{-1}(D) = f^{-1}(A) \cap f^{-1}(B)$ for some $A \in \boldsymbol{A}$

$(f^{-1}(D))^c = f^{-1}(B) \setminus f^{-1}(D) = f^{-1}(B) \setminus [f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)] = (f^{-1}(A))^c \cap f^{-1}(B) \in f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$

Let $f^{-1}(D_n) \in f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$ for each $n \in N$. Then $f^{-1}(D_n) = f^{-1}(A_t) \cap f^{-1}(B)$ for some $t \in N$

$\bigcup_{n\in N} f^{-1}(D_n) \in \bigcup_{n\in N} [f^{-1}(A_t) \cap f^{-1}(B)] = \bigcup_{n\in N} [f^{-1}(A_t)] \cap f^{-1}(B) \in f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$ which follows that $f^{-1}(\boldsymbol{A}) \cap f^{-1}(B)$ is sigma algebra of $f^{-1}(B)$.

IV. CONCLUSION

In this paper we introduced sigma algebra of soft sets by considering the inverse images of function defined on the set of real numbers. The properties of these sigma algebras are studied. It is observed that if $\boldsymbol{A}$ is sigma algebra then $f^{-1}(\boldsymbol{A})$ is also sigma algebra.

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