Operational calculus in two variables and product of special functions

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ABSTRACT
In this paper, we establish few operational relations between the original and the image for two dimensional Laplace transform whose kernel involves the product of the general multivariable polynomials $S_{m_1,\cdots,m_R}^{l_1,\cdots,l_R}(x_1,\cdots,x_R)$, a general class of multivariable polynomials $S_{N_1}^{M_{r+1},\cdots,M_{r'}}(-y_{r+1},\cdots,-y_{r'})$, A-function of one variable and the multivariable Aleph-function. The importance of the present document lies in the fact that it unifies and extends the results of a large number of authors. At the end we shall see several corollaries.

KEYWORDS : Aleph-function of several variables, A-function, general classes of multivariable polynomials, Bivariate Laplace transform, generalized Lauricella function, Aleph-function.

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1 .Introduction and preliminaries.

Chaurasia and Godika [3] have studied several relations between the original and the image for two dimensional Laplace tranforms whose kernel involves the product of classes of multivariable polynomials, Fox's H-function [5] and the multivariable H-function defined by Srivastava and Panda [20,21]. The aim of this paper is evaluated several relations between the original and the image for two dimensional Laplace tranforms whose kernel involves the product of classes of multivariable polynomials, A-function [6] and the multivariable Aleph-function defined by Ayant [1].

The integral equation (Ditkin and Prudnikov [4]) is defined by

$$F(w,v)=wv\int_0^\infty\int_0^\infty exp(-wx-vy)f(x,y)\mathrm{d}x\mathrm{d}y,\ Re(w,v)>0 \tag{1.1}$$

The formula (1.1) represents the Laplace-Carson tranforms of a function $f(x,y)$.

$F(w,v)$ and $f(x,y)$ are said to be operationally related to each other, $F(w,v)$ is called the image and $f(x,y)$ the original.

Symbolically we can write

$$F(w,v)=L^2\{f(x,y)\} \text{ or } f(x,y)=L^{-2}\{F(w,v)\}. \tag{1.2}$$

The A- function , introduced by Gautam et al [6], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$A(z)=A_{P,Q}^{M,N}\left(\mathrm{z}\ \middle|\ \begin{matrix}(\mathrm{a}_j,A'_j)_{N,N+1},(a_j,A'_j)_P\\(\mathrm{b}_j,B'_j)_{M,M+1},(b_j,B'_j)_Q\end{matrix}\right)=\frac{1}{2\pi\omega}\int_L\ \Omega_{P,Q}^{M,N}(s)z^s\mathrm{d}s \tag{1.3}$$

for all z different to 0 and

$$\Omega_{P,Q}^{M,N}(s)=\frac{\prod_{j=1}^M\Gamma(b_j-B'_js)\prod_{j=1}^N\Gamma(1-a_j+A'_js)}{\prod_{j=N+1}^p\Gamma(a_j-A'_js)\prod_{j=M+1}^q\Gamma(1-b_j+B'_js)} \tag{1.4}$$

The serie representation of A-function is obtained by Gautam and Asgar [6].

$$A(z)=\sum_{G=1}^M\sum_{g=0}^\infty\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_Gg!} \tag{1.5}$$

where

$$\eta_{G,g} = \frac{b_g + G}{\beta_g} \tag{1.6}$$

The Aleph-function of several variables is an extension the multivariable I-function defined by Sharma and Ahmad [14] , itself is a generalisation of G and H-functions of several variables defined by Srivastava et Panda [20,21]. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function of $r-$ variables throughout our present study and will be defined and represented as follows (see Ayant [1]).

We have : $\aleph(z_1,\cdots,z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|c} z_1 & [(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot & \cdot \\ \cdot & \cdots\cdots\cdots\cdots , \\ \cdot & \\ z_r & \end{array}\right.$

$$[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] : [(c_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;$$

$$[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}] : [(d_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;$$

$$\left.\begin{array}{l} [(c_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \cdot \\ [(d_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right) = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.7}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.8}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.9}$$

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0,$$

with $k=1,\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)}$ (1.10)

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

The generalized polynomials of multivariables defined by Srivastava [17, p.185, Eq.(7)]], is given in the following manner :

$$S_{m_1,\cdots,m_R}^{l_1,\cdots,l_R}[x_1,\cdots,x_R] = \sum_{K_1=0}^{[m_1/l_1]}\cdots\sum_{K_R=0}^{[m_R/l_R]}\frac{(-m_1)_{l_1K_1}}{K_1!}\cdots\frac{(-m_R)_{l_RK_R}}{K_R!}A[m_1,K_1;\cdots;m_R,K_R]x_1^{K_1}\cdots x_R^{K_R} \tag{1.11}$$

where $l_1,\cdots,l_R$ are arbitrary positive integers and the coefficients $A[m_1,K_1;\cdots;m_R,K_R]$ are arbitrary constants, real or complex.

We shall note

$$a_R = \frac{(-m_1)_{l_1 K_1}}{K_1!} \cdots \frac{(-m_R)_{l_R K_R}}{K_R!} A[m_1, K_1; \cdots; m_R, K_R] \tag{1.12}$$

A general class of multivariable polynomials defined by Srivastava and Garg [19] are defined as follows

$$S_{N_1}^{M_{r+1},\cdots,M_{r'}}(-y_{r+1},\cdots,-y_{r'}) = \sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'} M_j\alpha_j \leqslant N_1} (-N_1)_{\sum_{j=r+1}^{r'} M_j\alpha_j} B(\alpha_{r+1},\cdots,\alpha_{r'}) \prod_{j=r+1}^{r'} \frac{(-y_j)^{\alpha_j}}{\alpha_j!} \tag{1.13}$$

where $M_{r+1},\cdots,M_{r'}$ are arbitrary positive integers and the coefficients $B(\alpha_{r+1},\cdots,\alpha_{r'})$ are arbitrary constants, real or complex, where $\alpha_j \geqslant 0$ for $j = r+1,\cdots,r'$.

We shall note

$$b_{r'} = \frac{(-N_1)_{\sum_{j=r+1}^{r'} M_j\alpha_j} B(\alpha_{r+1},\cdots,\alpha_{r'})}{\prod_{j=r+1}^{r'} \alpha_j!} \tag{1.14}$$

If we take $R = 1$ in the equation (1.11), the class of multivariable polynomials $S_{m_1,\cdots,m_R}^{\mathfrak{l}_1,\cdots,\mathfrak{l}_{\mathfrak{R}}}[.]$ reduces to the general class of polynomials $S_{m_1}^{l_1}[.]$ defined by Srivastava [16].

By suitably specializing the parameters, the polynomials $S_{m_1}^{l_1}[.]$ can be reduced to other classical orthogonal polynomials. Similarly, we obtain a special cases of a general class of multivariable polynomials by specializing the parameters.

For convenience, we shall use the following notations in this paper.

$$V = m_1, n_1; \cdots; m_r, n_r \tag{1.15}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.16}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,\mathfrak{n}}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}\} : \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\},$$

$$\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}; \cdots; \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.17}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\}, \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\} \;; \cdots;$$

$$\{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.18}$$

2. Main formula.

The following theorem of this paper provides a key formula from which we get many other theorems by specializing the parameters. We have

Theorem

$$w^{-\frac{1}{2}}(wv)^{\frac{\beta}{2}-\gamma'+1} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[m_1/l_1]} \cdots \sum_{K_R=0}^{[m_R/l_R]} \sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'} M_j\alpha_j \leqslant N_1} \Omega_{P,Q}^{M,N}(\eta_{G,g}) \frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!} a_R b_{r'} (-)^{\sum_{j=r+1}^{r'} h_j\alpha_j}$$

$$(vw)^{h\eta_{G,g}+\sum_{i=1}^{R}K_iH_i+\sum_{j=r+1}^{r'}h_j\alpha_j}\aleph\left(z_1(\sqrt{wv})^{h_1},\cdots,z_r(\sqrt{wv})^{h_r}\right)=$$

$$L^2\left[\frac{(4xy)^{\gamma'-\frac{\beta+1}{2}}}{\sqrt{\pi y}}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[m_1/l_1]}\cdots\sum_{K_R=0}^{[m_R/l_R]}\sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'}M_j\alpha_j\leqslant N_1}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}\right.$$

$$a_R b_{r'}\,(-)^{\sum_{j=r+1}^{r'}h_j\alpha_j}\,(4xy)^{\frac{-h\eta_{G,g}-\sum_{i=1}^{R}K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j}{2}}$$

$$\aleph_{p_i+1,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}\mathrm{z}_1(2\sqrt{xy})^{-h_1} & (2\gamma'-b\eta-h\eta_{G,g}-\sum_{i=1}^{R}K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j;h_1,\cdots,h_r),A\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \mathrm{B}\\ \mathrm{z}_r(2\sqrt{xy})^{-h_r} & \end{array}\right) \tag{2.1}$$

Provided

$h>0,H_i>0,h_j>0$ for $i=1,\cdots,R;j=r+1,\cdots,r'$. $Re(\gamma')>0;Re(w)\geqslant 0$; $h_k>0$ for $k=1,\cdots,r$

$$Re(\beta+h\eta_{G,g})+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0 \text{ and } Re(\gamma'-\beta-h\eta_{G,g})-\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{1-c_j^{(i)}}{\gamma_j^{(i)}}\right)<\frac{3}{4}$$

$$|argz|<|arg(\Omega)z|<\frac{1}{2}\eta\pi,\xi^*=0,\eta>0$$

$$\text{where }\Omega=\prod_{j=1}^{P}\{A_j'\}^{A_j'}\prod_{j=1}^{Q}\{B_j'\}^{-B_j'}\quad\text{and}\quad\xi^*=Im\Big(\sum_{j=1}^{P}A_j'-\sum_{j=1}^{Q}B_j'\Big)$$

$$\text{and}\quad\eta=Re\left(\sum_{j=1}^{N}A_j'-\sum_{j=N+1}^{P}A_j'+\sum_{j=1}^{M}B_j'-\sum_{j=M+1}^{Q}B_j'\right)$$

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, $k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.10) and the multiple series in the left-hand side of (2.1) converge absolutely.

Proof

The Laplace transform of the product of classes of multivariable polynomials, A-function and the multivariable Aleph-function is given by.

$$L\left\{t^{b\eta^{-1}}\,A_{P,Q}^{M,N}\left(\mathrm{t}^h\left|\begin{array}{c}(\mathrm{a}_j,A_j')_{N,N+1},(a_j,A_j')_P\\(\mathrm{b}_j,B_j')_{M,M+1},(b_j,B_j')_Q\end{array}\right.\right)S_{m_1,\cdots,m_R}^{l_1,\cdots,l_R}[t_1^{H_1},\cdots,t_R^{H_R}]\,S_{N_1}^{M_{r+1},\cdots,M_{r'}}(-t_{r+1}^{h_{r+1}},\cdots,-t_{r'}^{h_{r'}})\right.$$

$$\left.\aleph_{p_i,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c}\mathrm{z}_1t^{h_1} & \mathrm{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \\ \mathrm{z}_rt^{h_r} & (1\text{-}\beta-h\eta_{G,g}-\sum_{i=1}^{R}K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j;h_1,\cdots,h_r),B\end{array}\right)\right\}$$

$$w^{-\beta}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[m_1/l_1]}\cdots\sum_{K_R=0}^{[m_R/l_R]}\sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'}M_j\alpha_j\leqslant N_1}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}a_R b_{r'}(-)^{\sum_{j=r+1}^{r'}h_j\alpha_j}$$

$$w^{-h\eta_{G,g}-\sum_{i=1}^{R}K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j}\aleph(z_1w^{-h_1},\cdots,z_rw^{-h_r}) \tag{2.2}$$

Provided that

$h>0, H_i>0, h_j>0$ for $i=1,\cdots,R; j=r+1,\cdots,r'$. $Re(\gamma')>0; Re(w)\geqslant 0$; $h_k>0$ for $k=1,\cdots,r$

$Re(\beta+h\eta_{G,g})+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$ and $Re(\eta+h\eta_{G,g})+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{1-c_j^{(i)}}{\gamma_j^{(i)}}\right)<0$

$|argz|<|arg(\Omega)z|<\frac{1}{2}\eta\pi, \xi^*=0, \eta>0$

where $\Omega=\prod_{j=1}^{P}\{A'_j\}^{A'_j}\prod_{j=1}^{Q}\{B'_j\}^{-B'_j}$ and $\xi^*=Im(\sum_{j=1}^{P}A'_j-\sum_{j=1}^{Q}B'_j)$

and $\eta=Re\left(\sum_{j=1}^{N}A'_j-\sum_{j=N+1}^{P}A'_j+\sum_{j=1}^{M}B'_j-\sum_{j=M+1}^{Q}B'_j\right)$

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, $k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.10) and the multiple series in the left-hand side of (2.2) converge absolutely.

The result in (2.2) can be obtained by the help of (1.5), (1.11) and (1.13) respectively and a result obtained by Chaurasia [2].

Now we replace w by $(vw)^{-\frac{1}{2}}$ and multiply both sides of (2.2) by $w^{-\frac{1}{2}}(vw)^{1-\gamma'}$ and interpreting it with the help of a known result of Ditkin [4], we obtain

$$(4xy)^{\frac{\gamma'}{2}-\frac{1}{4}}\int_0^\infty t^{\beta-\gamma'-\frac{1}{2}}J_{2\gamma'-1}\left[64(xyt^2)\right]^{\frac{1}{4}}A_{P,Q}^{M,N}\left(\mathrm{t}^h\ \middle|\ \begin{matrix}(\mathrm{a}_j,A'_j)_{N,N+1},(a_j,A'_j)_P\\(\mathrm{b}_j,B'_j)_{M,M+1},(b_j,B'_j)_Q\end{matrix}\right)$$

$$S_{m_1,\cdots,m_R}^{l_1,\cdots,l_R}[t_1^{H_1},\cdots,t_R^{H_R}]\,S_{N_1}^{M_{r+1},\cdots,M_{r'}}(-t_{r+1}^{h_{r+1}},\cdots,-t_{r'}^{h_{r'}})$$

$$\aleph_{p_i,q_i+1,\tau_i;R:W}^{0,\mathrm{n}:V}\left(\begin{matrix}\mathrm{z}_1t^{h_1}\\\cdot\\\cdot\\\cdot\\\mathrm{z}_rt^{h_r}\end{matrix}\ \middle|\ \begin{matrix}\mathrm{A}\\\cdot\\\cdot\\(1\text{-}\sigma-\beta-h\eta_{G,g}-\sum_{i=1}^{R}K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j;h_1,\cdots,h_r),B\end{matrix}\right)\mathrm{dt}=$$

$$w^{-\frac{1}{2}}(wv)^{\frac{\beta}{2}-\gamma'+1}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[m_1/l_1]}\cdots\sum_{K_R=0}^{[m_R/l_R]}\sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'}M_j\alpha_j\leqslant N_1}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}a_R b_{r'}(-)^{\sum_{j=r+1}^{r'}h_j\alpha_j}$$

$$(vw)^{\frac{h\eta_{G,g}+\sum_{i=1}^{R}K_iH_i+\sum_{j=r+1}^{r'}h_j\alpha_j}{2}}\aleph\left(z_1(\sqrt{wv})^{h_1},\cdots,z_r(\sqrt{wv})^{h_r}\right) \tag{2.3}$$

Now, evaluating the left-hand side of (2.3) by the process mentioned in (2.2), we obtain the desired result (2.1).

3. Corollaries.

If we take

$$B(\alpha_{r+1},\cdots,\alpha_{r'})=\frac{\prod_{j=1}^{E_1}(e_j)_{\alpha_{r+1}\theta_j^{(r+1)}+\cdots+\alpha_{r'}\theta_j^{(r')}}\prod_{j=1}^{U_1^{r+1}}(u_j^{(r+1)})_{\alpha_{r+1}\phi_j^{(r+1)}}\cdots\prod_{j=1}^{U_1^{r'}}(u_j^{(r')})_{\alpha_{r'}\phi_j^{(r')}}}{\prod_{j=1}^{G_1}(g_j)_{\alpha_{r+1}\psi_j^{(r+1)}+\cdots+\alpha_{r'}\psi_j^{(r')}}\prod_{j=1}^{V_1^{r+1}}(v_j^{(r+1)})_{\alpha_{r+1}\tau_j^{(r+1)}}\cdots\prod_{j=1}^{V_1^{r'}}(v_j^{(r')})_{\alpha_{r'}\tau_j^{(r')}}} \tag{3.1}$$

in (2.2), $S_{N_1}^{M_{r+1},\cdots,M_{r'}}(-t_{r+1}^{h_{r+1}},\cdots,-t_{r'}^{h_{r'}})$reduces to the generalized Lauricella function defined by Srivastava-Daoust [18] as follows :

$$S_{N_1}^{M_{r+1},\cdots,M_{r'}}(-t_{r+1}^{h_{r+1}},\cdots,-t_{r'}^{h_{r'}})= F_{G_1:V_1^{r+1},\cdots,V_1^{r'}}^{1+E_1:U_1^{r+1},\cdots,U_1^{r'}}$$

$$\left(\begin{array}{c|c} -t^{h_{r+1}} & (-N_1:M_{r+1},\cdots,M_{r'}),[(e);\theta^{(r+1)},\cdots,\theta^{(r')}]:[(u^{(r+1)});\phi^{(r+1)}];\cdots;[(u^{(r')});\phi^{(r')}] \\ \cdot & \cdot \\ \cdot & [(g);\psi^{(r+1)},\cdots,\psi^{(r')}]:[(v^{(r+1)});\tau^{(r+1)}];\cdots;[(v^{(r')});\tau^{(r)}] \\ -t^{h_{r'}} & \end{array}\right) \tag{3.2}$$

We have the following corollaries

Corollary 1

$$w^{-\frac{1}{2}}(wv)^{\frac{\beta}{2}-\gamma'+1}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[m_1/l_1]}\cdots\sum_{K_R=0}^{[m_R/l_R]}\sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'}M_j\alpha_j\leqslant N_1}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}a_R b_{r'}(-)^{\sum_{j=r+1}^{r'}h_j\alpha_j}$$

$$(vw)^{h\eta_{G,g}+\sum_{i=1}^{R}K_iH_i+\sum_{j=r+1}^{r'}h_j\alpha_j}\aleph\left(z_1(\sqrt{wv})^{h_1},\cdots,z_r(\sqrt{wv})^{h_r}\right)=$$

$$L^2\left[\frac{(4xy)^{\gamma'-\frac{\beta+1}{2}}}{\sqrt{\pi y}}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[m_1/l_1]}\cdots\sum_{K_R=0}^{[m_R/l_R]}\sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'}M_j\alpha_j\leqslant N_1}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}\right.$$

$$a_R b_{r'}(-)^{\sum_{j=r+1}^{r'}h_j\alpha_j}(4xy)^{\frac{-h\eta_{G,g}-\sum_{i=1}^{R}K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j}{2}}$$

$$\aleph_{p_i+1,q_i,\tau_i;R:W}^{0,n:V}\left(\begin{array}{c|c} z_1(2\sqrt{xy})^{-h_1} & (2\gamma'-b\eta-h\eta_{G,g}-\sum_{i=1}^{R}K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j;h_1,\cdots,h_r),A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B} \\ z_r(2\sqrt{xy})^{-h_r} & \end{array}\right) \tag{3.3}$$

where $b_{r'}$ is defined by (1.14) and $B(\alpha_{r+1},\cdots,\alpha_{r'})$ is defined by (3.1), under the same conditions that (2.1).

Corollary 2

$$L\left\{t^{\beta-1}\ A_{P,Q}^{M,N}\left(\ \mathrm{t}^h\ \middle|\ \begin{matrix}(\mathrm{a}_j,A'_j)_{N,N+1},(a_j,A'_j)_P\\ (\mathrm{b}_j,B'_j)_{M,M+1},(b_j,B'_j)_Q\end{matrix}\right)S_{m_1,\cdots,m_R}^{l_1,\cdots,l_R}[t_1^{H_1},\cdots,t_R^{H_R}]\ F_{G_1:V_1^{r+1},\cdots,V_1^{r'}}^{1+E_1:U_1^{r+1},\cdots,U_1^{r'}}\right.$$

$$\left(\begin{matrix}-\mathrm{t}^{h_{r+1}}\\ \cdot\\ \cdot\\ -\mathrm{t}^{h_{r'}}\end{matrix}\ \middle|\ \begin{matrix}(-\mathrm{N}_1:M_{r+1},\cdots,M_{r'}),[(e);\theta^{(r+1)},\cdots,\theta^{(r')}]:[(u^{(r+1)});\phi^{(r+1)}];\cdots;[(u^{(r')});\phi^{(r')}]\\ \cdot\\ [(\mathrm{g});\psi^{(r+1)},\cdots,\psi^{(r')}]:[(v^{(r+1)});\tau^{(r+1)}];\cdots;[(v^{(r')});\tau^{(r)}]\end{matrix}\right)$$

$$\left.\aleph_{p_i,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1t^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{h_r}\end{matrix}\ \middle|\ \begin{matrix}\mathrm{A}\\ \cdot\\ \cdot\\ (1\text{-}\beta-h\eta_{G,g}-\sum_{i=1}^R K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j;h_1,\cdots,h_r),B\end{matrix}\right)\right\}=$$

$$w^{-\beta}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[m_1/l_1]}\cdots\sum_{K_R=0}^{[m_R/l_R]}\ \sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'}M_j\alpha_j\leqslant N_1}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}\ a_R\ (-)^{\sum_{j=r+1}^{r'}h_j\alpha_j}$$

$$b_{r'}w^{-h\eta_{G,g}-\sum_{i=1}^R K_iH_i-\sum_{j=r+1}^{r'}h_j\alpha_j}\ \aleph(z_1w^{-h_1},\cdots,z_rw^{-h_r}) \tag{3.4}$$

where $b_{r'}$ is defined by (1.14) and $B(\alpha_{r+1},\cdots,\alpha_{r'})$ is defined by (3.1), under the same conditions that (2.2).

Consider the above corollary but the multivariable polynomials $S_{m_1,\cdots,m_R}^{l_1,\cdots,l_R}[.]$ vanishes, we obtain the following result.

Corollary 3

$$L\left\{t^{\beta-1}\ A_{P,Q}^{M,N}\left(\ \mathrm{t}^h\ \middle|\ \begin{matrix}(\mathrm{a}_j,A'_j)_{N,N+1},(a_j,A'_j)_P\\ (\mathrm{b}_j,B'_j)_{M,M+1},(b_j,B'_j)_Q\end{matrix}\right)F_{G_1:V_1^{r+1},\cdots,V_1^{r'}}^{1+E_1:U_1^{r+1},\cdots,U_1^{r'}}\right.$$

$$\left(\begin{matrix}-\mathrm{t}^{h_{r+1}}\\ \cdot\\ \cdot\\ -\mathrm{t}^{h_{r'}}\end{matrix}\ \middle|\ \begin{matrix}(-\mathrm{N}_1:M_{r+1},\cdots,M_{r'}),[(e);\theta^{(r+1)},\cdots,\theta^{(r')}]:[(u^{(r+1)});\phi^{(r+1)}];\cdots;[(u^{(r')});\phi^{(r')}]\\ \cdot\\ [(\mathrm{g});\psi^{(r+1)},\cdots,\psi^{(r')}]:[(v^{(r+1)});\tau^{(r+1)}];\cdots;[(v^{(r')});\tau^{(r)}]\end{matrix}\right)$$

$$\left.\aleph_{p_i,q_i+1,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1t^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rt^{h_r}\end{matrix}\ \middle|\ \begin{matrix}\mathrm{A}\\ \cdot\\ \cdot\\ (1\text{-}\beta-h\eta_{G,g}-\sum_{j=r+1}^{r'}h_j\alpha_j;h_1,\cdots,h_r),B\end{matrix}\right)\right\}=$$

$$w^{-\beta}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\ \sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'}M_j\alpha_j\leqslant N_1}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}\ (-)^{\sum_{j=r+1}^{r'}h_j\alpha_j}$$

$$b_{r'}w^{-h\eta_{G,g}-\sum_{j=r+1}^{r'}h_j\alpha_j}\ \aleph(z_1w^{-h_1},\cdots,z_rw^{-h_r}) \tag{3.5}$$

under the same notations and conditions that (2.2).

Now, the multivariable Aleph-function reduces to Aleph-function of one variable defined by Sudland [22,23], we have

Corollary 4

$$L\left\{t^{\beta-1} A_{P,Q}^{M,N}\left(t^h \left| \begin{array}{c} (\mathrm{a}_j, A'_j)_{N,N+1}, (a_j, A'_j)_P \\ (\mathrm{b}_j, B'_j)_{M,M+1}, (b_j, B'_j)_Q \end{array}\right.\right) F_{G_1:V_1^{r+1},\cdots,V_1^{r'}}^{1+E_1:U_1^{r+1},\cdots,U_1^{r'}}\right.$$

$$\left(\begin{array}{c} \text{-t}^{h_{r+1}} \\ \cdot \\ \cdot \\ \text{-t}^{h_{r'}} \end{array}\left| \begin{array}{c} (\text{-N}_1 : M_{r+1},\cdots,M_{r'}), [(e);\theta^{(r+1)},\cdots,\theta^{(r')}] : [(u^{(r+1)});\phi^{(r+1)}];\cdots;[(u^{(r')});\phi^{(r')}] \\ \cdot \\ [(\mathrm{g});\psi^{(r+1)},\cdots,\psi^{(r')}] : [(v^{(r+1)});\tau^{(r+1)}];\cdots;[(v^{(r')});\tau^{(r)}] \end{array}\right.\right)$$

$$\left.\aleph_{p_{i(1)},q_{i(1)}+1,\tau_{i(1)};R_1}^{m_1,n_1}\left(z_1 t^{h_1}\left| \begin{array}{c} ((\mathrm{c}_j;\gamma_j)_{1,\mathfrak{n}}, [\tau_i(c_{ji};\gamma_{ji})]_{\mathfrak{n}+1,p_i;R} \\ \cdot \\ (1\text{-}\beta - h\eta_{G,g} - \sum_{j=r+1}^{r'} h_j\alpha_j; h_1), (d_j;\delta_j)_{1,\mathfrak{n}}, [\tau_i(d_{ji};\delta_{ji})]_{m+1,q_i;R} \end{array}\right.\right)\right\} =$$

$$w^{-\beta}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{\alpha_{r+1},\cdots,\alpha_{r'}=0}^{\sum_{j=r+1}^{r'} M_j\alpha_j \leqslant N_1}\Omega_{P,Q}^{M,N}(\eta_{G,g})\frac{z^{\eta_{G,g}}(-)^g}{\delta_G g!}\ (-)^{\sum_{j=r+1}^{r'} h_j\alpha_j} b_{r'} w^{-h\eta_{G,g}-\sum_{j=r+1}^{r'} h_j\alpha_j}$$

$$\aleph_{p_{i(1)},q_{i(1)},\tau_{i(1)};R_1}^{m_1,n_1}\left(z_1 w^{-h_1}\left| \begin{array}{c} ((\mathrm{c}_j;\gamma_j)_{1,\mathfrak{n}}, [\tau_i(c_{ji};\gamma_{ji})]_{\mathfrak{n}+1,p_i;R} \\ \cdot \\ (\mathrm{d}_j;\delta_j)_{1,\mathfrak{n}}, [\tau_i(d_{ji};\delta_{ji})]_{m+1,q_i;R} \end{array}\right.\right) \qquad (3.6)$$

Provided that

$h > 0, h_j > 0$ for $j = r+1,\cdots,r'$. $Re(\gamma') > 0; Re(w) \geqslant 0\ ; h_1 > 0$

$Re(\beta + h\eta_{G,g}) + h_1 \min_{1\leqslant j\leqslant m_1} Re\left(\frac{d_j^{(1)}}{\delta_j^{(1)}}\right) > 0$ and $Re(\underline{\eta} + h\eta_{G,g}) + h_1 \min_{1\leqslant j\leqslant m_1} Re\left(\frac{1-c_j^{(1)}}{\gamma_j^{(1)}}\right) < 0$

$|argz| < |arg(\Omega)z| < \frac{1}{2}\eta\pi, \xi^* = 0, \eta > 0$

where $\Omega = \prod_{j=1}^{P}\{A'_j\}^{A'_j}\prod_{j=1}^{Q}\{B'_j\}^{-B'_j}$ and $\xi^* = Im(\sum_{j=1}^{P} A'_j - \sum_{j=1}^{Q} B'_j)$

and $\eta = Re\left(\sum_{j=1}^{N} A'_j - \sum_{j=N+1}^{P} A'_j + \sum_{j=1}^{M} B'_j - \sum_{j=M+1}^{Q} B'_j\right)$

$|argz_1| < \frac{1}{2}\pi\left(\sum_{j=1}^{n_1}\gamma_j^{(1)} - \tau_{i^{(1)}}\sum_{j=n_1+1}^{p_{i^{(1)}}}\gamma_{ji^{(1)}}^{(1)} + \sum_{j=1}^{m_1}\delta_j^{(1)} - \tau_{i^{(1)}}\sum_{j=m_1+1}^{q_{i^{(1)}}}\delta_{ji^{(1)}}^{(1)}\right) > 0$, and the multiple series in the left-hand side of (3.3) converge absolutely.

Remarks

By the similar methods, we obtain the same relations with the Aleph-function of two variables [13], the I-function of two variables ([8],[15]), the multivariable I-function ([9],[11]), the I-function of one variable [12], the multivariable A-function [7], the A-function [6] and the modified multivariable H-function [10].
The formulae (2.1), (2.2) and (2.3) can be extended of any finite number of classes of multivariable polynomials and multivariable Aleph-functions.

4. Conclusion.

In this paper, we evaluate the image for two dimensional Laplace tranforms whose kernel involves the product of classes of multivariable polynomials, A-function and the multivariable Aleph-function. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions, the orthogonal polynomials of one and several variables can be obtained.

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