Some fixed point theorems in Banach spaces with application to semilinear Cauchy problem

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Abstract
In this paper, we present a fixed point theorem for a new type of contractive mapping in Banach spaces. We also consider a semi-linear Cauchy problem in Banach spaces and prove the existence of its solution by using our fixed point theorem.

Keywords: Fixed points, non-expansive mapping, sequence of iterates, convex, uniformaly convex, compact, normal structure, diameter of a set, strongly continuous semigroup, mild solution, Cauchy problem.

1 Introduction

Historically the study of fixed point theory began in 1912 with a theorem given by famous Dutch mathematician L. E. Y. Brouwer. This is the most famous and important theorem on the topological fixed point property. It can be formulated as; The closed unit Ball $\mathbb{B}^n \in \mathbb{R}^n$ has the topological fixed point property. He also proved the fixed point theorems for a square, a sphere and their n-dimensional counterparts. Brouwer's theorem has many applications in analysis, differential equation and generally in proving all kinds of so-called existence theorems for many types of equations. Its discovery had a tremendous influence in the development of several branches of mathematics, especially algebraic topology. An important generalization of Brouwer's theorem was discovered in 1930 by J. Schauder it may be stated as follows; any nonempty, compact convex subset C of a Banach Space has the topological fixed point property. The compactness condition on subset is a stronger one.

Throughout this paper, unless otherwise stated, let E be a Banach Space with norm $\|\cdot\|$. Let C be the nonempty closed convex subset of E.
Let E denote the Banach Space. A mapping $f\colon E \to E$ is said to be non-expansive if $\|fu - fv\| \leq \|u - v\|$ for all $u, v \in E$. Many fixed point theorems for non-expansive mappings have been derived in recent years. For related results Browder and Petryshan [3], Belluse and Kirk [1], Daiz and Metcalf [4], Kirk [5] and Petryshyn and Williamson [9]. The object of this paper is to improve some fixed point theorems using a symmetric rational fraction. We shall be concerned with a mapping, which satisfies the following contractive condition. Let f be a mapping of E into itself, such that

$$\|fu - fv\| \leq \left[\frac{\|u - fu\|\|u - fv\| + \|v - fv\|\|v - fu\|}{\|u - fv\| + \|v - fu\|}\right] \quad \forall\, u, v \in E, u \neq v. \tag{1.1}$$

It is natural to begin our application of fixed point methods with existence and uniqueness of solution of certain first order initial value problems. In particular, we seek to solution to

$$\begin{aligned} u'(t) &= f(t, u(t)),\ t \in I \\ u(0) &= u_0 \end{aligned} \tag{1.2}$$

where $f\colon I \times E \to E$ and $I = [0, b]$. Notice that (1.2) is a system of first order equations because f takes values on E. We begin our analysis of (1.2) by assuming that $f\colon I \times E \to E$. Then evidently, $u \in C^1(I)$ (The Banach space of functions u whose first derivative is continuous on I and equipped with the norm $|u|_1 = \max\{\sup_{t\in I} |u(t)|, \quad \sup_{t\in I} |u'(t)|\})$. Solves (1.2) iff $u \in C(I)$ (The Banach Space of functions u, continuous on I and equipped with the norm $|u|_0 = \sup_{t\in I} |u(t)|$) solves

$$u(t) = u_0 + \int_0^t f(s, u(s))ds. \tag{1.3}$$

Define an integral operator $T\colon C(I) \to C(I)$ by

$$Tu(t) = u_0 + \int_0^t f(s, u(s))ds. \tag{1.4}$$

Then the equivalence above is expressed briefly by u solves (1.2) iff $u = Tu, \quad T\colon C(I) \to C(I)$. In other words, classical solution to (1.2) are fixed points of the integral operator T.

2 Preliminaries

Definition 2.1. [2] Let E be a real (complex) vector space. A norm on E is a mapping $\|\cdot\|\colon E \times E \to \mathbb{R}_+$ having the following properties
(n_1) $\|u\| = 0 \ \Leftrightarrow\ u = 0$, the null element of E;
(n_2) $\|\lambda u\| = |\lambda|\ \|u\|$, for any $u \in E$ and any scalar λ;
(n_3) $\|u + v\| \leq \|u\| + \|v\|, \forall\ u, v \in E$ ("the triangle inequality").
The pair $(E, \|\cdot\|)$ is called normed linear space.

Definition 2.2. [2] Let u_n be a sequence in a Banach space E. We say that the sequence u_n
(a) is convergent to $u \in E$ (denote by $u_n \to u$) if $lim_{n\to\infty}\|u_n - u\| = 0$,
(b) fundamental or Cauchy sequence if for any $\epsilon > 0$, there is an N such that $\|u_m - u_n\| < \epsilon$, for all $\quad n, m > N$.

Definition 2.3. [2] A normed space $(E, \|\cdot\|)$ is called complete if any Cauchy sequence in E is convergent.

Definition 2.4. [2] A Banach Space is a normed space which is complete .

Definition 2.5. [2] Let $f\colon E \to E$ is a self map. We say that $u \in E$ is a fixed point of f if $f(u) = u$ and denote by F_f the set of all fixed points of f.

Example 2.6. If $E = \mathbb{R}$ and $f(x) = x$, then $F_f = \mathbb{R}$.

Definition 2.7. [2] A mapping $f\colon E \to E$ is called
(1) Lipschitzian (or L-Lipschitzian) if there exists $L > 0$ such that $\|fu - fv\| \leq L\|u - v\|$ for all$u, v \in E$,
(2) contraction (or a-Contraction) if f is a-Contraction with $a \in [0, 1)$,
(3) non-expansive if f is 1-Lipschitzian, i.e, $\|fu - fv\| \leq \|u - v\|$, for all $u, v \in E, u \neq v$,
(4) contractive if $\|fu - fv\| < \|u - v\|$, for all $u, v \in E, u \neq v$,
(5) isometry if $\|fu - fv\| = \|u - v\|$, for all $u, v \in E$.

Theorem 2.8. *(Banach Contraction Principle)*
Let f be a contraction on a Banach space E. Then f has a unique fixed point $\tilde{u} \in E$.

Definition 2.9. [2] A subset C of a real normed space is called bounded if there exists $M > 0$ such that $\|u\| \leq M$, for all $u \in C$.

Definition 2.10. [10] For a bounded set $C \subset E$, the diameter of C denoted by $\delta(C)$ is defined as

$$\delta(C) = \sup\{\|u - v\|; u, v \in C\}.$$

Definition 2.11. [2] A subset C of a Banach space E is convex, if for any pair of points $u, v \in C$, the closed segement with the extremities u, v, that is, the set $\|\lambda u + (1-\lambda)v\|$ such that $0 \leq \lambda \leq 1$ is contained in C.

Definition 2.12. [10] The intersection of all convex subset of E containing C is called convex hull of C and is usually denoted by $C_0(C)$ and the closed convex hull of C is denoted by $\overline{C_0}(C)$.

Let $O(u) = \{u, fu, f^2u, ...\}$ where $f\colon C \to C$ and $f(C) = \{fu : \ u \in C\}$.

Definition 2.13. [2] A Banach space $(E, \|\cdot\|)$ is called uniformaly convex if, given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $u, v \in E$ satisfying $\|u\| \leq 1, \quad \|v\| \leq 1$, and $\|u - v\| \geq \epsilon$, we have $\frac{1}{2}\|u + v\| < 1 - \delta$.

Definition 2.14. [10] A bounded convex subset C of a Banach space is said to have a normal structure, if for each convex subset H of C with more than one point, there is a point $u \in H$ such that $\sup\{\|u - v\| : v \in H\} < \delta(H)$.

Definition 2.15. For $f\colon E \to E$ and $n \in \mathbb{N}$, we denote by f^n the nth iterate of f, namely, $fo.....of$ n times (f^0 is the identity map).

Theorem 2.16. *Let E be a Banach space and let $f\colon E \to E$, if f^n (the nth iterate of f), is a contraction, for some $n \geq 1$. Then f has a unique fixed point $\tilde{u} \in E$.*

Proof. Let $\tilde{u}$ be the unique fixed point of f^n, (By Banach contraction principle) then $f^n(f(\tilde{u})) = f(f^n(\tilde{u})) = f(\tilde{u})$, which implies $f(\tilde{u}) = \tilde{u}$. Since a fixed point of f is clearly a fixed point of f^n, we have uniqueness as well. ∎

Lemma 2.17. *[10] Let C be a subset of a Banach space E and $f\colon C \to C$, such that for $u \in C$, f satisfies (1.1). Then*

$$\|f^n u - f^{n+1}u\| \leq \|f^{n-1}u - f^n u\|$$

and for any positive integers m and n,

$$\|f^n u - f^m u\| \leq \|u - fu\|.$$

Proof.

$$\|f^n u - f^{n+1}u\| \leq \left[\frac{\|f^{n-1}u - f^n u\|\|f^{n-1}u - f^{n+1}u\| + \|f^n u - f^{n+1}u\|\|f^n u - f^n u\|}{\|f^{n-1}u - f^{n+1}u\| + \|f^n u - f^n u\|}\right].$$

Thus

$$\|f^n u - f^{n+1}u \leq \|f^{n-1}u - f^n u\| \leq .... \leq \|u - fu\|.$$

Now

$$\begin{aligned}\|f^n u - f^m u\| &\leq \left[\frac{\|f^{n-1}u - f^n u\|\|f^{n-1}u - f^m u\| + \|f^{m-1}u - f^m u\|\|f^{m-1}u - f^n u\|}{\|f^{n-1}u - f^m u\| + \|f^{m-1}u - f^n u\|}\right]\\ &\leq \left[\frac{\|u - fu\|\|f^{n-1}u - f^m u\| + \|u - fu\|\|f^{m-1}u - f^n u\|}{\|f^{n-1}u - f^m u\| + \|f^{m-1}u - f^n u\|}\right]\end{aligned}$$

$i.e, \|f^n u - f^m u\| \leq \|u - fu\|.$

∎

Theorem 2.18. *[10] Let C be a nonempty, bounded, closed, convex subset of a reflexive Banach space E and let C have normal structure. If $f: C \to C$ is continuous and satisfies (1.1), then f has a unique fixed point in C.*

Proof. Since E is a reflexive Banach space, every descending chain of nonempty closed convex subsets of E has nonempty intersection. Hence, by Zorn's lemma to obtain a sunset C_1 of C minimal with respect to being closed, convex and invariant under f.

If $\delta(C_1) = 0$, then result is obvious. Suppose $\delta(C_1) > 0$.
Since C has a normal structure, there is a point $v \in C_1$, such that $\sup\{\|u - v\| : u \in C_1\} \leq r < \delta(C_1)$.

Thus, $\|u - fv\| \leq r$ and by Lemma(2.17), $\delta(\overline{O(v)}) \leq r$.

Let $H_1 = \{u \in C_1 : \delta(O(u)) \leq r\}$ and $H_2 = \overline{C_0}(f(H_1))$. Then H_2 is closed, convex and nonempty. Let $g \in H_2$. Then three cases arise to consider;

Case(i) Let $g = fh$ for some $h \in H_1$.
Then $\|g - fg\| = \|fh - f^2h \leq \|h - fh\| \leq r$

Hence $g \in H_1$ and $fg \in H_2$.

Case(ii) Let $g = \sum_{i=1}^{n} \lambda_i f h_i,\ h_i \in H_1,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1$. Then

$$\begin{aligned} \|fg - g\| &= \|fg - \sum_{i=1}^{n} \lambda_i f h_i\| \\ &\leq \sum_{i=1}^{n} \lambda_i \|f^2 h_i - f h_i\| \leq r. \end{aligned}$$

so $\|g - fg\| \leq r,\ g \in H_1$ and $fg \in H_2$.

Case(iii) Let g is the limit of terms of the form $\sum_{i=1}^{n} \lambda_i f h_i$,
where $\lambda_i \geq 0,\ h_i \in H_1,\ \sum_{i=1}^{n} \lambda_i = 1$.
Then for any such terms we have

$\|fg - g\| \leq \|fg - \sum_{i=1}^{n} \lambda_i f^2 h_i\| + \|\sum_{i=1}^{n} \lambda_i f^2 h_i - \sum_{i=1}^{n} \lambda_i f h_i\| + \|\sum_{i=1}^{n} \lambda_i f h_i - g\| \leq r.$

Thus $g \in H_1$ and $fg \in H_2$.
Since C_1 is minimal, $C_1 = H_2$.

But

$$\begin{aligned} \delta(H_2) &= \delta(\overline{C_0}(f(H_1))) = \delta(f(H_1)) \\ &= \sup\{\|fu - fv\| : u, v \in H_1\} \\ &\leq r < \delta(C_1). \end{aligned}$$

Thus $\delta(C_1) > 0$ leads to a contradiction. Hence, $\delta(C_1) = 0$ and f has a fixed point, say $\tilde{u}$ in C.
From (1.1) it can be easily seen that $\tilde{u}$ is unique. ∎

Corollary 2.19. *If C be a convex subset of a Banach space E and $f: C \to C$ is continuous and satisfies (1.1), then f has a unique fixed point in C if any one of the following conditions is satisfied;*
(i) C is closed and bounded and E is a uniformly convex banach space Or
(ii) C is compact.

3 Main Result

Definition 3.1. [6] Let E be a Banach space. A one-parameter family $S(t)$ $(t \geq 0)$ of bounded linear operators on E is said to be a strongly continuous semigroup if
(i) $S(0) = I$ (identity operator on E);
(ii) $S(t + s) = S(t)S(s)$ for any $t, s \geq 0$;
(iii)$\lim_{t \to 0} S(t)u = u$ for any $u \in E$ (strong continuity).

Definition 3.2. [6] The Linear operator T of domain

$$D(T) = \{u \in E : lim_{t\to 0} \frac{S(t)u - u}{t} \ \ exists\}$$

defind by $Tu = lim_{t\to 0} \frac{S(t)u-u}{t}$, $\forall\ u \in D(T)$ is the infinitesimal generator of the semigroup $S(t)$.

Proposition 3.3. *[6] T is a closed linear operator with dense domain. For every fixed $u \in D(T)$, the map $t \to S(t)u$ belongs to $C^1([0,\infty), D(T))$ and $\frac{d}{dt}S(t)u = TS(t)u = S(t)Tu$.*

Consider the following semilinear Cauchy problem in Banach space E

$$\begin{aligned} &u'(t) = Tu(t) + f(t, u(t)),\ 0 < t \le b \\ &u(0) = u_0 \in E \end{aligned} \tag{3.1}$$

where T is the infinitesimal generator of a strongly continuous semigroup $S(t)$, and $f \colon [0, b] \times E \to E$ is continuous and uniformaly Lipschitz continuous on E with Lipschitz constant $L \ge 0$.

Definition 3.4. [6] A function $u \colon [0, b] \to E$ is said to be a classical solution to (3.1) if it is differentiable on $[0, b]$ and (3.1) is satisfied on $[o, b]$.
if u is a classical solution, it is necessarily unique, and is given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s, u(s))ds \tag{3.2}$$

This can be easily proved integrating in ds on $[0, t]$ the derivative with respect to s of the differentiable function $S(t - s)u(s)$ and using (3.1). Notice that above (Riemann) integral is well defined, since if $u \in (C[0, b], E)$ the map $t \to f(t, u(t))$ belongs to $C([0, b], E)$ as well. In particular, if $f \in C([0, b], E)$, then f is Riemann integrable on $[0, b]$.

Definition 3.5. [6] A function $u \colon [0, b] \to E$ is said to be a mild solution to (3.1) if it is continuous on $[0, b]$ and satisfies the integral equation (3.2).

Theorem 3.6. *[6] For any $u_0 \in E$ the Cauchy problem (3.1) has a unique solution. Moreover the map $u_0 \to u(t)$ is Lipschitz continuous from E into $C([0, b], E)$.*

Proof. Given that $u_0 \in E$. Define a map $\psi \colon C([0, b], E) \to C([0, b], E)$ by

$$\psi(u)(t) = S(t)u_0 + \int_o^t S(t-s)f(s, u(s))ds.$$

Then we have

$$\|\psi(u)(t) - \psi(v)(t)\|_E \le LMt\|u - v\|_{C([0,b],E)}$$

where $M = \sup_{t\in[0,b]} \|S(t)\|$.
By an inductive argument, we have

$$\|\psi^n(u)(t) - \psi^n(v)(t)\|_E \le \frac{(LMb)^n}{n!}\|u - v\|_{C([0,b],E)}.$$

Hence for any $n \in \mathbb{N}$, ψ^n is contraction, so by theorem (2.16) ψ has a unique fixed point in $C([0, b], E)$ which is clearly the desired mild solution to the Cauch problem (3.1).
Further, Let v be the unique mild solution corresponding to the initial value v_0. Then

$$\|u(t) - v(t)\|_E \le M\|u_0 - v_0\|_E + LM \int_0^t \|u(s) - v(s)\|_E ds.$$

By the Gronwall Lemma (see [6]), we get

$$\|u(t) - v(t)\|_E \le Me^{LMb}\|u_0 - v_0\|_E,\ \forall\ t \in [0, b]$$

which proves the Lipschitz continuity of the map $u_o \to u(t)$.

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