Integral involving multivariable Aleph-function with applications

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ABSTRACT
In an attempt to unify some early results in the theory of the hypergeometric function type, we establish new results involving the product of Gauss's hypergeometric function, generalized hypergeometric function and multivariable Aleph-function and uses it in a solution of problem of heat conduction. An expansion formula of very general nature has also been obtained. Finally, we shall the particular cases of the Aleph-function of two variables defined by Sharma [8], the I-function of two variables defined by Sharma et al [7] and the multivariable H-function defined by Srivastava et al [9,10].

Keywords :multivariable Aleph-function, Aleph-function of two variables, I-function of two variables, heat conduction, Aleph-function, multivariable H-function, generalized hypergeometric function.

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1.Introduction and preliminaries.

The multivariable Aleph-function is an extension of the multivariable I-function recently defined by C.K. Sharma and Ahmad [6] , itself is a generalization of the multivariable H-function defined by Srivastava et al [9,10]. The multivariable Aleph-function is defined by means of the multiple contour integral :

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$

$$[(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$$
$$\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,q_i}]:$$

$$\left.\begin{array}{l}[(c_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(c_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(d_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(d_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1,\cdots,r, i = 1,\cdots,R \ , i^{(k)} = 1,\cdots,R^{(k)} \tag{1.4}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r) = 0(\ |z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\ ), max(\ |z_1|,\cdots,|z_r|\ ) \rightarrow 0$$

$$\aleph(z_1,\cdots,z_r) = 0(\ |z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\ ), min(\ |z_1|,\cdots,|z_r|\ ) \rightarrow \infty$$

where $k = 1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1,\cdots,n_k$$

For convenience, we will use the following notations in this section.

$$V = m_1,n_1;\cdots;m_r,n_r \tag{1.5}$$

$$\mathrm{W} = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.6}$$

$$A = \{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\}:\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\}$$

$$\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.7}$$

$$B = \{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\}:\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\};\cdots;$$

$$\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.8}$$

The contracted form concerning the multivariable Aleph-function writes :

$$\aleph(z_1,\cdots,z_r) = \aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} \end{array}\right) \tag{1.9}$$

The generalized hypergeometric function is defined by :

$$ {}_uF_v\left[\begin{array}{c} (\mathrm{a}_u) \\ \cdot \\ (\mathrm{b}_v) \end{array} \middle| \mathrm{c}z^l \right] = \sum_{k=0}^{\infty} \frac{[(a_u)]_k}{[(b_u)]_k} \frac{c^k z^k}{k!} \tag{1.10} $$

where $|cz| < 1$

$[(a_u)]_k$ denotes $(a_1)_k, \cdots, (a_u)_k$ and $[(b_v)]_k$ denotes $(b_1)_k, \cdots, (b_v)_k$

2. Multiplication formula

For a convenience, we let $\{\Delta(N, a_j); \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}\}_{1,\mathfrak{n}}$ stand for the array of parameters :

$\{\Delta(N, a_1); \alpha_1^{(1)}, \cdots, \alpha_1^{(r)}\}, \cdots, \{\Delta(N, a_\mathfrak{n}); \alpha_\mathfrak{n}^{(1)}, \cdots, \alpha_\mathfrak{n}^{(r)}\}$and $\{\Delta(N, a_1); \alpha_1^{(1)}, \cdots, \alpha_1^{(r)}\}$ for

$\left(\frac{a_1}{N}; \alpha_1^{(1)}, \cdots, \alpha_1^{(r)}\right), \cdots, \left(\frac{a_1 + N - 1}{N}; \alpha_1^{(1)}, \cdots, \alpha_1^{(r)}\right)$, and so on.

Throughout this paper we shall note

$$ \mathbf{V} = Nm_1, Nn_1; \cdots; Nm_r, Nn_r \tag{2.1} $$

$$ \mathbf{W} = Np_{i^{(1)}}, Nq_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \cdots; Np_{i^{(r)}}, Nq_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{2.2} $$

$$ \Delta(A) = \{(\Delta(a_j, N); \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,\mathfrak{n}}\}, \{\tau_i(\Delta(a_{ji}, N); \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}\}: \{(\Delta(c_j^{(1)}, N); \gamma_j^{(1)})_{1,n_1}\} $$

$$, \{\tau_{i^{(1)}}(\Delta(c_{ji^{(1)}}^{(1)}, N); \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}; \cdots; \{(\Delta(c_j^{(r)}, N); \gamma_j^{(r)})_{1,n_r}\}, \{\tau_{i^{(r)}}(\Delta(c_{ji^{(r)}}^{(r)}, N); \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{2.3} $$

$$ \Delta(B) = \{\tau_i(\Delta(b_{ji}, N); \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\}: \{(\Delta(d_j^{(1)}, N); \delta_j^{(1)})_{1,m_1}\}, \{\tau_{i^{(1)}}(\Delta(d_{ji^{(1)}}^{(1)}, N); \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\} $$

$$; \cdots; \{(\Delta(d_j^{(r)}, N); \delta_j^{(r)})_{1,m_r}\}, \{\tau_{i^{(r)}}(\Delta(d_{ji^{(r)}}^{(r)}, N); \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{2.4} $$

$$ M = (1 - N)\left[\mathfrak{n} - \frac{p_i + q_i}{2} + m_1 + n_1 - \frac{p_{i^{(1)}} + q_{i^{(1)}}}{2} + \cdots + m_r + n_r - \frac{p_{i^{(r)}} + q_{i^{(r)}}}{2}\right] \tag{2.5} $$

$$ \mu = \left[\frac{p_i - q_i}{2} + \frac{p_{i^{(1)}} - q_{i^{(1)}}}{2} + \cdots + \frac{p_{i^{(r)}} - q_{i^{(r)}}}{2} + \tau_i \sum_{j=1}^{q_i} b_{ji} + \tau_{i^{(1)}} \sum_{j=1}^{q_{i^{(1)}}} d_{ji^{(1)}}^{(1)} + \cdots + \tau_{i^{(r)}} \sum_{j=1}^{q_{i^{(r)}}} d_{ji^{(r)}}^{(r)} - \sum_{j=1}^{\mathfrak{n}} a_j \right. $$

$$ \left. - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} a_{ji} - \sum_{j=1}^{n_1} c_j^{(1)} - \tau_{i^{(1)}} \sum_{j=n_1+1}^{p_{i^{(1)}}} c_{ji^{(1)}}^{(1)} - \cdots - \sum_{j=1}^{n_r} c_j^{(r)} - \tau_{i^{(r)}} \sum_{j=n_r+1}^{p_{i^{(r)}}} c_{ji^{(r)}}^{(r)} \right] \tag{2.6} $$

Lemma 1 (multiplication formula)

By using the above notations, we have the following result

$$\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} Z_1 z^{\lambda_1/N} & A \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r z^{\lambda_r/N} & B \end{array}\right) = (2\pi)^M N^{\mu+2} \aleph^{0,N\mathfrak{n}:\mathbf{V}}_{Np_i,Nq_i,\tau_i;R:\mathbf{W}}\left(\begin{array}{c|c} z^{\lambda_1} Z_1^N N^{NU_1} & \Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ z^{\lambda_r} Z_r^N N^{NU_r} & \Delta(B) \end{array}\right) \quad (2.7)$$

where $\Delta(A), \Delta(B), M$ and μ are defined by (2.3), (2.4), (2.5) and (2.6) respectively and

$$U_i^{(k)} = \tau_i \sum_{j=1}^{p_i} \alpha_{ji}^{(k)} + \tau_{i^{(k)}} \sum_{j=1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} < 0 \quad (2.8)$$

Proof

We can be proved by making use of the multiplication formula of the Gamma function formula in the definition of the Aleph-function of several function given by (1.1)

3. Integral

Lemma 2 (Erdelyi [4], page 9)

$$\int_0^t z^{\rho-1}(t-z)^{\beta-1}\mathrm{d}z = t^{\rho+\beta-1}B(\rho,\beta) \quad (3.1)$$

where $t > 0, Re(\rho) > 0, Re(\beta) > 0$

Lemme 3

$$\int_0^t z^{\rho-1}(t-z)^{\beta-1} \aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} Z_1 z^{\lambda_1/N} & A \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r z^{\lambda_r/N} & B \end{array}\right)\mathrm{d}z = (2\pi)^N N^{\mu+2}\Gamma(\beta) t^{\rho+\beta-1}$$

$$\aleph^{0,N\mathfrak{n}+1:\mathbf{V}}_{Np_i+1,Nq_i+1,\tau_i;R:\mathbf{W}}\left(\begin{array}{c|c} t^{\lambda_1} Z_1^N N^{NU_1} & (1-\rho:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ t^{\lambda_r} Z_r^N N^{NU_r} & (1-\beta-\rho:\lambda_1,\cdots,\lambda_r),\Delta(B) \end{array}\right) \quad (3.2)$$

provided that

$$Re(\beta) > 0, \lambda_1 > 0, \cdots, \lambda_r > 0, N \in \mathbb{N}^* ; Re(\rho) + \frac{1}{N}\sum_{i=1}^r \lambda_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$$

$|argZ_k| < \frac{1}{2}A_i^{(k)}\pi$, $k = 1,\cdots,r$ where $A_i^{(k)}$ is given in (1.4) and

$$U_i^{(k)} = \tau_i \sum_{j=1}^{p_i} \alpha_{ji}^{(k)} + \tau_{i^{(k)}} \sum_{j=1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} < 0$$

Proof

The formula (3.2) can be obtained by replacing the multivariable Aleph-function (with $N=1$) on the left hande side by its equivalent contour integral (1.1), changing the order of integrations, which is justified due to the absolute convergence of the integral, evaluating the inner integral with the help of the lemma 2 and applying the multiplication formula for multivariable Aleph-function with the help of lemma 1.

Theorem

$$\int_0^t z^{\rho-1}(t-z)^{\beta-1}\,{}_2F_1[\alpha,\tau;\beta:t-z]{}_uF_v\left[\begin{array}{c|c}(\mathrm{a}_u) & \\ . & \mathrm{c}z^l \\ (\mathrm{b}_v) & \end{array}\right]\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c}\mathrm{Z}_1 z^{\lambda_1/N} & \mathrm{A} \\ . & . \\ . & . \\ \mathrm{Z}_r z^{\lambda_r/N} & \mathrm{B}\end{array}\right)\mathrm{d}z=$$

$$(2\pi)^N N^{\mu+2}\Gamma(\beta)t^{\rho+\beta-1}\sum_{h,g=0}^{\infty}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}\frac{(\alpha)_g(\tau)_g t^g}{g!}$$

$$\aleph^{0,N\mathfrak{n}+1:\mathbf{V}}_{Np_i+1,Nq_i+1,\tau_i;R:\mathbf{W}}\left(\begin{array}{c|c}\mathrm{t}^{\lambda_1}Z_1^N N^{NU_1} & (1\text{-}\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ . & . \\ . & . \\ \mathrm{t}^{\lambda_r}Z_r^N N^{NU_r} & (1\text{-}\beta-\rho-hl-g:\lambda_1,\cdots,\lambda_r),\Delta(B)\end{array}\right) \quad (3.3)$$

provided that

$Re(\beta)>0,\lambda_1>0,\cdots,\lambda_r>0,N\in\mathbb{N}^*$; $Re(\rho)+\dfrac{1}{N}\displaystyle\sum_{i=1}^r\lambda_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0;|c|<1$

$|argZ_k|<\dfrac{1}{2}A_i^{(k)}\pi$, $k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.4) and

$$U_i^{(k)}=\tau_i\sum_{j=1}^{p_i}\alpha_{ji}^{(k)}+\tau_{i^{(k)}}\sum_{j=1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\tau_{i^{(k)}}\sum_{j=1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}<0$$

Proof

To prove (3.3), we expand the generalized hypergeometric function ${}_uF_v[.]$ in serie with help of (1.10), the Gauss's hypergeometric function ${}_2F_1[.]$ in serie and we interchange the order of summations and z-integral (which is permissible under the conditions stated), see Carslaw ([2], page 173,eq(74.1)). Now we use the lemma 3 and we obtain the desired result.

Puting $t=1$ in (3.3), replacing the multivariable Aleph-function on the right hand side by its equivalent contour integral (1.1), changing the order of integration and summation, evaluating the inner summation with the help of Gauss's theorem [4, page 61], we obtain

Corollary 1

$$\int_0^1 z^{\rho-1}(1-z)^{\beta-1}\,{}_2F_1[\alpha,\tau;\beta:t-z]{}_uF_v\left[\begin{array}{c|c}(\mathrm{a}_u) & \\ . & \mathrm{c}z^l \\ (\mathrm{b}_v) & \end{array}\right]$$

$$\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{Z}_1 z^{\lambda_1/N} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r z^{\lambda_r/N} & \mathrm{B} \end{array}\right)\mathrm{d}z = (2\pi)^N N^{\mu+2}\ \Gamma(\beta)\sum_{h=0}^{\infty}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}$$

$$\aleph_{Np_i+2,Nq_i+2,\tau_i;R:\mathbf{W}}^{0,N\mathfrak{n}+2:\mathbf{V}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-hl:\lambda_1,\cdots,\lambda_r),(1-\rho-\beta+\alpha+\tau-hl:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r^N N^{NU_r} & (1\text{-}\beta-\rho+\alpha-hl:\lambda_1,\cdots,\lambda_r),(1-\beta-\rho+\tau-hl:\lambda_1,\cdots,\lambda_r),\Delta(B) \end{array}\right) \quad (3.4)$$

Replacing α by $-k$ (k is a positive integer), τ by $1+\alpha+\beta+k$, β by $1+\alpha$ in (3.4) and changing ${}_2F_1[.]$ into Jacobi polynomials, we have

Corollary 2

$$\int_0^1 z^{\rho-1}(1-z)^{\beta-1}(1-z)^{\alpha}P_k^{(\alpha,\beta)}(z)\ {}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right]\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{Z}_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r\left\{\frac{1+z}{2}\right\}^{\lambda_r/N} & \mathrm{B} \end{array}\right)\mathrm{d}z =$$

$$(2\pi)^N N^{\mu+2}\ 2^{\rho+\alpha}\frac{\Gamma(1+\alpha+k)}{k!}\sum_{h=0}^{\infty}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}$$

$$\aleph_{Np_i+2,Nq_i+2,\tau_i;R:\mathbf{W}}^{0,N\mathfrak{n}+2:\mathbf{V}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho+\beta-hl:\lambda_1,\cdots,\lambda_r),(1-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r^N N^{NU_r} & (\text{-}\alpha-\rho-l-hl:\lambda_1,\cdots,\lambda_r),(1+\beta-\rho+k-hl:\lambda_1,\cdots,\lambda_r),\Delta(B) \end{array}\right) \quad (3.5)$$

under the same conditions that (3.3)

4. Expansion formula

We shall establish the following expansion formula

$$(1+z)^{\rho-1}\ {}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right]\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{Z}_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r\left\{\frac{1+z}{2}\right\}^{\lambda_r/N} & \mathrm{B} \end{array}\right) = (2\pi)^N N^{\mu+2} 2^{\rho-1}$$

$$\sum_{h,s=0}^{\infty}\frac{(1+\alpha+\beta+2s)\Gamma(1+\alpha+\beta+s)}{\Gamma(1+\beta+s)}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}P_s^{(\alpha,\beta)}(z)$$

$$\aleph^{0,Nn+2:\mathbf{V}}_{Np_i+2,Nq_i+2,\tau_i;R:\mathbf{W}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-\beta-hl:\lambda_1,\cdots,\lambda_r),(1-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_r^N N^{NU_r} & (1+\mathrm{s}\text{-}\rho-hl:\lambda_1,\cdots,\lambda_r),(-s-\alpha-\beta-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(B) \end{array}\right) \quad (4.1)$$

provided

$$Re(\beta)>0,\lambda_1>0,\cdots,\lambda_r>0,N\in\mathbb{N}^*\,;Re(\rho)+\frac{1}{N}\sum_{i=1}^{r}\lambda_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0;|c|<1$$

$|argZ_k|<\frac{1}{2}A_i^{(k)}\pi$, $k=1,\cdots,r$ where $A_i^{(k)}$ is given in (1.4) and

$$U_i^{(k)}=\ \tau_i\sum_{j=1}^{p_i}\alpha_{ji}^{(k)}\ +\tau_{i^{(k)}}\sum_{j=1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}\ -\tau_{i^{(k)}}\sum_{j=1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}<0$$

Proof

Let

$$f(z)=\ (1+z)^{\rho-1}\,{}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right]\aleph^{0,\mathrm{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{Z}_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_r\left\{\frac{1+z}{2}\right\}^{\lambda_r/N} & \mathrm{B} \end{array}\right)=\sum_{r=0}^{\infty}A_rP_r^{(\alpha,\beta)}(z),-1<z<1 \quad (4.2)$$

The equation (4.2) is valid since $f(z)$ is continuous and bounded variation in the open interval $(-1,1)$ with $\rho\geqslant 1$. Multiply both sides of (4.2) by $(1-z)^\alpha(1+z)^\beta P_k^{(\alpha,\beta)}(z)$ and integrate with respect to z from -1 to 1, use the corollary 2 and the orthogonality property of Jacobi polynomials [5, page 285(5) and (9)], we obtain

$$A_k=(2\pi)^N N^{\mu+2}2^{\rho-1}\ \frac{(1+\alpha+\beta+2k)\Gamma(1+\alpha+\beta+k)}{\Gamma(1+\beta+k)}\ \ P_s^{(\alpha,\beta)}(z)\sum_{h=0}^{\infty}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}$$

$$\aleph^{0,Nn+2:\mathbf{V}}_{Np_i+2,Nq_i+2,\tau_i;R:\mathbf{W}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-\beta-hl:\lambda_1,\cdots,\lambda_r),(1-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_r^N N^{NU_r} & (1+\mathrm{k}\text{-}\rho-hl:\lambda_1,\cdots,\lambda_r),(-k-\alpha-\beta-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(B) \end{array}\right) \quad (4.3)$$

on substituting the value A_k from (4.3) in (4.2), we obtain the result (4.1).

5. Heat conduction

As an exemple of the application of this multivariable Aleph-function in applied mathematics we shall consider the problem of determining a function $\theta(z,t)$ representing the temperature in a non-homogeneous bar with ends at $X=\pm 1$ in which the thermal diffusivity is proportional to $(1-z^2)$ and if the lateral surface on the bar is insulated, it satisfies the partial differential equation of heat conduction see Churchill ([3], page 197,(8)).

$$\frac{\partial\theta}{\partial t}=b\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\theta}{\partial x}\right] \tag{5.1}$$

where b is a constant, provided thermal coefficient is constant. The boundary conditions of the problem are that both ends of a bar at $z=\pm 1$ are also insulated because the conductivity vanishes there ; and the initial conditions :

$$\theta(z,0)=f(z),-1<z<1 \tag{5.2}$$

The solution of (5.1) to obtained is

$$\theta(z,t)=\ (2\pi)^N N^{\mu+2}2^{\rho-1}\sum_{n,h=0}^{\infty}(2n+1)e^{-bn(n+1)t}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}P_n(z)$$

$$\aleph^{0,N\mathfrak{n}+2:\mathbf{V}}_{Np_i+2,Nq_i+2,\tau_i;R:\mathbf{W}}\left(\begin{array}{c|c} Z_1^N N^{NU_1} & (1\text{-}\rho-hl:\lambda_1,\cdots,\lambda_r),(1-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(A)\\ \cdot & \cdot\\ \cdot & \cdot\\ Z_r^N N^{NU_r} & (1\text{+n-}\rho-hl:\lambda_1,\cdots,\lambda_r),(-\rho-n-hl:\lambda_1,\cdots,\lambda_r),\Delta(B)\end{array}\right) \tag{5.3}$$

under the same conditions that (3.5) with $\alpha=\beta=0$

Proof of the problem

The solution of the problem can be written as Churchill [3, page 198 (8)].

$$\theta(z,t)=\sum_{n=0}^{\infty}R_n e^{-\lambda n(n+1)}P_n(z) \tag{5.4}$$

Because of the condition (5.2), the coefficient of R_n must be chosen to satisfy the relation

$$f(z)=\sum_{n=0}^{\infty}R_nP_n(z) \tag{5.5}$$

If we take

$$f(z)=\ (1+z)^{\rho-1}\ {}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right]\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} Z_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & A\\ \cdot & \cdot\\ \cdot & \cdot\\ Z_r\left\{\frac{1+z}{2}\right\}^{\lambda_r/N} & B\end{array}\right) \tag{5.6}$$

we get

$$\sum_{n=0}^{\infty}R_nP_n(z)=(1+z)^{\rho-1}\ {}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right]\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} Z_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & A\\ \cdot & \cdot\\ \cdot & \cdot\\ Z_r\left\{\frac{1+z}{2}\right\}^{\lambda_r/N} & B\end{array}\right) \tag{5.7}$$

Multiply both sides of (5.7) by $P_k(z)$ and integrating with respect to z between the limits -1 to 1, we have

$$\sum_{n=0}^{\infty} R_n \int_{-1}^{1} P_k(z)P_n(z)\mathrm{d}z = \int_{-1}^{1} (1+z)^{\rho-1} P_k(z)\ {}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right] \aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{Z}_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_r\left\{\frac{1+z}{2}\right\}^{\lambda_r/N} & \mathrm{B} \end{array}\right)\mathrm{d}z \quad (5.8)$$

Now use (3.5) with $\alpha = \beta = 0$ and orthogonality property of Legendre's polynomials [5, page 277, (13) and (14)], we get

$$R_n = (2\pi)^N N^{\mu+2} 2^{\rho} \frac{2k+1}{2} \sum_{h=0}^{\infty} \frac{[(a_u)]_h}{[(b_u)]_h} \frac{c^h t^{lh}}{h!} P_n(z)$$

$$\aleph^{0,N\mathfrak{n}+2:\mathbf{V}}_{Np_i+2,Nq_i+2,\tau_i;R:\mathbf{W}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho - hl:\lambda_1,\cdots,\lambda_r),(1-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_r^N N^{NU_r} & (1\text{+n-}\rho - hl:\lambda_1,\cdots,\lambda_r),(-\rho-n-hl:\lambda_1,\cdots,\lambda_r),\Delta(B) \end{array}\right) \quad (5.9)$$

Now with the help of (5.4) and (5.9), we obtain the result (5.3).

6. Aleph-function of two variables

If $r = 2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [8], we have :

Expansion formula

$$(1+z)^{\rho-1}\ {}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right] \aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{Z}_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_2\left\{\frac{1+z}{2}\right\}^{\lambda_2/N} & \mathrm{B} \end{array}\right) = (2\pi)^N N^{\mu+2} 2^{\rho-1}$$

$$\sum_{h,s=0}^{\infty} \frac{(1+\alpha+\beta+2s)\Gamma(1+\alpha+\beta+s)}{\Gamma(1+\beta+s)} \frac{[(a_u)]_h}{[(b_u)]_h} \frac{c^h t^{lh}}{h!} P_s^{(\alpha,\beta)}(z)$$

$$\aleph^{0,N\mathfrak{n}+2:\mathbf{V}}_{Np_i+2,Nq_i+2,\tau_i;R:\mathbf{W}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-\beta - hl:\lambda_1,\lambda_2),(1-\rho-hl:\lambda_1,\lambda_2),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{Z}_2^N N^{NU_2} & (1\text{+s-}\rho - hl:\lambda_1,\lambda_2),(-s-\alpha-\beta-\rho-hl:\lambda_1,\lambda_2),\Delta(B) \end{array}\right) \quad (6.1)$$

under the same conditions and notations that (4.1) with $r = 2$.

The solution of the heat conduction problem is

$$\theta(z,t) = (2\pi)^N N^{\mu+2} 2^{\rho-1} \sum_{n,h=0}^{\infty} (2n+1) e^{-bn(n+1)t} \frac{[(a_u)_h]}{[(b_u)_h]} \frac{c^h}{h!} P_n(z)$$

$$\aleph_{Np_i+2,Nq_i+2,\tau_i;R:\mathbf{W}}^{0,N\mathfrak{n}+2:\mathbf{V}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-hl:\lambda_1,\lambda_2),(1-\rho-hl:\lambda_1,\lambda_2),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_2^N N^{NU_2} & (1\text{+n-}\rho-hl:\lambda_1,\lambda_2),(-\rho-n-hl:\lambda_1,\lambda_2),\Delta(B) \end{array}\right) \tag{6.2}$$

under the same conditions and notations that (5.3) with $r=2$.

7. I-function of two variables

If $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$, the Aleph-function of two variables reduces to I-function of two variables defined by Sharma et al [7]. We have

Expansion formula

$$(1+z)^{\rho-1}\,{}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right]I_{p_i,q_i;R:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{Z}_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_2\left\{\frac{1+z}{2}\right\}^{\lambda_2/N} & \mathrm{B} \end{array}\right)=(2\pi)^N N^{\mu+2}2^{\rho-1}$$

$$\sum_{h,s=0}^{\infty}\frac{(1+\alpha+\beta+2s)\Gamma(1+\alpha+\beta+s)}{\Gamma(1+\beta+s)}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}P_s^{(\alpha,\beta)}(z)$$

$$I_{Np_i+2,Nq_i+2;R:\mathbf{W}}^{0,N\mathfrak{n}+2:\mathbf{V}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-\beta-hl:\lambda_1,\lambda_2),(1-\rho-hl:\lambda_1,\lambda_2),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_2^N N^{NU_2} & (1\text{+s-}\rho-hl:\lambda_1,\lambda_2),(-s-\alpha-\beta-\rho-hl:\lambda_1,\lambda_2),\Delta(B) \end{array}\right) \tag{7.1}$$

under the same conditions and notations that (4.1) with $r=2$ with $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$

The solution of the heat conduction problem is

$$\theta(z,t)=\;(2\pi)^N N^{\mu+2}2^{\rho-1}\sum_{n,h=0}^{\infty}(2n+1)e^{-bn(n+1)t}\frac{[(a_u)_h]}{[(b_u)_h]}\frac{c^h}{h!}P_n(z)$$

$$I_{Np_i+2,Nq_i+2;R:\mathbf{W}}^{0,N\mathfrak{n}+2:\mathbf{V}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-hl:\lambda_1,\lambda_2),(1-\rho-hl:\lambda_1,\lambda_2),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_2^N N^{NU_2} & (1\text{+n-}\rho-hl:\lambda_1,\lambda_2),(-\rho-n-hl:\lambda_1,\lambda_2),\Delta(B) \end{array}\right) \tag{7.2}$$

under the same conditions and notations that (5.3) with $r=2$ with $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$

8. Multivariable H-function

If the multivariable Aleph-function reduces to multivariable H-function defined by Srivastava et al [9,10], we have

Expansion formula

$$(1+z)^{\rho-1}\ {}_uF_v\left[c\left\{\frac{1+z}{2}\right\}^l\right]H^{0,\mathfrak{n}:V}_{p,q:W}\left(\begin{array}{c|c} \mathrm{Z}_1\left\{\frac{1+z}{2}\right\}^{\lambda_1/N} & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r\left\{\frac{1+z}{2}\right\}^{\lambda_r/N} & \mathrm{B} \end{array}\right) = (2\pi)^N N^{\mu+2}2^{\rho-1}$$

$$\sum_{h,s=0}^{\infty}\frac{(1+\alpha+\beta+2s)\Gamma(1+\alpha+\beta+s)}{\Gamma(1+\beta+s)}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}P_s^{(\alpha,\beta)}(z)$$

$$H^{0,N\mathfrak{n}+2:\mathbf{V}}_{Np+2,Nq+2:\mathbf{W}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-\beta-hl:\lambda_1,\cdots,\lambda_r),(1-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r^N N^{NU_r} & (1\text{+s-}\rho-hl:\lambda_1,\cdots,\lambda_r),(-s-\alpha-\beta-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(B) \end{array}\right) \quad (8.1)$$

under the same conditions and notations that (4.1) with $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$ and $R=R^{(1)}=\cdots,R^{(r)}=1$

The solution of the heat conduction problem is

$$\theta(z,t)=\ (2\pi)^N N^{\mu+2}2^{\rho-1}\sum_{n,h=0}^{\infty}(2n+1)e^{-bn(n+1)t}\frac{[(a_u)]_h}{[(b_u)]_h}\frac{c^h t^{lh}}{h!}P_n(z)$$

$$H^{0,N\mathfrak{n}+2:\mathbf{V}}_{Np+2,Nq+2:\mathbf{W}}\left(\begin{array}{c|c} \mathrm{Z}_1^N N^{NU_1} & (1\text{-}\rho-hl:\lambda_1,\cdots,\lambda_r),(1-\rho-hl:\lambda_1,\cdots,\lambda_r),\Delta(A) \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r^N N^{NU_r} & (1\text{+n-}\rho-hl:\lambda_1,\cdots,\lambda_r),(-\rho-n-hl:\lambda_1,\cdots,\lambda_r),\Delta(B) \end{array}\right)$$

under the same conditions and notations that (5.3) with $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$ and $R=R^{(1)}=\cdots,R^{(r)}=1$.

8. Conclusion

Specializing the parameters of the Aleph-function and the multivariable we can obtain a large number of results involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics, in particular, the problem of heat conduction in non homogeneous bar and the expansion formula. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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