Some Fixed Point Results in B-Metric Spaces

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Abstract:
In this paper we have established a fixed point theorem for a generalized Kannan and Chatterjee type contractive mapping in a b-metric spaces. Our result generalizes the result of Kir and Kiziltune[10].

1.0 Introduction:

The concept of b-metic pace was introduced by Bakhtin [2] and Czerwik [7](1993). This generalization of metric space was done to deal some problems particularly the problem of the convergence of measurable functions with respect of the measure. After Czerwik ([7],[8]) many papers have been published containing fixed point results on b-metric spaces for single value and multivalued functions ([1], [2], [4],[6],[7],[8], etc.

The extension of Banach's fixed point theorem for contractive mappings has been done in many directions. Banach's contraction mapping theorem states that if the metric space (X,d) is complete then the contraction mapping T(Say) has a unique fixed point. We know that every contraction mapping is uniformly continuous and hence continuous. A natural question arises whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

Kannan [9] established the following result in which above question has been answered in the affirmative.

If $T:X\rightarrow X$ where (X,d) is complete metric space satisfy the inequality:

$$d(Tx,Ty) \leq a[d(x,Tx),d(y,Ty)] \qquad (1.0.1)$$

where $a\in [0,½)$ and $x, y\in X$. Then T has unique fixed point. The mapping (1.0.1) is called Kannan type mapping.

A similar contractive conditions has been introduced by Chatterjee [6] as follows:

If $T:X\rightarrow X$, where (X,d) is a complete metric space, satisfies the inequality

$$d(Tx,Ty) \leq b[d(x,Ty)+d(x,Tx)] \qquad (1.0.2)$$

where $b\in [0,½)$ and $x,y\epsilon X$, then T has a unique fixed point. The mapping satisfying (1.0.2) is called Chatterjee type (or c-type) mapping.

Later on Moradi and Alimohammad [11] extended the Kannan type mapping in complete metric space and generalized metric spaces as follows:

Let (X,d) be a complete metric space and $T,S:S\rightarrow X$ be mappings such that T is continuous, one to one and subsequentially converge nt. If $\lambda \in [0,½)$ and

$$d(TSx,TSy) \leq \lambda\, [d(Tx,TSx)+d(Ty,TSx)\ ,\ (x,y \epsilon X) \qquad (1.0.3)$$

then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0\epsilon X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point and also establish the following:

Let (X,d) be a complete generalized metric space and $T,S:X\rightarrow X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\lambda \in [0,½)$ and

$$d(TSx,TSy) \leq \lambda\, [d(Tx,TSy)+d(Ty,TSx)\ ,\ (x,y \epsilon X) \qquad (1.0.4)$$

then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0\epsilon X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

In this paper we have extended idea of Moradi and Alimohammadi [11] to b-metric space and also extended the Chatterjee type (C-type) mappings in b-metric space analogue to Moradi and Alimohammadi [11].

1.1 Preliminaries

Definition (1.1.1): Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : XxX \rightarrow R_+$ is said to be a b-metric on X if the following conditions hold:

(i) d(x, y) = 0 if and only if x = y;
(ii) d(x, y) = d(y; x) for all x; y ϵ X;
(iii) d(x, y) ≤ s (d(x; z) + d(z; y)) for all x; y; z ϵ X.
The pair (X; d) is called a b-metric space.

We observe that if s = 1, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when s > 1. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example (1.1.2): Let X = {1, 0, - 1}. Define d : X x X → R^+ by d(x, y) = d(y, x) for all x, y ϵ X; d(x, x) = 0; x ϵ X and d(-1, 0) = 3; d(-1, 1) = d(0,1) = 1. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0):

It is easy to verify that s = $^3/_2$.

Definition (3.1.6): Let (X, d) be a b-metric space, x ϵ X and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if d(x_n, x) = 0. We denote this by $\lim_{n\to\infty} x_n = x \; as \; x_n \to x (n \to \infty)$ x_n .
(ii) (x_n) is Cauchy if and only if $\lim_{m,n\to\infty} d(x_n, x_m) = 0$
(iii) (X; d) is complete if and only if every Cauchy sequence in X is convergent.

Proposition (1.1.7) :(See, remark 3.1 in [4]) In a b−metric space (X, d) the following assertions hold:
(i) a convergent sequence has a unique limit,
(ii) each convergent sequence is Cauchy,
(iii)in general, a b−metric is not continuous.

In general a b−metric function d for k > 1 is not jointly continuous in all of its two variables. Following is an example of a b−metric which is not continuous.
Example. (1.1.8) Let X = N U{∞} and D : X × X → R defined by

$$D\,(m,n) = \begin{cases} 0, \; if \; m = n \\ \left|\frac{1}{m} - \frac{1}{n}\right|, \; if \; m, n \; are \; even \; or \; mn = \infty \\ 5, \; if \; m, n \; are \; odd \; and \; m \neq n \\ 2, \; otherwise \end{cases}$$

Then it is easy to see that for all m, n, p ϵ X, we have
D(m, p) ≤ 3(D(m, n) + D(n, p)).
Thus, (X,D) is a b−metric space with k = 3. If $x_n = 2_n$, for each n ϵ N, then
D(2n, ∞) = $\to 0 \; as \; n \to \infty$
that is $x_n \to \infty \; but \; D\,(x_n, 1) = 2 \nrightarrow D\,(\infty, 1) \; as \; n \to \infty$

Definition (1.1.9) : Let (X, d) be a b−metric space. If Y is a nonempty subset of X, then the closure Y of Y is the set of limits of all convergent sequences of points in Y , i.e.,

Y = {x ϵ X : there exists a sequence {x_n} in Y such that $\lim_{n\to\infty} x_n$ = x}

Definition. (1.1.10): Let (X, d) be a b−metric space. Then a subset Y ⊂ X is called closed if and only if for each sequence {x_n} in Y which converges to an element x, we have x ϵ Y (i.e., Y = Ȳ).

Definition (1.1.11): Let (X, d) be a b-metric space and let T : X →X be a given mapping. We say that T is continuous at x_0 ϵ X if for every sequence (x_n) in X, we have $x_n \to x_0 \; as \; n \to \infty \Rightarrow T(x_n) \longrightarrow T(x_0) \; as \; n \to \infty$.
If T is continuous at each point x0 2 X, then we say that T is continuous on X.

1.2 Main Results:

Theorem (1.2.1): Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $T,S:X \rightarrow X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\lambda \in [0,½)$ such that

$$d(TSx,TSy) \leq \lambda [d(Tx,TSx)+d(Ty,TSx)] , (x,y \in X) \qquad (1.2.1)$$

then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$, the sequence of Iterates $\{S^n x_0\}$ converges to this fixed point.

Proof: Let x_0 be an arbitrary point in X. we define sequence $\{x_n\}$ as $x_{n+1}=Sx_n$: n=1,2,…………

Then by using the given condition (1.2.1) we have:

$$d(Tx_n,Tx_{n+1}) = d(TSx_{n-1},TSx_n)$$
$$\leq \lambda [d(Tx_{n-1},TSx_{n-1})+d(Tx_n,TSx_n)]$$

or, $d(Tx_n,Tx_{n+1}) \leq \lambda [d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})]$

or, $d(Tx_n,Tx_{n+1}) \leq \left(\frac{\lambda}{1-\lambda}\right) d(Tx_{n-1},Tx_n)$

similarly,

$$d(Tx_{n+1},Tx_{n+2}) \leq \left(\frac{\lambda}{1-\lambda}\right) d(Tx_n,Tx_{n+1})$$
$$\leq \left(\frac{\lambda}{1-\lambda}\right)^2 d(Tx_{n-1},Tx_n)$$
$$\vdots$$
$$\leq \left(\frac{\lambda}{1-\lambda}\right)^{n+1} d(Tx_0,Tx_1)$$
$$= (p)^{n+1} d(Tx_0,Tx_1) \text{ where } p=\left(\frac{\lambda}{1-\lambda}\right).$$

Now we shall show that $\{Tx_n\}$ is a Cauchy sequence.

For this let m,n>0 with m>n, then

$$d(Tx_m,Tx_n) \leq sd(Tx_n,Tx_{n+1})+s^2d(Tx_{n+1},Tx_{n+2})+s^3d(Tx_{n+2},Tx_{n+3})+\ldots\ldots\ldots\ldots+s^m d(Tx_{m-1},Tx_m)$$
$$\leq sp^n d(Tx_0,Tx_1)+s^2p^{n+1}d(Tx_0,Tx_1)+ \ldots\ldots\ldots\ldots+s^m p^{n+m-1}d(Tx_0,Tx_1)$$
$$\leq sp^n d(Tx_0,Tx_1)[1+sp+(sp)^2+ \ldots\ldots\ldots\ldots+(sp)^{m-1}]$$
$$\leq sp^n d(Tx_0,Tx_1)\left[\frac{1-(sp)^{n-(m-1)}}{1-sp}\right] \qquad (1.2.2)$$

Letting $m,n \rightarrow \infty$ in (1.2.2), we have $\{Tx_n\}$ is a Cauchy sequence and since X is a complete b-metric space there exists $v \in X$ such that:

$$\lim_{n\rightarrow\infty} Tx_n = v \qquad (1.2.3)$$

Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So there exists $u \in X$ and $\{x_{n(k)}\}_{k=1}^{\infty}$ such that $\lim_{n\rightarrow\infty} Tx_{n(k)} = Tu$

By (1.2.3) we have Tu=v. So,

$$d(TSu,Tu) \leq s[d(TSu,TS^{n(k)}x_0) + d(TS^{n(k)}x_0,Tu)]$$
$$\leq s[d(TSu,TS^{n(k)}x_0) + s[d(TS^{n(k)}x_0,Tu)]$$
$$\leq s\lambda[d(Tu,TSu) + d(TS^{n(k)-1}x_0,TS^{n(k)}x_0)] + s[d(TS^{n(k)}x_0,Tu)]$$

$$\Rightarrow d(TSu,Tu) \leq \left(\frac{s\lambda}{1-s\lambda}\right) p^{n(k)-1} d(Tx_0,Tx_1) + \left(\frac{s}{1-s\lambda}\right) d(Tx_{n(k)},Tu) \qquad (1.2.4)$$

Letting k→ ∞ in (1.2.4), we get d(TSu,Tu)=0.
Since T is one to one Su=u. So u is a fixed point.
Uniqueness: If possible let u and v be two fixed point of S. then Su=u and Sv=v where u ≠ v.
Now,

$$\begin{aligned} d(u,v)=d(TSu,TSv) &\leq \lambda\,[d(Tu,TSu)+d(Tv,TSv)] \\ &= \lambda\,[d(Tu,Tu)+d(Tv,Tv)] \\ &= 0 \end{aligned}$$

i.e. d(TSu,TSv) = 0, which gives Su=Sv, as T is one to one and u=v. Hence our supposition is wrong and u=v i.e. S has a unique fixed point.

Further if T is sequentially convergent, we have $\lim_{n\to\infty} x_n = u$ and this shows that $\{s_n\}$ converges to the fixed point of S.

Corollary (1.2.2) Theorem 2 of [10]: let (X,d) be a complete b-metric space with constant s ≥ 1 and define the sequence $\{x_n\}_{n=1}$ as $x_n = Tx_{n-1} = T^n x_0$, n= 1,2,…….. Let T:X→ X be a mapping for which there exists λ ϵ [0,½) such that

$$d(Tx,Ty) \leq \lambda\,[d(x,Tx)+d(y,Ty)] \text{ , for all } x,y\epsilon X. \qquad (1.2.5)$$

Then there exists xϵX such that x_n→X such that $x_n \to x^*$ and x^* is unique fixed point of T.

Proof: By putting T=I, identity map in theorem (1.2.1) we get:

$$d(Sx,Sy) \leq \lambda\,[d(x,Sx)+d(y,Sy)]$$

and the hence the corollary follows.
Remarks: By above it is clear that the theorem (1.2.1) proved above is more general than the result established by Kir and Kizilthune [10].
Our next theorem is about Chatterjee type contraction in b-metric spaces.
Theorem (1.2.3): Let (X,d) be a complete b-metric space and T,S:X→X be mappings such that T is continuous, one to one and subsequentially convergent. If λ ϵ [0,½), s ≥ 1 satisfying the condition

$$d(TSx,TSy) \leq \lambda\,[d(Tx,TSy)+d(Ty,TSx)] \qquad (1.2.6)$$

for all x,yϵX.
Then S has a unique fixed point. Also if T is sequentially convergent then for every x_0ϵX, the sequence of iterates $\{s^n x_0\}$ converges to this fixed point.
Proof: Let x_0 be an arbitrary point in X. We define the sequence $\{x_n\}$ by $s_{n+1}=sx_n$ (or, equivalently $x_n=\{s^n x_0\}$, n= 1,2,3,……..
Then using the given condition (1) we have:

$$\begin{aligned} d(Tx_n,Tx_{n+1}) &= d(TSx_{n-1},TSx_n) \\ &\leq \lambda\,[d(Tx_{n-1},TSx_n)+d(Tx_n,TSx_{n-1})] \end{aligned}$$

$$\begin{aligned} \text{Or, } d(Tx_n,Tx_{n+1}) &= \lambda\,[d(Tx_{n-1},Tx_{n+1})+d(Tx_n,Tx_n)] \\ &\leq \lambda s\,[d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})] \end{aligned}$$

$$\text{Or, } d(Tx_n,Tx_{n+1}) \leq \frac{\lambda s}{1-\lambda s} d(Tx_{n-1},Tx_n)$$

Similarly,

$$\begin{aligned} d(Tx_n,Tx_{n+1}) &\leq \left(\frac{\lambda s}{1-\lambda s}\right) d(Tx_n,Tx_{n+1}) \\ &\leq \left(\frac{\lambda s}{1-\lambda s}\right)^2 d(Tx_{n-1},Tx_n) \\ &\vdots \\ &\leq \left(\frac{\lambda s}{1-\lambda s}\right)^{n+1} d(Tx_0,Tx_1) \end{aligned}$$

$= p^{n+1} d(Tx_0, Tx_1)$ where p= $\left(\dfrac{\lambda s}{1-\lambda s}\right)$

Now we shall show that {Tx_n} is a cauchy sequence.
For this let m,n>0 with m>n, then

$d(Tx_m, Tx_n) \leq sd(Tx_n, Tx_{n+1}) + s^2 d(Tx_{n+1}, Tx_{n+2}) + s^3 d(Tx_{n+2}, Tx_{n+3}) + \ldots\ldots.. + s^m d(Tx_{m-1}, Tx_m)$

$\leq sp^n d(Tx_0, Tx_1) + s^2 p^{n+1} d(Tx_0, Tx_1) + s^3 p^{n+2} d(Tx_0, Tx_1) + \ldots\ldots.. + s^m p^{n+m-1} d(Tx_0, Tx_1)$

$\leq sp^n d(Tx_0, Tx_1)[1 + sp + (sp)^2 + \ldots\ldots.(sp)^{m-1}]$

$$\leq sp^n d(Tx_0, Tx_1)\left[\frac{1-(sk)^{n-m-1}}{1-sk}\right] \qquad (1.2.6)$$

Letting m,n→ ∞ in (2), we have {Tx_n} is a Cauchy sequence and since X is a complete b-metric space, there exists vϵX such that

$$\lim_{n\to\infty} Tx_n = v \qquad (1.2.7)$$

Since T is subsequentially convergent {s_n} has a convergent subsequence, so there exists uϵX and $\{Tx_{n(k)}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} Tx_{n(k)} = Tu$.
By (3) we have Tu=v, so,

$d(TSu, Tu) \leq s[d(TSu, TS^{n(k)}x_0) + d(TS^{n(k)}x_0, Tu)]$

$\leq s[d(TSu, TS^{n(k)}x_0)] + s[d(TS^{n(k)}x_0, Tu)]$

$$\leq s\lambda[d(Tu, TS^{n(k)}x_0) + d(TS^{n(k)-1}x_0, TSu)] + sd(TS^{n(k)}x_0, Tu) \qquad (1.2.8)$$

Letting k→∞ in (1.2.8) we have

$d(TSu, Tu) \leq s\lambda[d(Tu, Tu) + d(Tu, TSu)] + sd(Tu, Tu)$

$d(TSu, Tu) \leq s\lambda d(Tu, TSu)$, which is a contradiction and hence $d(TSu, Tu) = 0$ Thus TSu=Tu i.e. Su=u

Uniqueness: If possible let u anv v be two fixed point of S, then Su=u and Sv=v where u ≠ v.
Now,

$d(u, v) = d(Tu, Tv) = d(TSu, TSv) \leq \lambda[d(Tu, TSv) + d(Tv, TSu)]$

$= \lambda[d(Tu, Tv) + d(Tv, Tu)]$

$= 2\lambda d(Tu, Tv)$

i.e. $d(Tu, Tv) \leq 2\lambda d(Tu, Tv)$, which is only possible when d(Tu,Tv)=0, which gives u=v as T is one to one. Hence S has a unique fixed point.

Further if T is sequentially convergent, we have $\lim_{n\to\infty} x_n = u$ and this shows that {x_n} converges to the fixed point of S.

Corollary (1.2.4): Theorem 3 of [10]: Let (X,d) be a complete b-metric space and define the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ as $x_n = Tx_{n-1} = T^n x_0$, n=1,2,3, …….. Let T:X→X be a mapping under the terms λ ϵ [0,½), such that:

$$d(Tx, Ty) \leq \lambda[d(x, Ty) + d(y, Tx)] \text{ for all } x,y \in X \qquad (1.2.9)$$

Then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is the unique fixed point of T.
Proof: By putting T=I in theorem (3.2.3) we have get

$$d(Sx, Sy) \leq \lambda[d(x, Sy) + d(y, Sx)]$$

and hence the corollary follow.
Remark (1.2.5): The theorem (1.2.3) proved above is more general than that the theorem (3) proved by Kir & Kiziltune [10]

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