Application of special functions in one dimensional advective diffusion problem

$F.Y.\ AYANT^{1}$

1 Teacher in High School , France

ABSTRACT
In the present paper, we construct an one dimensional advective diffusion model problem to evaluate the substance concentration distribution along $x-$axis between $x=-1$ and $x=1$. Here, the diffusivity of the substance and the velocity of the solvent flow are both considered as a variable. Kumar et al [4] use the product of the class of multivariable polynomials and the multivariable H-function defined by Srivastava et al [11,12] to obtain the analytic solution. Then, we employ the product of the generalized hypergeometric function, class of multivariable polynomials and the multivariable Aleph-function to obtain an analytic formula of our problem. Finally, some particular cases will be given.

Keywords :Multivariable Aleph-function, , classes of multivariable polynomials, generalized hypergeometric function, advective-diffusion model problem, expansion formula, multivariable H-function., Aleph-function of two variables.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

An advective diffusion is a process where solute particles in a solvent are diffused and transported with the flow of solvent. If $C(x,t)$ represents be the concentration of the diffusing substance, D denotes the diffusivity of the substance, $\mathbf{u}$ is the velocity of the solvent flow and $F(x,t)$ is the externel source function, then the governing advective-diffusion equation is given by (see Harrison and Perry [3], Hanna et al [2], Lyons et al [5])

$$\frac{\partial C}{\partial t}+div(C\mathbf{u})=div(D\nabla C)+F \tag{1.1}$$

Here, we construct an one dimensional advective-diffusion model problem in which the diffusivity of the substance is considered as a variable of position x and the velocity of the solvent flow is supposed to be function of x. There is no generation or absorption of the solute in the solvent.

The class of multivariable polynomials defined by Srivastava [10], is given in the following manner :

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[y_1,\cdots,y_u]=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}A[N_1,K_1;\cdots;N_u,K_u]y_1^{K_1}\cdots y_u^{K_u} \tag{1.2}$$

The multivariable Aleph-function is an extension of the multivariable I-function recently defined by C.K. Sharma and Ahmad [7] , itself is a generalization of the multivariable H-function defined by Srivastava et al [11,12]. The multivariable Aleph-function is defined by means of the multiple contour integral :

We have : $\aleph(z_1,\cdots,z_r)=\aleph_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,q_i}]:$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ [(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.3}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.4}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.5}$$

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\ \text{with}\ k=1,\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)} \tag{1.6}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,)\,,max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,)\,,min(\,|z_1|,\cdots,|z_r|\,)\rightarrow \infty$$

where $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k$$

For convenience, we will use the following notations in this paper.

$$V=m_1,n_1;\cdots;m_r,n_r \tag{1.7}$$

$$\mathrm{W}=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.8}$$

$$A=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\}:\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\}$$

$$\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.9}$$

$$B = \{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\}\text{: } \{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}\ ;\cdots;$$

$$\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.10}$$

$$B' = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}A[N_1,K_1;\cdots;N_u,K_u] \tag{1.11}$$

2. Statement of the problem and governing equation

Let us consider an advective-diffusion process in which the solute particles in a solvent are diffused and transported with the variable velocity $\mathbf{u}$ of the solvent flow in the direction of the $x-$axis between the limits $x=-1$ to $x=1$ and that is supposed to be a function of position and proportional to $[1+\alpha-\gamma-\beta+(\alpha+\beta-\gamma+1)x]$, $Re(\alpha-\gamma)>-2$ $Re(\beta)>-1$, the diffusivity D of the substance (solute) is a function of position and is proportional to $(1-x^2)$. The externel source function is absent in the process that is there is no generation or absorption of the solute in the solvent. The desired function $C(x,t)$ represents the concentration distribution of the diffusing substance at the position x and at the time t. Then due to (1.1), we get an one dimensional advective-diffusion equation in the form given below :

$$\frac{\partial C}{\partial t}+[\lambda_1(\alpha-\gamma-\beta+1)+(\alpha+\beta-\gamma+1)x]+2x\lambda_2]\frac{\partial C}{\partial x}+\lambda_1(\alpha+\beta-\gamma+1)C-\lambda_2(1-x^2)\frac{\partial^2 C}{\partial x^2}=0 \tag{2.1}$$

where λ_1 and λ_2 are arbitrary non zero proportionality constants, $Re(\alpha-\gamma)>-2$ and $Re(\beta)>-1, t>0$ and $-1\leqslant x\leqslant 1$.

Let us suppose that the initial concentration of the solute in the solvent is given by

$$C(x,0)=f(x), -1\leqslant x\leqslant 1 \tag{2.2}$$

In (2.1), for simplicity, letting λ_1 and λ_2 , we get

$$\frac{1}{\lambda_1}\frac{\partial C}{\partial t}=(1-x^2)\frac{\partial^2 C}{\partial x^2}+[\beta+\gamma-\alpha-1-(\alpha+\beta-\gamma+3)x]\frac{\partial C}{\partial x}-(\alpha+\beta-\gamma+1)C \tag{2.3}$$

where $\lambda_1\neq 0,\ Re(\alpha-\gamma)>-2, Re(\beta)>-1, t>0$ where $-1\leqslant x\leqslant 1$

3. Main integral

In our work, we also require the following integral

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_Q^{(\mu,\upsilon)}(x)\,{}_MF_N\left[\begin{array}{c|c}(\mathrm{e}_M) & \\ . & \mathrm{y(1\text{-}x)}^{g}(1+x)^{w}\\ (\mathrm{f}_N) & \end{array}\right]$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\begin{pmatrix}\mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1}\\ \cdots\\ \cdots\\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u}\end{pmatrix}\aleph\begin{pmatrix}\mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdots\\ \cdots\\ \mathrm{Z}_r(1-x)^{h_r}(1+x)^{k_r}\end{pmatrix}\mathrm{d}x=$$

$$\frac{\Gamma(1+\mu+Q)2^{\rho+\sigma+1}}{\Gamma(1+\mu)\Gamma(1+Q)}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty}\sum_{k=0}^{Q}A'B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!}$$

$$\frac{(-Q)_k(1+\mu+\upsilon+Q)_k}{(1+\mu)_k k!}\, y_1^{K_1}\cdots y_u^{K_u} 2^{gq+wq+\sum_{i=1}^u (f_i+w_i)K_i}$$

$$\aleph^{0,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{Z}_1 2^{h_1+k_1} & (\text{-}\rho-k-gq-\sum_{i=1}^u f_iK_i : h_1,\cdots,h_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot \\ \mathrm{Z}_r 2^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\sigma-wq-\sum_{i=1}^u w_iK_i : k_1,\cdots,k_r), A \\ \cdot \\ \cdot \\ (\text{-1-k-}\sigma-\rho-(g+w)q-\sum_{i=1}^u (f_i+w_i)K_i : h_1+k_1,\cdots,h_r+k_r), B \end{array}\right) \quad (3.1)$$

Provided that

$min\{g,w,f_i,w_i,h_l,k_l\} > 0$ for $i=1,\cdots,u; l=1,\cdots,r$

$$Re\left(\rho+\sum_{i=1}^u K_if_i\right)+\sum_{i=1}^r h_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1 \text{ and}$$

$$Re\left(\sigma+\sum_{i=1}^u K_iw_i\right)+\sum_{i=1}^r k_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1; Re(\mu)>-1, Re(\upsilon)>-1$$

$|argZ_k| < \frac{1}{2^{h_i+k_i+1}}A_i^{(k)}\pi, i=1,\cdots,r$, where $A_i^{(k)}$ is defined by (1.6). $|y|<1$

Proof

To prove (3.1), first expressing the Jacobi polynomial $P_Q^{(\mu,\upsilon)}(.)$ in series , a class of multivariable polynomials defined by Srivastava $S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[.]$ in series with the help of (1.2), expressing the generalized hypergeometric in series and we interchange the order of summations and x-integral (which is permissible under the conditions stated). Expressing the $\aleph$-function of r-variables in Mellin-Barnes contour integral with the help of (1.3) and interchange the order of integrations which is justifiable due to absolute convergence of the integrals involved in the process. Now collecting the powers of $(1-x)$ and $(1+x)$ and evaluating the inner x-integral. Now, interpreting the Mellin-Barnes contour integral in multivariable $\aleph$-function, we obtain the desired result (3.1).

4. Solution of the problem and analysis

To solve the differential equation (2.3), we suppose that $C(x,t)=X(x)T(t)$ in it, we get

$$\frac{1}{\lambda_1 T}\frac{dC}{dt}=\frac{1}{X}(1-x^2)\frac{d^2X}{dx^2}+[\beta+\gamma-\alpha-1-(\alpha+\beta-\gamma+3)x]\frac{dX}{dx}-(\alpha+\beta-\gamma+1)X$$

$$=-(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon) \quad (4.1)$$

and

$$\frac{dT}{dt} = -\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)T \tag{4.2}$$

The solution of the equation (4.1) is $P_{\upsilon}^{(\alpha+\gamma+1,\beta)}(x)$ (see 7 page 258). Thus by (4.1) and (4.2), we obtain

$$X(x) = A_1 P_{\upsilon}^{(\alpha+\gamma+1,\beta)}(x), \, -1 \leqslant x \leqslant 1 \tag{4.3}$$

$$T(t) = A_2 exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big], t \geqslant 0 \tag{4.4}$$

The general solution of the equation (2.3) is :

$$C(x,t) = \sum_{\upsilon=0}^{\infty} A_{\upsilon} P_{\upsilon}^{(\alpha+\gamma+1,\beta)}(x) exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big], t \geqslant 0 \text{ and } -1 \leqslant x \leqslant 1 \tag{4.5}$$

Now making an appeal to (2.2) and taking the expression of (4.5) at $t = 0$, we have

$$f(x) = \sum_{\upsilon=0}^{\infty} A_{\upsilon} P_{\upsilon}^{(\alpha+\gamma+1,\beta)}(x) \tag{4.6}$$

Then to evaluate A_{υ}; multiply both sides of (4.6) by $(1-x)^{\alpha-\gamma+1}(1+x)^{\beta} P_{\mu}^{(\alpha-\gamma+1,\beta)}(x)$ and then integrate the two sides of (4.6) with respect x from -1 to 1 and use the orthogonality of Jacobi polynomials (see, [6, page 258]) therein, we get

$$A_{\upsilon} = \frac{2^{\gamma-\alpha-\beta-2}\Gamma(\upsilon+1)\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha+\beta-\gamma+\upsilon+2)}{\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha+\gamma+\upsilon+2)\Gamma(\beta+\upsilon+1)}$$

$$\int_{-1}^{1}(1-x)^{\alpha-\gamma+1}(1+x)^{\beta} P_{\upsilon}^{(\alpha+\gamma+1,\beta)}(x) f(x)\mathrm{d}x \tag{4.7}$$

$Re(\alpha-\gamma) > -2, Re(\beta) > -1, \forall \upsilon \in \mathbb{N}.$

Finally, with the aid of (4.5) and (4.7), we obtain the concentration distribution of the solute in the form

$$C(x,t) = 2^{\gamma-\alpha-\beta-2}\sum_{\upsilon=0}^{\infty}\frac{\Gamma(\upsilon+1)\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha+\beta-\gamma+\upsilon+2)}{\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha+\gamma+\upsilon+2)\Gamma(\beta+\upsilon+1)}$$

$$P_{\upsilon}^{(\alpha+\gamma+1,\beta)}(x) exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big] \int_{-1}^{1}(1-x)^{\alpha-\gamma+1}(1+x)^{\beta} P_{\upsilon}^{(\alpha+\gamma+1,\beta)}(x) f(x)\mathrm{d}x \tag{4.8}$$

provided $Re(\alpha-\gamma) > -2, Re(\beta) > -1, t > 0$ and $-1 \leqslant x \leqslant 1$

5.Example

Let the initial concentration of the solute in the solvent is given by

$$f(x) = (1-x)^{\rho}(1+x)^{\sigma} {}_M F_N \left[\begin{array}{c|c} (\mathrm{e}_M) & \\ . & \mathrm{y(1\text{-}x)}^{g}(1+x)^{w} \\ (\mathrm{f}_N) & \end{array} \right]$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c} \mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1} \\ \cdots \\ \cdots \\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u} \end{array}\right)\aleph\left(\begin{array}{c} \mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdots \\ \cdots \\ \mathrm{Z}_r(1-x)^{h_r}(1+x)^{k_r} \end{array}\right) \tag{5.1}$$

Substituting the value of $f(x)$ in (4.8), we obtain

$$C(x,t) = 2^{\rho+\sigma}\sum_{v=0}^{\infty}\sum_{k=0}^{v}\frac{\Gamma(\alpha+\beta-\gamma+2v+3)\Gamma(\alpha+\beta-\gamma+v+2+k)}{\Gamma(\alpha+\beta-\gamma+2v+2)\Gamma(\alpha-\gamma+k+2)}\frac{(-v)_k}{\Gamma(\beta+v+1)k!}$$

$$P_v^{(\alpha+\gamma+1,\beta)}(x)exp\big[\lambda_1(v+1)(1+\alpha+\beta-\gamma+v)t\big]F^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) \tag{5.2}$$

where

$$F^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty}B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!}\; y_1^{K_1}\cdots y_u^{K_u}2^{gq+wq+\sum_{i=1}^u(f_i+w_i)K_i}$$

$$\aleph_{p_i+2,q_i+1,\tau_i;R:W}^{0,\mathbf{n}+2:V}\left(\begin{array}{c} \mathrm{Z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_r 2^{h_r+k_r} \end{array}\middle| \begin{array}{c} (\gamma-\rho-\alpha-1-k-gq-\sum_{i=1}^u f_iK_i:h_1,\cdots,h_r), \\ \cdot \\ \cdot \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\sigma-\beta-wq-\sum_{i=1}^u w_iK_i:k_1,\cdots,k_r),A \\ \cdot \\ \cdot \\ (\gamma-2-k-\sigma-\rho-\alpha-\beta-(g+w)q-\sum_{i=1}^u(f_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r),B \end{array}\right) \tag{5.3}$$

Provided that

$min\{g,w,f_i,w_i,h_l,k_l\}>0$ for $i=1,\cdots,u;l=1,\cdots,r$

$$Re\left(\rho+\alpha-\gamma+\sum_{i=1}^u K_if_i\right)+\sum_{i=1}^r h_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1 \text{ and}$$

$$Re\left(\sigma+\beta+\sum_{i=1}^u K_iw_i\right)+\sum_{i=1}^r k_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1; Re(\beta)>-1$$

$|argZ_k| < \frac{1}{2^{h_i+k_i+1}}A_i^{(k)}\pi, i=1,\cdots,r$, where $A_i^{(k)}$ is defined by (1.6). $|y|<1$

6. Particular cases

(ii) If $\gamma=\alpha+\beta+1$ in (2.3), then the solvent moves with the constant velocity in the direction of $\mathbf{u}$ and the the equation (2.3) reduces to

$$\frac{1}{\lambda_1}\frac{\partial C}{\partial t} = (1-x^2)\frac{\partial^2 C}{\partial x^2} + 2(\beta - x)\frac{\partial C}{\partial x} \tag{6.1}$$

and the solution is

$$C(x,t) = 2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{k=0}^{\upsilon}\frac{\Gamma(2\upsilon+2)\Gamma(\upsilon+1+k)}{\Gamma(2\upsilon+1)\Gamma(1-\beta+k)}\frac{(-\upsilon)_k}{\Gamma(\beta+\upsilon+1)k!}$$

$$P_{\upsilon}^{(-\beta,\beta)}(x)exp\left[\lambda_1(\upsilon+1)\upsilon t\right]F^{\alpha,\beta,\alpha+1,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) \tag{6.2}$$

where

$$F^{\alpha,\beta,\alpha+1,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty}B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!}\ y_1^{K_1}\cdots y_u^{K_u}2^{gq+wq+\sum_{i=1}^{u}(f_i+w_i)K_i}$$

$$\aleph_{p_i+2,q_i+1,\tau_i;R:W}^{0,\mathbf{n}+2:V}\left(\begin{array}{c|c} \mathrm{Z}_1 2^{h_1+k_1} & (\beta-\rho-k-gq-\sum_{i=1}^{u}f_iK_i:h_1,\cdots,h_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdots \\ \mathrm{Z}_r 2^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\sigma-\beta-wq-\sum_{i=1}^{u}w_iK_i:k_1,\cdots,k_r),A \\ \cdot \\ \cdot \\ (\text{-k-1-}\sigma-\rho-(g+w)q-\sum_{i=1}^{u}(f_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r),B \end{array}\right) \tag{6.3}$$

Provided that

$min\{g,w,f_i,w_i,h_l,k_l\} > 0$ for $i=1,\cdots,u; l=1,\cdots,r$

$Re\left(\rho-\beta+\sum_{i=1}^{u}K_if_i\right)+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$ and

$Re\left(\sigma+\beta+\sum_{i=1}^{u}K_iw_i\right)+\sum_{i=1}^{r}k_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$

$|argZ_k| < \frac{1}{2^{h_i+k_i+1}}A_i^{(k)}\pi, i=1,\cdots,r$, where $A_i^{(k)}$ is defined by (1.6). $|y|<1$

(ii) If we put $\gamma=\alpha+1$ and $\beta=0$ in (2.3), the medium is stationary that is $\mathbf{u}=0$ and the equation (2.3) reduces to the following equation

$$\frac{1}{\lambda_1}\frac{\partial C}{\partial t} = (1-x^2)\frac{\partial^2 C}{\partial x^2} - 2x\frac{\partial C}{\partial x} \tag{6.4}$$

and the solution is

$$C(x,t)=2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{k=0}^{\upsilon}\frac{\Gamma(2\upsilon+2)\Gamma(\upsilon+1+k)}{\Gamma(2\upsilon+1)\Gamma(1-\beta+k)}\frac{(-\upsilon)_k}{\Gamma(\upsilon+1)k!}$$

$$P_\upsilon(x)exp\big[\lambda_1(\upsilon+1)\upsilon t\big]F^{\alpha,0,\alpha+1,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) \tag{6.5}$$

where

$$F^{\alpha,0,\alpha+1,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty}B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!}y_1^{K_1}\cdots y_u^{K_u}2^{gq+wq+\sum_{i=1}^u(f_i+w_i)K_i}$$

$$\aleph^{0,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c}Z_1 2^{h_1+k_1}\\ \cdot\\ \cdot\\ \cdot\\ Z_r 2^{h_r+k_r}\end{array}\middle|\begin{array}{c}(\text{-}\rho-k-gq-\sum_{i=1}^u f_iK_i:h_1,\cdots,h_r),\\ \cdot\\ \cdot\\ \cdots\end{array}\right.$$

$$\left.\begin{array}{c}(\text{-}\sigma-wq-\sum_{i=1}^u w_iK_i:k_1,\cdots,k_r),A\\ \cdot\\ \cdot\\ (\text{-k-1-}\sigma-\rho-(g+w)q-\sum_{i=1}^u(f_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r),B\end{array}\right) \tag{6.6}$$

Provided that

$min\{g,w,f_i,w_i,h_l,k_l\}>0$ `for` $i=1,\cdots,u;l=1,\cdots,r$

$Re\left(\rho+\sum_{i=1}^u K_if_i\right)+\sum_{i=1}^r h_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$ and $Re\left(\sigma+\sum_{i=1}^u K_iw_i\right)+\sum_{i=1}^r k_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$

$|argZ_k|<\frac{1}{2^{h_i+k_i+1}}A_i^{(k)}\pi, i=1,\cdots,r$, where $A_i^{(k)}$ is defined by (1.16). $|y|<1$

7. Multivariable H-function

If the multivariable Aleph-function reduces to multivariable H-function defined by Srivastava et al [11,12], the solution of our problem is :

$$C(x,t)=2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{k=0}^{\upsilon}\frac{\Gamma(\alpha+\beta-\gamma+2\upsilon+3)\Gamma(\alpha+\beta-\gamma+\upsilon+2+k)}{\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha-\gamma+k+2)}\frac{(-\upsilon)_k}{\Gamma(\beta+\upsilon+1)k!}$$

$$P_\upsilon^{(\alpha+\gamma+1,\beta)}(x)exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big]F_1^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) \tag{7.1}$$

where

$$F^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty} B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!} y_1^{K_1}\cdots y_u^{K_u} 2^{gq+wq+\sum_{i=1}^u (f_i+w_i)K_i}$$

$$H_{p+2,q+1:W}^{0,\mathbf{n}+2:V}\left(\begin{array}{c} Z_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ Z_r 2^{h_r+k_r} \end{array}\middle| \begin{array}{c} (\gamma-\rho-\alpha-1-k-gq-\sum_{i=1}^u f_iK_i : h_1,\cdots,h_r), \\ \cdot \\ \cdot \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\sigma-\beta-wq-\sum_{i=1}^u w_iK_i : k_1,\cdots,k_r), A \\ \cdot \\ \cdot \\ (\gamma-2-k-\sigma-\rho-\alpha-\beta-(g+w)q-\sum_{i=1}^u (f_i+w_i)K_i : h_1+k_1,\cdots,h_r+k_r), B \end{array}\right) \quad (7.2)$$

under the same conditions that (5.2)

8. Aleph-function of two variables

If $r=2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [9]. The solution of the problem is :

$$C(x,t) = 2^{\rho+\sigma}\sum_{v=0}^{\infty}\sum_{k=0}^{v}\frac{\Gamma(\alpha+\beta-\gamma+2v+3)\Gamma(\alpha+\beta-\gamma+v+2+k)}{\Gamma(\alpha+\beta-\gamma+2v+2)\Gamma(\alpha-\gamma+k+2)}\frac{(-v)_k}{\Gamma(\beta+v+1)k!}$$

$$P_v^{(\alpha+\gamma+1,\beta)}(x) exp\big[\lambda_1(v+1)(1+\alpha+\beta-\gamma+v)t\big] F^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) \quad (8.1)$$

where

$$F^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty} B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!} y_1^{K_1}\cdots y_u^{K_u} 2^{gq+wq+\sum_{i=1}^u (f_i+w_i)K_i}$$

$$\aleph_{p_i+2,q_i+1,\tau_i;R:W}^{0,\mathbf{n}+2:V}\left(\begin{array}{c} Z_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ Z_2 2^{h_2+k_2} \end{array}\middle| \begin{array}{c} (\gamma-\rho-\alpha-1-k-gq-\sum_{i=1}^u f_iK_i : h_1,h_2), \\ \cdot \\ \cdot \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\sigma-\beta-wq-\sum_{i=1}^u w_iK_i : k_1,k_2), A \\ \cdot \\ \cdot \\ (\gamma-2-k-\sigma-\rho-\alpha-\beta-(g+w)q-\sum_{i=1}^u (f_i+w_i)K_i : h_1+k_1,h_2+k_2), B \end{array}\right) \quad (8.2)$$

under the same conditions that (5.2) with $r=2$.

9. I-function of two variables

If $r=2$ and $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$, the multivariable Aleph-function reduces to I-funtion of two variables defined by Sharma et al [8]. The solution of the problem is :

$$C(x,t)=2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{k=0}^{\upsilon}\frac{\Gamma(\alpha+\beta-\gamma+2\upsilon+3)\Gamma(\alpha+\beta-\gamma+\upsilon+2+k)}{\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha-\gamma+k+2)}\frac{(-\upsilon)_k}{\Gamma(\beta+\upsilon+1)k!}$$

$$P_{\upsilon}^{(\alpha+\gamma+1,\beta)}(x)exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big]F^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) \quad (9.1)$$

where

$$F^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty}B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!}\; y_1^{K_1}\cdots y_u^{K_u}2^{gq+wq+\sum_{i=1}^{u}(f_i+w_i)K_i}$$

$$I_{p_i+2,q_i+1;R:W}^{0,\mathbf{n}+2:V}\left(\begin{array}{c}\mathrm{Z}_1 2^{h_1+k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{Z}_2 2^{h_2+k_2}\end{array}\middle|\begin{array}{c}(\gamma-\rho-\alpha-1-k-gq-\sum_{i=1}^{u}f_iK_i:h_1,h_2),\\ \cdot\\ \cdot\\ \cdots\end{array}\right.$$

$$\left.\begin{array}{c}(\text{-}\sigma-\beta-wq-\sum_{i=1}^{u}w_iK_i:k_1,k_2),A\\ \cdot\\ \cdot\\ (\gamma-2-k-\sigma-\rho-\alpha-\beta-(g+w)q-\sum_{i=1}^{u}(f_i+w_i)K_i:h_1+k_1,h_2+k_2),B\end{array}\right) \quad (9.2)$$

under the same conditions that (5.2) with $r=2$ and $\tau_i,\tau_{i'},\tau_{i''}\rightarrow 1$.

6. Conclusion

Specializing the parameters of the multivariable Aleph-function, the generalized hypergeometric function and the multivariable polynomials, we can obtain a large number of results of problem of advective-diffusion process involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

References

[1] Ayant F.Y. An integral associated with the Aleph-functions of several variables. *International Journal of Mathematics Trends and Technology (IJMTT)*. 2016 Vol 31 (3), page 142-154.

[2] Hanna S.R. Et al. Review of atmospheric diffusion models for regulatory applications,WMO Tech. Note no. 1777, WMO, Geneva, 1982, 37.

[3] Harisson R.M. and Perry R.Handbook of air pollution analysis, 2nd edn. Chapman and Hall and Methuen, New York, 1986

[4] Kumar H. and Satyarth P.S.Y. Application of Generalized Polynomials of Several Variables and Multivariable H-Function in one Dimensional Advective Diffusion Problem. Bull. Pure. Appl. Math. Vol 4, no 2, 2010, page 353-362.

[5] Lyons T.J and Scott W.D. Principles of Air Pollution Meteorology, CBS. Publisher and Distributers (P) LTD, New Delhi, 1992.

[6] Rainville E.D. Specials functions. Mac Millan Co. New York 1960; Reprinted Chelsea Publ. Co Bronx. New York,

1971.

[7] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[8] Sharma C.K. and mishra P.L. On the I-function of two variables and its properties. Acta Ciencia Indica Math , 1991 Vol 17, page 667-672.

[9] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page 1-13.

[10] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page 183-191.

[11] Srivastava H.M. and Panda R. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), page.119-137.

[12] Srivastava H.M. and Panda R. Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25(1976), page.167-197.