Generalized differential operators involving multivariable Aleph-function

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ABSTRACT
Recently, Thakur and Rajwade use the generalized differential operators concerning the multivariable H-function defined by Srivastava and Panda [11,12] . In this paper we use differential operators, we to derive three formulas of multivariable Aleph-function. At the end, we shall give several remarks.

Keywords :Multivariable Aleph-function, fractional differential operator, multivariable H-function., Aleph-function of two variables, I-function of two variables.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20.

1. Intdroduction

The fractional derivative of special function of one and more variables is important such as in the evaluation of series [10], the derivation of generating function [9] and the solution of differential equation [4] motivated by these and many other avenues of application. The fractional differential operators $D^m_{k,\alpha,x}$ and ${}_\alpha D^\mu_x$ are much used in the theory of special function of one and more variables.
Mishra [3] has defined the fractional derivative operators in the following manner.

$$D^n_{k,\alpha,x}\left(x^\mu\right)=\prod_{r=0}^{p-1}\frac{\Gamma(\mu+rk+1)}{\Gamma(\mu+rk-\alpha+1)}x^{\mu+nk} \tag{1.1}$$

We use the binomial expansion in the following manner

$$\left(ax^\mu+b\right)^\lambda=b^\lambda\sum_{l=0}^{\infty}\binom{\lambda}{l}\left(\frac{ax^\mu}{b}\right)^l \text{ where } \left|\frac{ax^\mu}{b}\right|<1 \tag{1.2}$$

The familiar differential operator ${}_\alpha D^\mu_x$ is defined by [5]

$${}_\alpha D^\mu_x f(x)=\begin{cases}\frac{1}{\Gamma(-\mu)}\int_\alpha^x(x-t)^{-\mu-1}f(t)\mathrm{d}t,[Re(\mu)<0]\\ \\ \frac{d^m}{dx^m}\ {}_\alpha D^{\mu-m}_x f(x),[0\leqslant Re(\mu)<m]\end{cases} \tag{1.3}$$

where m is a positive integer.
For $\alpha=0$(1.3), we define the classical Riemann-Liouville fractional of order μ (or $-\mu$), when $\alpha\to\infty$(1.3) may be identified with the definition of the Weyl fractional derivative of order μ (or $-\mu$) , see [2, chap.13 ; 8] the special case of fractional calculus operator ${}_\alpha D^\mu_x$ when $\alpha=0$ is written as D^μ_x thus we have $D^\mu_x={}_0D^\mu_x$.
In this paper, we obtain several fractional derivative formulas involving multivariable Aleph-function. This function is an extension of the multivariable I-function recently defined by C.K. Sharma and Ahmad [7] , itself is a generalization of the multivariable H-function defined by Srivastava et al [11,12]. The multivariable Aleph-function is defined by means of the multiple contour integral :

$$\text{We have : }\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right|$$

$$\begin{array}{l}[(\mathrm{a}_j;\alpha^{(1)}_j,\cdots,\alpha^{(r)}_j)_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha^{(1)}_{ji},\cdots,\alpha^{(r)}_{ji})_{\mathfrak{n}+1,p_i}]:\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ ,[\tau_i(b_{ji};\beta^{(1)}_{ji},\cdots,\beta^{(r)}_{ji})_{1,q_i}]:\end{array}$$

$$\left. \begin{matrix} [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{matrix} \right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.4}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.5}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.6}$$

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\ \text{with}\ k=1,\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)} \tag{1.7}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\to 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\to\infty$$

where $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k$$

For convenience, we will use the following notations in this paper.

$$V=m_1,n_1;\cdots;m_r,n_r \tag{1.8}$$

$$\mathrm{W}=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{1.9}$$

$$A = \{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\}, \{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\}$$

$$\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$B = \{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\},\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\},\cdots,$$

$$\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

Throughout the present paper we assume that the convergence and existence condition corresponding appropriately to the ones detained above are satisfied by each of the various Aleph-function involved in our results which are presented in the following sections.

2. Main results

In this section, we shall prove our main formula on fractional differential operator involving multivariable Aleph-function.

Theorem 1

$$D_{k,\alpha,x}^{p}\left\{x^t\left(x^{\upsilon_1}+a\right)^{\lambda}\left(b-x^{\upsilon_2}\right)^{-\delta}\aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{z}_1 x^{\rho_1}\left(x^{\upsilon_1}+a\right)^{\sigma_1}\left(b-x^{\upsilon_2}\right)^{-\delta_1} \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\rho_r}\left(x^{\upsilon_1}+a\right)^{\sigma_r}\left(b-x^{\upsilon_2}\right)^{-\delta_r}\end{pmatrix}\right\}$$

$$= a^{\lambda}b^{-\delta}x^{t+pk}\sum_{l,m=0}^{\infty}\frac{\left(\frac{x^{\upsilon_1}}{a}\right)^l\left(\frac{x^{\upsilon_2}}{b}\right)^m}{l!m!}\aleph_{p_i+p+2,q_i+p+2,\tau_i;R:W}^{0,\mathfrak{n}+p+2:V}\left(\begin{matrix} \mathrm{z}_1 x^{\rho_1}a^{\sigma_1}b^{-\delta_1} \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\rho_r}a^{\sigma_r}b^{-\delta_r}\end{matrix}\right|$$

$$\left.\begin{matrix}(\text{-}\lambda;\sigma_1,\cdots,\sigma_r),(1-\delta-m;\delta_1,\cdots,\delta_r),(-t-jk-\upsilon_1 l-\upsilon_2 m;\rho_1,\cdots,,\rho_r)_{0,p-1},A \\ \cdot \\ \cdot \\ (\text{-}\lambda+1;\sigma_1,\cdots,\sigma_r),(1-\delta;\delta_1,\cdots,\delta_r),(\alpha-t-jk-\upsilon_1 l-\upsilon_2 m;\rho_1,\cdots,\rho_r)_{0,p-1},B\end{matrix}\right) \tag{2.1}$$

Provided that

$$min\{\upsilon_1,\upsilon_2,\rho_i,\sigma_i,\delta_i\} > 0 (i=1,\cdots,r)$$

$$max\left\{\left|arg\left(\frac{x^{\upsilon_1}}{a}\right)\right|,\left|arg\left(\frac{x^{\upsilon_2}}{b}\right)\right|\right\} < \pi$$

$$Re(t)+\sum_{i=1}^{r}\rho_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$$

and $|arg x_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.7).

proof

To prove the fractional derivative formula (2.1), we first replace the multivariable Aleph-function occuring therein by its Mellin-Barnes contour integral given by (1.4) and collect the power of $x\left(x^{\upsilon_1}+a\right)\left(b-x^{\upsilon_2}\right)$. Now making use of the following binomial expansions (1.2) two times and apply the formula (1.1). We interpret the resulting Mellin-Barnes contour integrals as a multivariable aleph-function, we get the desired result.

Theorem 2

$$D^{p}_{k,\alpha,x}\left\{x^{t}\left(x^{\upsilon_1}+a\right)^{\lambda_1}\left(b-x^{\upsilon_2}\right)^{-\delta_1}\left(x^{\upsilon_3}+c\right)^{\lambda_2}\left(d-x^{\upsilon_4}\right)^{-\delta_2}\right.$$

$$\left.\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1}\left(x^{\upsilon_1}+a\right)^{\sigma_1}\left(b-x^{\upsilon_2}\right)^{-\delta_1}\left(x^{\upsilon_3}+c\right)^{\sigma_1}\left(d-x^{\upsilon_4}\right)^{-\delta_1} \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\rho_r}\left(x^{\upsilon_1}+a\right)^{\sigma_r}\left(b-x^{\upsilon_2}\right)^{-\delta_r}\left(x^{\upsilon_3}+c\right)^{\sigma_r}\left(d-x^{\upsilon_4}\right)^{-\delta_r} \end{array}\right)\right\}$$

$$=a^{\lambda_1}b^{-\delta_1}c^{\lambda_2}d^{-\delta_2}x^{t+pk}\sum_{k,l,m,n=0}^{\infty}\frac{\left(\frac{x^{\upsilon_1}}{a}\right)^{k}\left(\frac{x^{\upsilon_2}}{b}\right)^{l}\left(\frac{x^{\upsilon_3}}{c}\right)^{m}\left(\frac{x^{\upsilon_4}}{d}\right)^{n}}{k!l!m!n!}$$

$$\aleph^{0,\mathfrak{n}+p+2:V}_{p_i+p+2,q_i+p+2,\tau_i;R:W}\left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1}a^{\sigma_1}b^{-\delta_1}c^{\sigma_1}d^{-\delta_1} \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\rho_r}a^{\sigma_r}b^{-\delta_r}c^{\sigma_r}d^{-\delta_r} \end{array}\left|\begin{array}{c} (\text{-}\lambda;\sigma_1,\cdots,\sigma_r),(1-\delta-k-m;\delta_1,\cdots,\delta_r), \\ \cdot \\ \cdot \\ (\text{-}\lambda+1;\sigma_1,\cdots,\sigma_r),(1-\delta;\delta_1,\cdots,\delta_r), \end{array}\right.\right.$$

$$\left.\begin{array}{c} (\text{-t-jk-}\upsilon_1 k-\upsilon_2 l-\upsilon_3 m-\upsilon_4 n;\rho_1,\cdots,\rho_r)_{0,p-1},A \\ \cdot \\ \cdot \\ (\alpha-t-jk-\upsilon_1 k-\upsilon_2 l-\upsilon_3 m-\upsilon_4 n;\rho_1,\cdots,\rho_r)_{0,p-1},B \end{array}\right) \quad (2.2)$$

Provided that

$min\{\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,\rho_i,\sigma_i,\delta_i\}>0(i=1,\cdots,r)$

$$max\left\{\left|arg\left(\frac{x^{\upsilon_1}}{a}\right)\right|,\left|arg\left(\frac{x^{\upsilon_2}}{b}\right)\right|,\left|arg\left(\frac{x^{\upsilon_3}}{c}\right)\right|,\left|arg\left(\frac{x^{\upsilon_4}}{d}\right)\right|\right\}<\pi$$

$$Re(t)+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$$

and $|argx_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.7).

Proof

To prove the fractional derivative formula (2.2), we first replace the multivariable Aleph-function occuring therein by its Mellin-Barnes contour integral given by (1.4) and collect the power of $x\left(x^{\upsilon_1}+a\right)\left(b-x^{\upsilon_2}\right)\left(x^{\upsilon_3}+c\right)\left(d-x^{\upsilon_4}\right)$. Now making use of the following binomial expansions (1.2) four times and apply the formula (1.1). We interpret the resulting Mellin-Barnes contour integrals as a multivariable aleph-function, we get the desired result.

Theorem 3

$$D^{p}_{k,\alpha,x}D^{\mu}_{y}\left\{x^{t}y^{\lambda}\left(x^{\upsilon_1}+a\right)^{\lambda_1}\left(b-x^{\upsilon_2}\right)^{-\delta_1}\left(y^{\upsilon_3}+c\right)^{h}\left(d-y^{\upsilon_4}\right)^{-g}\right.$$

$$\left.\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\begin{pmatrix} \mathrm{z}_1x^{\rho_1}y^{\lambda_1}\left(x^{\upsilon_1}+a\right)^{\sigma_1}\left(b-x^{\upsilon_2}\right)^{-\delta_1}\left(y^{\upsilon_3}+c\right)^{h_1}\left(d-y^{\upsilon_4}\right)^{-g_1} \\ \cdot \\ \cdot \\ \mathrm{z}_rx^{\rho_r}y^{\lambda_r}\left(x^{\upsilon_1}+a\right)^{\sigma_r}\left(b-x^{\upsilon_2}\right)^{-\delta_r}\left(y^{\upsilon_3}+c\right)^{h_r}\left(d-y^{\upsilon_4}\right)^{-g_r}\end{pmatrix}\right\}$$

$$=a^{\lambda}b^{-\delta}c^{h}d^{-g}x^{t+pk}y^{\lambda-\mu}\sum_{l,m,r,s=0}^{\infty}\frac{\left(\frac{x^{\upsilon_1}}{a}\right)^{l}\left(\frac{x^{\upsilon_2}}{b}\right)^{m}\left(\frac{y^{\upsilon_3}}{c}\right)^{r}\left(\frac{y^{\upsilon_4}}{d}\right)^{s}}{l!m!r!s!}$$

$$\aleph^{0,\mathfrak{n}+p+5:V}_{p_i+p+5,q_i+p+5,\tau_i;R:W}\left(\begin{matrix} \mathrm{z}_1x^{\rho_1}y^{\lambda_1}a^{\sigma_1}b^{-\delta_1}c^{h_1}d^{-g_1} \\ \cdot \\ \cdot \\ \mathrm{z}_rx^{\rho_r}y^{\lambda_r}a^{\sigma_r}b^{-\delta_r}c^{h_r}d^{-g_r}\end{matrix}\ \middle|\ \begin{matrix}(\text{-}\lambda;\sigma_1,\cdots,\sigma_r),(1-\delta-m;\delta_1,\cdots,\delta_r), \\ \cdot \\ \cdot \\ (\text{-}\lambda+1;\sigma_1,\cdots,\sigma_r),(1-\delta;\delta_1,\cdots,\delta_r),\end{matrix}\right.$$

$$\begin{matrix}(\text{-h;h}_1,\cdots,h_r),(1-g-t;g_1,\cdots,g_r),(-\lambda-r\upsilon_3-s\upsilon_4;\lambda_1,\cdots,\lambda_r), \\ \cdot \\ \cdot \\ (\text{-h+r;h}_1,\cdots,h_r),(1-g;g_1,\cdots,g_r),(-\lambda+\mu-r\upsilon_3-s\upsilon_4;\lambda_1,\cdots,\lambda_r),\end{matrix}$$

$$\left.\begin{matrix}(\text{-t-jk-}\upsilon_1l-\upsilon_2m;k_1,\cdots,k_r)_{0,p-1},A \\ \cdot \\ \cdot \\ (\alpha-t-jk-\upsilon_1k-\upsilon_2l;k_1,\cdots,k_r)_{0,p-1},B\end{matrix}\right) \tag{2.3}$$

Provided that

$min\{\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,\rho_i,\sigma_i,\delta_i,h_i,g_i\}>0(i=1,\cdots,r)$

$$max\left\{\left|arg\left(\frac{x^{\upsilon_1}}{a}\right)\right|,\left|arg\left(\frac{x^{\upsilon_2}}{b}\right)\right|,\left|arg\left(\frac{y^{\upsilon_3}}{c}\right)\right|,\left|arg\left(\frac{y^{\upsilon_4}}{d}\right)\right|\right\}<\pi$$

$$Re(t)+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1\,,\ Re(\lambda)+\sum_{i=1}^{r}\lambda_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$$

$|argx_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.7).

Proof

To prove the fractional derivative formula (2.3), we first replace the multivariable Aleph-function occuring therein by its Mellin-Barnes contour integral given by (1.4) and collect the power of $xy\left(x^{\upsilon_1}+a\right)\left(b-x^{\upsilon_2}\right)\left(x^{\upsilon_3}+c\right)\left(d-x^{\upsilon_4}\right)$. Now making use of the following binomial expansions (1.2) four times and apply the formula (1.1) two times. We interpret the resulting Mellin-Barnes contour integrals as a multivariable aleph-function, we get the desired result.

Theorem 4

$$D^{p}_{k,\alpha,x}D^{\mu}_{y}D^{\eta}_{w}\left\{x^{t}y^{\lambda}w^{\gamma}\left(x^{\upsilon_1}+a\right)^{\sigma}\left(b-x^{\upsilon_2}\right)^{-\delta}\left(x^{\upsilon_3}+c\right)^{h}\left(d-x^{\upsilon_4}\right)^{-g}\left(w^{\upsilon_5}+e\right)^{\theta}(f-w^{\upsilon_6})^{-\phi}\right.$$

$$\aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c} \mathrm{z}_1x^{\rho_1}y^{\lambda_1}w^{\gamma_1}\left(x^{\upsilon_1}+a\right)^{\sigma_1}\left(b-x^{\upsilon_2}\right)^{-\delta_1}\left(y^{\upsilon_3}+c\right)^{h_1}\left(d-y^{\upsilon_4}\right)^{-g_1}\left(w^{\upsilon_5}+e\right)^{\theta_1}(f-w^{\upsilon_6})^{-\phi_1} \\ \cdot \\ \cdot \\ \mathrm{z}_rx^{\rho_r}y^{\lambda_r}w^{\gamma_r}\left(x^{\upsilon_1}+a\right)^{\sigma_r}\left(b-x^{\upsilon_2}\right)^{-\delta_r}\left(y^{\upsilon_3}+c\right)^{h_r}\left(d-y^{\upsilon_4}\right)^{-g_r}\left(w^{\upsilon_5}+e\right)^{\theta_r}(f-w^{\upsilon_6})^{-\phi_r} \end{array}\right)$$

$$=a^{\lambda}b^{-\delta}c^{h}d^{-g}e^{\theta}f^{-\phi}x^{t+pk}y^{\lambda-\mu}w^{\gamma-\eta}\sum_{k,l,m,n,r,s=0}^{\infty}\frac{\left(\frac{x^{\upsilon_1}}{a}\right)^{k}\left(\frac{x^{\upsilon_2}}{b}\right)^{l}\left(\frac{y^{\upsilon_3}}{c}\right)^{m}\left(\frac{y^{\upsilon_4}}{d}\right)^{n}\left(\frac{w^{\upsilon_5}}{e}\right)^{r}\left(\frac{w^{\upsilon_6}}{f}\right)^{s}}{k!l!m!n!r!s!}$$

$$\aleph^{0,\mathfrak{n}+p+7:V}_{p_i+p+7,q_i+p+7,\tau_i;R:W}\left(\begin{array}{c} \mathrm{z}_1x^{\rho_1}y^{\lambda_1}a^{\sigma_1}b^{-\delta_1}c^{h_1}d^{-g_1}e^{\theta_1}f^{-\phi_1} \\ \cdot \\ \cdot \\ \mathrm{z}_rx^{\rho_r}y^{\lambda_r}a^{\sigma_r}b^{-\delta_r}c^{h_r}d^{-g_r}e^{\theta_r}f^{-\phi_r} \end{array}\left|\begin{array}{c} (\text{-}\lambda;\sigma_1,\cdots,\sigma_r),(1-\delta-m;\delta_1,\cdots,\delta_r), \\ \cdot \\ \cdot \\ (\text{-}\lambda+1;\sigma_1,\cdots,\sigma_r),(1-\delta;\delta_1,\cdots,\delta_r), \end{array}\right.\right.$$

$$\begin{array}{c} (\text{-h};\mathrm{h}_1,\cdots,h_r),(1-g-t;g_1,\cdots,g_r),(-\theta;\theta_1,\cdots,\theta_r),(-\phi;\phi_1,\cdots,\phi_r), \\ \cdot \\ \cdot \\ (\text{-h+r};\mathrm{h}_1,\cdots,h_r),(1-g;g_1,\cdots,g_r),(-\theta+r;\theta_1,\cdots,\theta_r),(-\phi+s;\phi_1,\cdots,\phi_r), \end{array}$$

$$\left.\begin{array}{c} (\text{-}\lambda-r\upsilon_3-s\upsilon_4-\theta\upsilon_5-\phi\upsilon_6;\ \lambda_1,\cdots,\lambda_r),(-t-jk-\upsilon_1l-\upsilon_2m;k_1,\cdots,k_r)_{0,p-1},A \\ \cdot \\ \cdot \\ (\text{-}\lambda+\mu-r\upsilon_3-s\upsilon_4-\theta\upsilon_5-\phi\upsilon_6;\ \lambda_1,\cdots,\lambda_r),(\alpha-t-jk-\upsilon_1l-\upsilon_2m;k_1,\cdots,k_r)_{0,p-1},B \end{array}\right) \tag{2.4}$$

Provided that

$$min\{\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,\upsilon_5,\upsilon_6,\rho_i,\sigma_i,\delta_i,h_i,g_i,\theta_i,\phi_i\}>0(i=1,\cdots,r)$$

$$max\left\{\left|arg\left(\frac{x^{\upsilon_1}}{a}\right)\right|,\left|arg\left(\frac{x^{\upsilon_2}}{b}\right)\right|,\left|arg\left(\frac{y^{\upsilon_3}}{c}\right)\right|,\left|arg\left(\frac{y^{\upsilon_4}}{d}\right)\right|,\left|arg\left(\frac{w^{\upsilon_5}}{e}\right)\right|,\left|arg\left(\frac{w^{\upsilon_6}}{f}\right)\right|\right\}<\pi$$

$$Re(t)+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1,\ Re(\lambda)+\sum_{i=1}^{r}\lambda_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$$

$$Re(\gamma) + \sum_{i=1}^{r} \gamma_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$$

and $|arg x_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (1.7).

Proof

To prove the fractional derivative formula (2.3), we first replace the multivariable Aleph-function occuring therein by its Mellin-Barnes contour integral given by (1.4) and collect the power of the expression $xyw\,(x^{\upsilon_1}+a)\,(b-x^{\upsilon_2})\,(x^{\upsilon_3}+c)\,(d-x^{\upsilon_4})\,(w^{\upsilon_5}+e)(f-w^{\upsilon_6})$. Now making use of the following binomial expansions (1.2) six times and apply the formula (1.1) three times. We interpret the resulting Mellin-Barnes contour integrals as a multivariable aleph-function, we get the desired result.

Remarks :

We obtain the same relations with the multivariable H-function defined by Srivastava and Panda [8,9], see Thakur and Rajwade [13] for more details.

If $r = 2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [6], and we obtain the same relations.

If $r = 2$ and $\tau_i, \tau_{i'}, \tau_{i''} \rightarrow 1$, the multivariable Aleph-function reduces to I-function of two variables defined by Sharma and Mishra [8] and we have the similar formulae.

3. Conclusion

Specializing the parameters of the multivariable Aleph-function, we can obtain a large number of news and knowns fractional derivatives involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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