On multiple integral relations involving the multivariable A-function

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ABSTRACT
In this paper, first we obtain a finite integral involving the product of the H-function of two variables and the multivariable A-function defined by Gautam et al [2]. Next, with the application of one of its special cases and lemma due to Srivastava et al [7] we obtain two interesting and general multiple integral relations involving the multivariable A-function, Fox's H-function and two arbitrary functions f and g. These integral relations are quite general in nature. Again by suitably specializing the functions f anf g occuring in the main integral relations, we have also evaluated two multiple integrals which are new and quite general in nature. We shall the particular case concerning the multivariable H-function defined by Srivastava et al [8,9].

Keywords: Multivariable A-function, multiple integrals, multivariable H-function, H-function of two variables, arbitrary function, Fox's H-function

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1. Introduction

The multivariable A-function defined by Gautam et al [2] is an extension of the multivariable H-function defined by Srivastava et al [8,9]. The multivariable A-function is defined in term of multiple Mellin-Barnes type integral.

$$A(Z_1,\cdots,Z_s) = A^{\mathbf{m},\mathbf{n}:m_1,n_1;\cdots;m_r,n_r}_{\mathbf{p},\mathbf{q}:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} Z_1 & (a_j;A_j^{(1)},\cdots,A_j^{(s)})_{1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ Z_s & (b_j;B_j^{(1)},\cdots,B_j^{(s)})_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (c_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(s)},C_j^{(s)})_{1,p_s} \\ \\ (d_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(s)},D_j^{(s)})_{1,q_s} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\phi'(t_1,\cdots,t_s)\prod_{i=1}^{s}\theta'_i(t_i)Z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.2}$$

where $\phi'(t_1,\cdots,t_s),\theta'_i(t_i), i=1,\cdots,s$ are given by :

$$\phi'(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{\mathbf{m}}\Gamma(b_j-\sum_{i=1}^{s}B_j^{(i)}t_i)\prod_{j=1}^{\mathbf{n}}\Gamma(1-a_j+\sum_{i=1}^{s}A_j^{(i)}t_j)}{\prod_{j=\mathbf{n}+1}^{\mathbf{p}}\Gamma(a_j-\sum_{i=1}^{s}A_j^{(i)}t_j)\prod_{j=\mathbf{m}+1}^{\mathbf{q}}\Gamma\left(1-b_j+\sum_{i=1}^{s}B_j^{(i)}t_j\right)} \tag{1.3}$$

$$\theta'_i(t_i) = \frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+C_j^{(i)}t_i)\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-D_j^{(i)}t_i)}{\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-C_j^{(i)}t_i)\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+D_j^{(i)}t_i)} \tag{1.4}$$

Here $\mathbf{m},\mathbf{n},\mathbf{p},\mathbf{q},m_i,n_i,p_i,c_i\in\mathbb{N}^*;i=1,\cdots,r\;;a_j,b_j,c_j^{(i)},d_j^{(i)},A_j^{(i)},B_j^{(i)},C_j^{(i)},D_j^{(i)}\in\mathbb{C}$

The multiple integral defining the A-function of s-variables converges absolutely if :

$$|arg(\Omega_i')Z_k|<\frac{1}{2}\eta_k'\pi,\xi'^*=0,\eta_i'>0 \tag{1.5}$$

$$\Omega_i'=\prod_{j=1}^{\mathbf{p}}\{A_j^{(i)}\}^{A_j^{(i)}}\prod_{j=1}^{\mathbf{q}}\{B_j^{(i)}\}^{-B_j^{(i)}}\prod_{j=1}^{q_i}\{D_j^{(i)}\}^{D_j^{(i)}}\prod_{j=1}^{p_i}\{C_j^{(i)}\}^{-C_j^{(i)}};i=1,\cdots,s \tag{1.6}$$

$$\xi_i'^*=Im(\sum_{j=1}^{\mathbf{p}}A_j^{(i)}-\sum_{j=1}^{\mathbf{q}}B_j^{(i)}+\sum_{j=1}^{q_i}D_j^{(i)}-\sum_{j=1}^{p_i}C_j^{(i)});i=1,\cdots,s \tag{1.7}$$

$$\eta_i'=Re\left(\sum_{j=1}^{\mathbf{n}}A_j^{(i)}-\sum_{j=\mathbf{n}+1}^{\mathbf{p}}A_j^{(i)}+\sum_{j=1}^{\mathbf{m}}B_j^{(i)}-\sum_{j=\mathbf{m}+1}^{\mathbf{q}}B_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}+\sum_{j=1}^{n_i}C_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\right)$$

$$i=1,\cdots,s \tag{1.8}$$

In this paper, we shall note.

$$X=m_1,n_1;\cdots;m_s,n_s\quad:Y=p_1,q_1;\cdots;p_s,q_s \tag{1.9}$$

$$\mathbb{A}=(a_j;A_j^{(1)},\cdots,A_j^{(s)})_{1,p}:\ (\mathrm{c}_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(s)},C_j^{(s)})_{1,p_s} \tag{1.10}$$

$$\mathbb{B}=(b_j;B_j^{(1)},\cdots,B_j^{(s)})_{1,q}:\ (\mathrm{d}_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(s)},D_j^{(s)})_{1,q_s} \tag{1.11}$$

2. Required results

The following results will be required in establishing our main integral relations :

Lemma 1 (Srivastava et al [7] ,1981) :

Let the functions $f(x)$ and $g(x)$ be integrable over the semi interval $(0,\infty)$ and define.

$$F(R)=\int_0^{\frac{\pi}{2}}h(R,\theta)\,\mathrm{d}\theta \tag{2.1}$$

where $h(R,\theta)$ is an integrable function of two variables in the rectangular region $0\leqslant R\leqslant\infty,0\leqslant\theta\leqslant\frac{\pi}{2}$, then

$$\int_0^\infty\int_0^\infty f(x^2+y^2)h\left\{(x^2+y^2)^{1/2}tan^{-1}(y/x)\right\}\mathrm{d}x\mathrm{d}y=\frac{1}{2}\int_0^\infty f(t)F(\sqrt{t})\mathrm{d}t \tag{2.2}$$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty(x^2+y^2)^{1/2}f(x^2+y^2+z^2)g\left[tan^{-1}\left\{(x^2+y^2)^{1/2}/z\right\}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=\int_0^\infty\int_0^\infty f(u^2+v^2)F\left\{(x^2+y^2)^{1/2}\right\}g\left\{tan^{-1}(v/u)\right\}\mathrm{d}u\mathrm{d}v \tag{2.3}$$

provided that the various integrals involved are absolutely convergent.

Lemma 2 (Kalla et al [5] ,1981) :

$$H'(at,bt) = H^{0,n_1';1,n_2';1,n_3'}_{p_1',q_1';p_2',q_2'+1;p_3',q_3'+1}\left(\begin{array}{c|c} \text{at} & (\text{a'}_j;A_j',A_j'')_{1,p_1'}:(e_j,E_j)_{1,p_2'};(g_j,G_j)_{1,p_3'} \\ \cdot & \cdot \\ \cdot & \\ \text{bt} & (\text{b'}_j;B_j',B_j'')_{1,q_1'}:(0,1),(f_j,F_j)_{1,q_2'};(0,1),(h_j,H_j)_{1,q_3'} \end{array}\right)$$

$$= \sum_{\mathbf{M}'=0}^{\infty} \phi(\mathbf{M}')\frac{(-at)^{\mathbf{M}'}}{\mathbf{M}'!} \tag{2.4}$$

where

$$\phi(\mathbf{M}') = \sum_{\mathbf{N}'=\mathbf{0}}^{\mathbf{M}'} \phi'(\mathbf{M}'-\mathbf{N}',\mathbf{N}')\theta_1'(\mathbf{M}')\theta_2'(\mathbf{N}')(b/a)^{\mathbf{N}'}\binom{\mathbf{M}'}{\mathbf{N}'} \tag{2.5}$$

$$\phi(s,t) = \frac{\prod_{j=1}^{n_1'}\Gamma(1-a_j'+A_j's+A_j''t)}{\prod_{j=n_1'+1}^{p_1'}\Gamma(a_j'-A_j's-A_j''t)\prod_{j=1}^{q_1'}\Gamma(1-b_j'+B_j's+B_j''t)} \tag{2.6}$$

$$\theta_1'(s) = \frac{\prod_{j=1}^{n_2'}\Gamma(1-e_j+E_js)}{\prod_{j=n_2'+1}^{p_2'}\Gamma(e_j-E_js)\prod_{j=1}^{q_2'}\Gamma(1-f_j+F_js)} \tag{2.7}$$

and

$$\theta_2'(t) = \frac{\prod_{j=1}^{n_3'}\Gamma(1-g_j+G_jt)}{\prod_{j=n_3'+1}^{p_3'}\Gamma(g_j-G_jt)\prod_{j=1}^{q_3'}\Gamma(1-h_j+H_jt)} \tag{2.8}$$

3. A useful integral

We obtain the following integral, which will be required in the next section :

$$\int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{(\cos^2\theta+e\sin^2\theta)^{\alpha+\beta}} H'\left(\begin{array}{c} \text{a}\frac{(\sin\theta)^{2c}(\cos\theta)^{2d}}{(\cos^2\theta+c\sin^2\theta)^{c+d}} \\ \cdot \\ \cdot \\ \text{b}\frac{(\sin\theta)^{2c}(\cos\theta)^{2d}}{(\cos^2\theta+c\sin^2\theta)^{c+d}} \end{array}\right) A\left(\begin{array}{c} \text{Z}_1\frac{(\sin\theta)^{2c_1}(\cos\theta)^{2d_1}}{(\cos^2\theta+e\sin^2\theta)^{c_1+d_1}} \\ \cdot \\ \cdot \\ \text{Z}_s\frac{(\sin\theta)^{2c_s}(\cos\theta)^{2d_s}}{(\cos^2\theta+e\sin^2\theta)^{c_s+d_s}} \end{array}\right) \mathrm{d}\theta = \frac{1}{2}\sum_{\mathbf{M}'=0}^{\infty}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}}{e^{\alpha+c\mathbf{M}'}\mathbf{M}'!}$$

$$A^{\mathbf{m},\mathbf{n}+2;X}_{\mathbf{p}+2,\mathbf{q}+1;Y}\left(\begin{array}{c|c} \text{Z}_1e^{-c_1} & (1\text{-}\alpha-c\mathbf{M}':c_1,\cdots,c_s),(1-\beta-d\mathbf{M}':d_1,\cdots,d_s),\mathbb{A} \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \cdot\ \cdot\ \cdot \\ \text{Z}_se^{-c_s} & (1-\alpha-\beta-(c+d)\mathbf{M}':c_1+d_1,\cdots,c_s+d_s),\mathbb{B} \end{array}\right) \tag{3.1}$$

provide that

$min\{Re(\alpha), Re(\beta), e, c, d, c_i, d_i\} > 0$ for $i = 1, \cdots, s$; $|arg(\Omega'_i)Z_k| < \frac{1}{2}\eta'_k\pi, \xi'^{*} = 0, \eta'_i > 0$

$$Re\left[\alpha + c\mathbf{M}' + \sum_{i=1}^{s} c_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{D_j^{(i)}}\right] > 0 \text{ and } Re\left[\beta + d\mathbf{M}' + \sum_{i=1}^{s} d_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{D_j^{(i)}}\right] > 0$$

The serie occuring on the right hand side of (3.1) is absolutely convergent

Proof

To prove (3.1), we first express the H-function of two variables in serie with the help of (2.4) and interchange the orders of summations and integrations (which is permissible under the conditions stated). Now, we express the multivariable A-function defined by Gautam et al [2] in Mellin-Barnes contour integral given by (1.2). Interchange the order of $\theta-$integral and $(t_1, \cdots, t_s)-$integrals . Now collect the power of $\cos\theta, \sin\theta$ and $(\cos^2\theta + e\sin^2\theta)$ and evaluate the $\theta-$integral with the help of result ([4], 1980, page 376, eq.(3.642)).

$$\int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{(\cos^2\theta + e\sin^2\theta)^{\alpha+\beta}} \mathrm{d}\theta = \frac{\Gamma(\alpha)\Gamma(\beta)}{2e^{\alpha}\Gamma(\alpha+\beta)} \tag{3.2}$$

where $e, Re(\alpha), \Re(\beta) > 0$

Finally, interpret the Mellin-Barnes contour integral to multivariable A-function with the help of (1.2), we obtain the desired result.

4. Fox's H-function

If in (3.1) we specialize some of the parameters of the $H'-$function of two variables by taking each $A''_j = B''_j = G_j = H_j = 1, n'_3 = p'_3$ and let $b \to 0, c \to 0, \dot{i} \to 0$ for $i = 1, \cdots, r$ therein we get the following result by vertue of a relation given by Goyal ([3], 1975, page 123 Eq.(3.5)) :

$$\int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{(\cos^2\theta + e\sin^2\theta)^{\alpha+\beta}} H_{p'_1,q'_1+1}^{1,n'_1}\left[\frac{a(\cos\theta)^{2d}}{(\cos^2\theta+e\sin^2\theta)}\ \middle|\ \begin{matrix}(\mathrm{a}'_j, A'_j)_{1,p'} \\ \cdot \\ (0,1),\ (\mathrm{b}'_j, B'_j)_{1,q'}\end{matrix}\right] A\begin{pmatrix} \mathrm{Z}_1 \frac{(\sin\theta)^{2c_1}(\cos\theta)^{2d_1}}{(\cos^2\theta+e\sin^2\theta)^{c_1+d_1}} \\ \cdot \\ \cdot \\ \mathrm{Z}_s \frac{(\sin\theta)^{2c_s}(\cos\theta)^{2d_s}}{(\cos^2\theta+e\sin^2\theta)^{c_s+d_s}} \end{pmatrix} \mathrm{d}\theta$$

$$= \frac{1}{2e^{\alpha}} \sum_{\mathbf{M}'=0}^{\infty} \phi(\mathbf{M}') \frac{(-a)^{\mathbf{M}'}\Gamma(\beta+\rho\mathbf{M}')}{\mathbf{M}'!} A_{\mathbf{p}+1,\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n}+1;X}\left(\begin{matrix} \mathrm{Z}_1 e^{-c_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Z}_s e^{-c_s}\end{matrix}\ \middle|\ \begin{matrix}(1\text{-}\alpha : c_1, \cdots, c_s), \mathbb{A} \\ \cdots \\ \cdots \\ (1-\alpha-\beta-d\mathbf{M}' : c_1, \cdots, c_s), \mathbb{B}\end{matrix}\right) \tag{4.1}$$

where

$$\phi'(\mathbf{M}') = \frac{\prod_{j=1}^{N}\Gamma(1-a'_j+A'_j\mathbf{M}')}{\prod_{j=N+1}^{P}\Gamma(a'_j-A'_j\mathbf{M}')\prod_{j=1}^{Q}\Gamma(1-b'_j+B'_j\mathbf{M})} \tag{4.2}$$

5. Main integral relations

$$\int_0^\infty \int_0^\infty \frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)}{(x^2+ey^2)^{\alpha+\beta}} H_{P,Q}^{1,N}\left[\frac{ax^{2d}}{(x^2+ey^2)^d}\right] f(x^2+y^2)$$

$$A_{\mathbf{p},\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1 y^{2c_1}\frac{(x^2+y^2)^{\lambda_1}}{(x^2+ey^2)^{c_1}} & \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_s {y'}^{2c_s}\frac{(x^2+y^2)^{\lambda_s}}{(x^2+ey^2)^{c_s}} & (1\text{-}\alpha; c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \mathrm{d}x\mathrm{d}y = \frac{1}{4e^\alpha}\sum_{\mathbf{M}'=0}^{\infty}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}\Gamma(\beta+\rho\mathbf{M}')}{\mathbf{M}'!}\int_0^\infty f(t)$$

$$A_{\mathbf{p},\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1 t^{\lambda_1}e^{-c_1} & \mathbb{A} \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{Z}_s t^{\lambda_s}e^{-c_s} & (1-\alpha-\beta-d\mathbf{M}': c_1,\cdots,c_s),\mathbb{B} \end{array}\right)\mathrm{d}t \tag{5.1}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{1/2}}{(x^2+ey^2)^{\alpha+\beta}} H_{P,Q}^{1,N}\left[\frac{ax^{2d}}{(x^2+ey^2)^d}\right] f(x^2+y^2+z^2)$$

$$g\left[tan^{-1}\{\frac{(x^2+y^2)^{1/2}}{z}\}\right] A_{\mathbf{p},\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1 y^{2c_1}\frac{(x^2+y^2+z^2)^{\lambda_1}}{(x^2+ey^2)^{c_1}} & \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_s {y'}^{2c_s}\frac{(x^2+y^2+z^2)^{\lambda_s}}{(x^2+ey^2)^{c_s}} & (1\text{-}\alpha; c_1,\cdots,c_s),\mathbb{B} \end{array}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$= \frac{1}{2e^\alpha}\sum_{\mathbf{M}'=0}^{\infty}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}\Gamma(\beta+\rho\mathbf{M}')}{\mathbf{M}'!}\int_0^\infty \int_0^\infty f(u^2+v^2) g\left[tan^{-1}\left(\frac{v}{u}\right)\right]$$

$$A_{\mathbf{p},\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1 (u^2+v^2)^{\lambda_1}e^{-c_1} & \mathbb{A} \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{Z}_s (u^2+v^2)^{\lambda_s}e^{-c_s} & (1-\alpha-\beta-d\mathbf{M}': c_1,\cdots,c_s),\mathbb{B} \end{array}\right)\mathrm{d}u\mathrm{d}v \tag{5.2}$$

$\phi'(\mathbf{M})$ is defined by (4.2),

provided that

$min\{Re(\alpha), Re(\beta), e, d, c_i, \lambda_i\} > 0$ for $i = 1,\cdots,s$; $|arg(\Omega'_i)Z_k| < \frac{1}{2}\eta'_k\pi, \xi'^* = 0, \eta'_i > 0$

and $Re\left[\alpha + d\mathbf{M}' + \sum_{i=1}^{s} c_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{D_j^{(i)}}\right] > 0$

The serie occuring on the right hand side of (5.1) and (5.2) are absolutely convergents. The function f and g are chosen that various integrals involved in (5.1) and (5.2) exist.

Proof of (4.1) and (4.2) .

To establish the integral relations (5.1) and (5.2), we take in (2.1)

$$h(R,\theta) = \frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{(\cos^2\theta + e\sin^2\theta)^{\alpha+\beta}} H^{1,n_1'}_{p_1',q_1'+1}\left[\frac{a(\cos\theta)^{2d}}{(\cos^2\theta + e\sin^2\theta)^d}\right]$$

$$A\left(\begin{array}{c|c} \mathrm{Z}_1\frac{(\sin\theta)^{2c_1}(\cos\theta)^{2d_1}}{(\cos^2\theta+e\sin^2\theta)^{c_1+d_1}} & \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_s\frac{(\sin\theta)^{2c_s}(\cos\theta)^{2d_s}}{(\cos^2\theta+e\sin^2\theta)^{c_s+d_s}} & (1\text{-}\alpha : c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \tag{5.3}$$

Now, we evaluate the resulting integral by means of (4.1) and arrive at the following result after algebric manipulations and simplifications :

$$F(R) = = \frac{1}{2e^\alpha}\sum_{\mathbf{M}'=0}^{\infty}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}\Gamma(\beta+\rho\mathbf{M}')}{\mathbf{M}'!}A^{\mathbf{m},\mathbf{n};X}_{\mathbf{p},\mathbf{q}+1;Y}\left(\begin{array}{c|c} \mathrm{Z}_1R^{2\lambda_1}e^{-c_1} & \mathbb{A} \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{Z}_sR^{2\lambda_s}e^{-c_s} & (1-\alpha : c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \tag{5.4}$$

Now substituting the values of $h(R,\theta)$ and $F(R)$ as given by (5.3) and (5.4) respectively in (2.2) and (2.3) in succession, we obtain the integral relations (5.1) and (5.2) after algebric manipulations and simplifications.

6. Applications

By suitable choosing the function f and g in the integral relations obtained in the previous section, a large number of interesting double and triple integrals can be evaluated. We shall obtain here only one double and one triple integral by way of illustration.

Thus if in (5.2), we set

$$g(t) = \frac{(\sin t)^{2\alpha-1}(\cos t)^{2\beta-1}}{(\cos^2 t + e\sin^2 t)^{\alpha+\beta}} H^{1,N'}_{P',Q'+1}\left[\mathrm{a}\frac{(\cos t)^{2d}}{(\cos^2 t+c\sin^2 t)^d}\ \middle|\ \begin{array}{c} (\mathrm{e}_j,E_j)_{1,P'} \\ \cdot \\ (0,1),\ (\mathrm{f}_j,F_j)_{1,Q'} \end{array}\right] \tag{6.1}$$

we arrive at the following integral relation on making use of (5.1) with $max\{c_1,\cdots,c_s\} \rightarrow 0$

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(xz)^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^\alpha(x^2+y^2+z^2)}{(x^2+ey^2)^{\alpha+\beta}[z^2+e(x^2+y^2)]^{c+d}} f(x^2+y^2+z^2)$$

$$H_{P,Q}^{1,N}\left[\frac{ax^{2d}}{(x^2+ey^2)^d}\right] H_{P',Q'+1}^{1,N'}\left[\begin{array}{c|c} \frac{az^{2d}}{[z^2+e(x^2+y^2)]^d} & \begin{array}{c}(\mathrm{e}_j,E_j)_{1,P'}\\ \cdot \\ (0,1),\ (\mathrm{f}_j,F_j)_{1,Q'}\end{array}\end{array}\right]$$

$$A_{\mathbf{p},\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1 y^{2c_1}\frac{(x^2+y^2+z^2)^{\lambda_1}}{(x^2+ey^2)^{c_1}} & \mathbb{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{Z}_s {y'}^{2c_s}\frac{(x^2+y^2+z^2)^{\lambda_s}}{(x^2+ey^2)^{c_s}} & (1\text{-}\alpha;c_1,\cdots,c_s),\mathbb{B}\end{array}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{\Gamma(\alpha)}{8e^{2\alpha}}\sum_{\mathbf{M}',\mathbf{M}''=0}^{\infty}\phi'(\mathbf{M}')\phi'(\mathbf{M}'')\frac{(-a)^{\mathbf{M}'+\mathbf{M}''}}{\mathbf{M}'!\mathbf{M}''!}$$

$$\frac{\Gamma(\beta+d\mathbf{M}')\Gamma(\beta+d\mathbf{M}'')}{\Gamma(\alpha+\beta+d\mathbf{M}'')}\int_0^\infty f(t)\, A_{\mathbf{p},\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1 t^{\lambda_1}e^{-c_1} & \mathbb{A}\\ \cdot & \cdots\\ \cdot & \cdots\\ \mathrm{Z}_s t^{\lambda_s}e^{-c_s} & (1-\alpha-\beta-d\mathbf{M}':c_1,\cdots,c_s),\mathbb{B}\end{array}\right)\mathrm{d}t \qquad (6.2)$$

provided that

$min\{Re(\alpha),Re(\beta),e,d,c_i,\lambda_i\}>0$ for $i=1,\cdots,s\,;|arg(\Omega_i')Z_k|<\frac{1}{2}\eta_k'\pi,\xi'^{*}=0,\eta_i'>0$

and $Re\left[\alpha+d\mathbf{M}'+\sum_{i=1}^{s}c_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{D_j^{(i)}}\right]>0$

The serie occuring on the right hand side of (5.1) and (5.2) are absolutely convergents. The function f is chosen that various integrals involved in (5.1) and (5.2) exist.

where

$$\phi'(\mathbf{M}')=\frac{\prod_{j=1}^{N}\Gamma(1-a_j'+A_j'\mathbf{M}')}{\prod_{j=N+1}^{P}\Gamma(a_j'-A_j'\mathbf{M}')\prod_{j=1}^{Q}\Gamma(1-b_j'+B_j'\mathbf{M})} \qquad (6.3)$$

and

$$\phi'(\mathbf{M}'')=\frac{\prod_{j=1}^{N'}\Gamma(1-e_j+E_j\mathbf{M}'')}{\prod_{j=N'+1}^{P'}\Gamma(e_j-E_j\mathbf{M}'')\prod_{j=1}^{Q'}\Gamma(1-f_j+F_j\mathbf{M}'')} \qquad (6.4)$$

Nest, we take

$$f(t)=H_{\mu,\upsilon}^{\lambda,0}\left[\begin{array}{c|c} \mathrm{t}^{\omega-1} & \begin{array}{c}(\mathrm{g}_j,G_j)_{1,\mu}\\ \cdot\\ (\mathrm{h}_j,H_j)_{1,\upsilon}\end{array}\end{array}\right] \qquad (6.5)$$

and evaluate the $t-$integrals occuring on the right hand sides of (5.1) and (5.2) with the help of formula obtained ([6], 1977 ,page 112, eq.(4.3)) and arrive at the following multiple integrals after algebric manipulations and simplifications.

$$\int_0^\infty \int_0^\infty \frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)}{(x^2+ey^2)^{\alpha+\beta}} H_{P,Q}^{1,N}\left[\frac{ax^{2d}}{(x^2+ey^2)^d}\right] H_{\mu,\upsilon}^{\lambda,0}\left[\xi(x^2+y^2)\right]$$

$$A_{\mathbf{p},\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1 y^{2c_1}\frac{(x^2+y^2)^{\lambda_1}}{(x^2+ey^2)^{c_1}} & \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_s {y'}^{2c_s}\frac{(x^2+y^2)^{\lambda_s}}{(x^2+ey^2)^{c_s}} & (1\text{-}\alpha; c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \mathrm{d}x\mathrm{d}y = \frac{\xi^{-\omega}}{4e^\alpha}\sum_{\mathbf{M}'=0}^{\infty}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}\Gamma(\beta+\rho\mathbf{M}')}{\mathbf{M}'!}$$

$$A_{\mathbf{p}+\upsilon,\mathbf{q}+\mu+1;Y}^{\mathbf{m},\mathbf{n}+\lambda;X}\left(\begin{array}{c|c} \mathrm{Z}_1\xi^{-\lambda_1}e^{-c_1} & (1\text{-h}_j-\xi H_j:\lambda_1 H_j,\cdots,\lambda_s H_j)_{1,\upsilon},\mathbb{A} \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{Z}_s\xi^{-\lambda_s}e^{-c_s} & (1\text{-g}_j-\xi G_j:\lambda_j G_j,\cdots,\lambda_s G_j)_{1,\mu},(1-\alpha-\beta-d\mathbf{M}':c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \quad (6.6)$$

provided that

$min\{Re(\alpha), Re(\beta), e, d, c_i, \lambda_i\} > 0$ for $i = 1,\cdots,s\,; |arg(\Omega_i')Z_k| < \frac{1}{2}\eta_k'\pi, \xi'^{*} = 0, \eta_i' > 0$

$Re\left[\alpha + d\mathbf{M}' + \sum_{i=1}^{s} c_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{D_j^{(i)}}\right] > 0$ and $Re\left[\xi + \min_{1\leqslant j\leqslant\lambda}\frac{h_j}{H_j} + \sum_{i=1}^{s}\lambda_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{D_j^{(i)}}\right] > 0$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{(xz)^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^\alpha(x^2+y^2+z^2)}{(x^2+ey^2)^{\alpha+\beta}[z^2+e(x^2+y^2)]^{c+d}} H_{\mu,\upsilon}^{\lambda,0}\left[\xi(x^2+y^2+z^2)\right]$$

$$H_{P,Q}^{1,N}\left[\frac{ax^{2d}}{(x^2+ey^2)^d}\right] H_{P',Q'+1}^{1,N'}\left[\begin{array}{c|c} \frac{az^{2d}}{[z^2+e(x^2+y^2)]^d} & \begin{array}{c}(\mathrm{e}_j,E_j)_{1,P'} \\ \cdot \\ (0,1),\ (\mathrm{f}_j,F_j)_{1,Q'}\end{array} \end{array}\right]$$

$$A_{\mathbf{p},\mathbf{q}+1;Y}^{\mathbf{m},\mathbf{n};X}\left(\begin{array}{c|c} \mathrm{Z}_1 y^{2c_1}\frac{(x^2+y^2+z^2)^{\lambda_1}}{(x^2+ey^2)^{c_1}} & \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_s {y'}^{2c_s}\frac{(x^2+y^2+z^2)^{\lambda_s}}{(x^2+ey^2)^{c_s}} & (1\text{-}\alpha; c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$\frac{\Gamma(\alpha)\xi^{-\omega}}{8e^{2\alpha}}\sum_{\mathbf{M}',\mathbf{M}''=0}^{\infty}\phi'(\mathbf{M}')\phi'(\mathbf{M}'')\frac{(-a)^{\mathbf{M}'+\mathbf{M}''}}{\mathbf{M}'!\mathbf{M}''!}\frac{\Gamma(\beta+d\mathbf{M}')\Gamma(\beta+d\mathbf{M}'')}{\Gamma(\alpha+\beta+d\mathbf{M}'')}$$

$$A^{\mathbf{m},\mathbf{n}+\lambda;X}_{\mathbf{p}+\upsilon,\mathbf{q}+\mu+1;Y}\left(\begin{array}{c|c} \mathrm{Z}_1\xi^{-\lambda_1}e^{-c_1} & (1\text{-h}_j-\xi H_j:\lambda_1 H_j,\cdots,\lambda_s H_j)_{1,\upsilon},\mathbb{A} \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{Z}_s\xi^{-\lambda_s}e^{-c_s} & (1\text{-g}_j-\xi G_j:\lambda_j G_j,\cdots,\lambda_s G_j)_{1,\mu},(1-\alpha-\beta-d\mathbf{M}':c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \quad (6.7)$$

provided that

$min\{Re(\alpha),Re(\beta),e,d,c_i,\lambda_i\}>0$ for $i=1,\cdots,s\,;|arg(\Omega_i')Z_k|<\frac{1}{2}\eta_k'\pi,\xi'^*=0,\eta_i'>0$

$Re\left[\alpha+d\mathbf{M}'+\sum_{i=1}^{s}c_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{D_j^{(i)}}\right]>0$ and $Re\left[\xi+\min_{1\leqslant j\leqslant\lambda}\frac{h_j}{H_j}+\sum_{i=1}^{s}\lambda_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{D_j^{(i)}}\right]>0$

7. Multivariable H-function

If the multivariable A-function reduces to multivariable H-function defined by Srivastava et al [8,9], we have the following integrals.

$$\int_0^\infty\int_0^\infty\frac{x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)}{(x^2+ey^2)^{\alpha+\beta}}H^{1,N}_{P,Q}\left[\frac{ax^{2d}}{(x^2+ey^2)^d}\right]H^{\lambda,0}_{\mu,\upsilon}\left[\xi(x^2+y^2)\right]$$

$$H^{0,\mathbf{n};X}_{\mathbf{p},\mathbf{q}+1;Y}\left(\begin{array}{c|c} \mathrm{Z}_1 y^{2c_1}\frac{(x^2+y^2)^{\lambda_1}}{(x^2+ey^2)^{c_1}} & \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_s y'^{2c_s}\frac{(x^2+y^2)^{\lambda_s}}{(x^2+ey^2)^{c_s}} & (1\text{-}\alpha;c_1,\cdots,c_s),\mathbb{B} \end{array}\right)\mathrm{d}x\mathrm{d}y=\frac{\xi^{-\omega}}{4e^{\alpha}}\sum_{\mathbf{M}'=0}^{\infty}\phi(\mathbf{M}')\frac{(-a)^{\mathbf{M}'}\Gamma(\beta+\rho\mathbf{M}')}{\mathbf{M}'!}$$

$$H^{0,\mathbf{n}+\lambda;X}_{\mathbf{p}+\upsilon,\mathbf{q}+\mu+1;Y}\left(\begin{array}{c|c} \mathrm{Z}_1\xi^{-\lambda_1}e^{-c_1} & (1\text{-h}_j-\xi H_j:\lambda_1 H_j,\cdots,\lambda_s H_j)_{1,\upsilon},\mathbb{A} \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{Z}_s\xi^{-\lambda_s}e^{-c_s} & (1\text{-g}_j-\xi G_j:\lambda_j G_j,\cdots,\lambda_s G_j)_{1,\mu},(1-\alpha-\beta-d\mathbf{M}':c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \quad (7.1)$$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{(xz)^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{\alpha}(x^2+y^2+z^2)}{(x^2+ey^2)^{\alpha+\beta}[z^2+e(x^2+y^2)]^{c+d}}H^{\lambda,0}_{\mu,\upsilon}\left[\xi(x^2+y^2+z^2)\right]$$

$$H^{1,N}_{P,Q}\left[\frac{ax^{2d}}{(x^2+ey^2)^d}\right]H^{1,N'}_{P',Q'+1}\left[\begin{array}{c|c} \frac{az^{2d}}{[z^2+e(x^2+y^2)]^d} & \begin{array}{c}(\mathrm{e}_j,E_j)_{1,P'}\\ \cdot \\ (0,1),\ (\mathrm{f}_j,F_j)_{1,Q'}\end{array} \end{array}\right]$$

$$H^{0,\mathbf{n};X}_{\mathbf{p},\mathbf{q}+1;Y}\left(\begin{array}{c|c} \mathrm{Z}_1 y^{2c_1}\frac{(x^2+y^2+z^2)^{\lambda_1}}{(x^2+ey^2)^{c_1}} & \mathbb{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_s {y'}^{2c_s}\frac{(x^2+y^2+z^2)^{\lambda_s}}{(x^2+ey^2)^{c_s}} & (1\text{-}\alpha;c_1,\cdots,c_s),\mathbb{B} \end{array}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$\frac{\Gamma(\alpha)\xi^{-\omega}}{8e^{2\alpha}}\sum_{\mathbf{M}',\mathbf{M}''=0}^{\infty}\phi'(\mathbf{M}')\phi'(\mathbf{M}'')\frac{(-a)^{\mathbf{M}'+\mathbf{M}''}}{\mathbf{M}'!\mathbf{M}''!}\frac{\Gamma(\beta+d\mathbf{M}')\Gamma(\beta+d\mathbf{M}'')}{\Gamma(\alpha+\beta+d\mathbf{M}'')}$$

$$H^{0,\mathbf{n}+\lambda;X}_{\mathbf{p}+\upsilon,\mathbf{q}+\mu+1;Y}\left(\begin{array}{c|c} \mathrm{Z}_1\xi^{-\lambda_1}e^{-c_1} & (1\text{-h}_j-\xi H_j:\lambda_1 H_j,\cdots,\lambda_s H_j)_{1,\upsilon},\mathbb{A} \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{Z}_s\xi^{-\lambda_s}e^{-c_s} & (1\text{-g}_j-\xi G_j:\lambda_j G_j,\cdots,\lambda_s G_j)_{1,\mu},(1-\alpha-\beta-d\mathbf{M}':c_1,\cdots,c_s),\mathbb{B} \end{array}\right) \quad (7.2)$$

Remark

If $\mathbf{n}=0$, we obtain the results of Garg [1]

8. Conclusion

The multiple integral relations (5.1) and (5.2) are quite general nature on account of the arbitrary nature of the functions f and g and also on account of the presence of Fox's H-functions and the multivariable A-function defined by Gautam et al [2]. A number of special cases of these relations can be obtained by suitably reducing the H-functions and multivariable A-function in terms of simpler special functions of one and several variables.

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