Common Fixed Point Theorem of New Type of Contraction in Complete Metric Space.

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Abstract:-In this paper we define new type of contraction and Prove common fixed point theorem for the sequence of mappings of this new type of contraction in Complete Metric Space.

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1. Introduction

Metric fixed Point theory has tremendous applications in many Branches of Science and Engineering field. Metric fixed point theory developed after the Polish Mathematician S.Banach [1] proved Banach Contraction principle, which states that a contraction map on a complete Metric Space has a unique fixed point. Now Fixed point theory has a vast literature. In 1968 R.Kanan [2] proved a fixed point theorem for self map. In 1971 Chatterjea [3] proved a fixed point theorem for a self map which is a modification of Kanan map.
In 1980 Khan M.S. [4] proved some fixed point theorems in Metric and Banach Space. Afterward Delbosko [5] defined set of all continuous functions $g : R_+^3 \to R_+$ which satisfying some properties and proved some fixed point results .In 1994AndrianConstantin [6] proved common fixed point theorem for two pair of weakly commuting maps. Now In this paper we define a new contraction and prove common fixed point theorems for sequence of maps.

2. Preliminaries

Definition 2.1:-Let X be a non-empty set . A mapping $d : X \times X \to R$ is said to be a Metric or a distance function if it satisfies following conditions.

1. $d(x, y)$ is non-negative.
2. $d(x, y) = 0$ if and only if x and y coincides i.e. x=y.
3. $d(x, y) = d(y, x)$ (Symmetry)
4. $d(x, y) \le d(x, z) + d(z, y)$ (Triangle inequality)

Then the function d is referred to as metric on X. And (X,d) or simply X is said to as Metric space.

Definition 2.2:- A Metric space (X,d) is said to be a complete Metric space if every Cauchy sequence in X converges to a point of X.

Definition 2.3:- If (X,d) be a complete Metric space and a function $F : X \to X$ is said to be a contraction map if

$$d(F(x), F(y)) \le \beta d(x, y)$$

For all $x, y \in X$ and for $0 < \beta < 1$

Definition 2.4:- Let $F : X \to X$, then $x \in X$ is said to be a fixed point of F if $F(x) = x$

Definition 2.5:-Let X be a Metric space and if F_1 and F_2 be any two maps. An element $a \in X$ is said to be a common fixed point of F_1 and F_2 if $F_1(a) = F_2(a)$

For ex:- If $F_1(x) = sin(x)$ *and* $F_2(x) = tan(x)$

Then 0 is the common fixed point F_1 *and* F_2 *Since* $F_1(0) = sin(0)$ *and* $F_2(0) = tan(0) = 0$

Theorem 2.1 (Kanan fixed point theorem in complete metric space) [2]
Let X be a Complete metric Space and $F : X \to X$ is a mapping such that,
$d(Fx, Fy) \leq \beta\ [d(x, Fx) + d(y, Fy)]$ for all $x, y \in X$
Where $\beta \in\ ]0,1[$ Then F has a unique fixed point.

Theorem 2.2[3]A mapping $F : X \to X$ where (X,d) is a Metric space is said to be C-Contraction if there is a some β s.t. $0 < \beta < \frac{1}{2}$ s.t. the following inequality holds

$$d(F_x, F_y) \leq \beta(d(x, F_y) + d(y, F_x))$$

If (X, d) be a complete Metricspace, then any C-contraction on X has a unique fixed point.

In 1981 Delbosco [5] defined the set G of all continuous mappings $\alpha : [0,\infty)^3 \to [0,\infty)$ which satisfies the following conditions

(i) $\alpha(1,1,1) = k < 1$

(ii) Let $a, b \geq 0$ be such that either $a \leq k(a,b,b)$ or $a \leq k(b,b,a)$ or $a \leq k(b,a,b)$.Then $a \leq kb$.

And Delbosco proved that for $P : X \to X$, $Q : X \to X$ on a Complete metric space (X,d)
Which satisfies the condition,

$d(Pa, Qb) \leq \alpha(d(a,b), d(a,Qb), d(b,Qb))$ (1)

Forall $a, b \in X$,whereα in G.Then P and Q have a unique common fixed point. Then in 1994 Adrian Constantin proved following result of common fixed point theorem.

Let P and Q be two self maps of a metric space (X,d) which satisfies following conditions

(i) $d(Pa, Qb) \leq \alpha(d(a,b), d(a, Pa), d(b, Qb))$

for all $a, b \in X$ and $\alpha \in G$

(ii)There is a point $v \in X$ so that P is continuous at v and Q is continuous at Pv,

(iii)There exists a point $a \in X$ s.t. the sequence $\{(P \circ S)^n(a)\} = \{(Q \circ P)^n\}$ has a subsequence $\{(QP)^{n_i}(a)\}$ converging to v.Then v'=Pu is the unique fixed point of P and Q.

And Delbosco proved common fixed point theorem for pair of two weakly commuting mappings also

Theorem 2.3[5]IfP and R be weakly commuting mappings and if Q and S be weakly commuting self mappings of a complete metric space (X,d) into itself which satisfies the following conditions

$d(Px, Qy) \leq \alpha(d(Rx, Sy), d(Rx, Px), d(Sy, Qy))$ for all x,y òX

for α òG.If the range of R contains the range of Q and the range of S contains range of P, and if one of P,Q,R and S is continuous ,then P,Q,R and S have a unique common fixed point z.

Now we modify Delboscos inequality for two maps given by equation (1) and prove some common fixed point theorems .

First we introduce β Let Ebe the set of all functions $\beta : [0,\infty)^3 \to [0,\infty)$ which satisfies

(i) β is continuous on the set $[0,\infty)^3$ (with respect to Euclidean metric on $[0,\infty)^3$)

(ii) $x \le ky$ for some k s.t. $0 \le k < 1$ whenever $x \le \beta(y,y,x)$ or $x \le \beta(y,x,y)$

or $x \le \beta(x,y,y)$ for all x,y $\in [0,\infty)$

(iii) $\beta(x,y,y) = 0$ iff x=y=0

Definition 2.6:- A mapping F on a metric space X into itself is said to be New contraction if it satisfies the following condition.

$d(Fa,Fb) \le \beta(d(a,b), d(a,Ta), d(b,Fb))$

for every a,b $\in$ X and some $\beta \in E$.

Example 2.1 :- A mapping $F : X \to X$ defined by

$d(Fa,Fb) \le \alpha \max\{d(Fa,a)+d(Fb,b), d(Fb,b)+d(a,b), d(Fa,a)+d(a,b)\}$

for all a,b in X and some $0 \le \alpha < \frac{1}{2}$ is New type contraction.

The map $\beta : R_+^3 \to R_+$ is defined as

$\beta(u,v,w) = \alpha \max\{u+v, v+w, u+w\}$

for every u,v,w $\in R_+$, *where* α is s.t. $0 \le \alpha < \frac{1}{2}$. Then as $\beta \in E$.

Clearly β is continuous. Also for $u \le \beta(u,v,v) = \alpha \max\{u+v, v+u, v+v\}$

There are two possibilities

Case (i) If $\max\{u+v, v+u, v+v\} = u+v$

$\therefore$ In this case

$u \le \frac{\alpha}{1-\alpha} \le kv$, where $k = \frac{\alpha}{1-\alpha}$ in [0,1)

Case (ii) If $\max\{u+v, v+u, v+v\} = v+v$

Case(ii) If $\therefore$ $u \le kv$ where $k = 2\alpha$, for $0 \le \alpha < 1$

$\therefore$ we have for $u \le \beta(v,u,v)$ or $u \le \beta(v,v,u)$ we obtain $u \le kv$ for k in $0 \le \alpha < 1$.

$d(Fa,Fb) \le \alpha \max\{d(Fa,a)+d(Fb,b), d(Fb,b)+d(a,b), d(Fa,a)+d(a,b)\}$

$= \beta(d(a,b), d(Fa,a), d(Fb,b))$

$\therefore$ by definition F is new type contraction.

3 Main Result

Now we prove a fixed point result for new type contraction.

Theorem 3.1 :-If (X,d) be a Complete Metric space and if F be mapping on X into itself which satisfies

$d(Fa,Fb) \le \beta(d(a,b), d(a,Ta), d(b,Fb))$ (3.1)

for all a,b $\in$ X and some $\beta \in X$.

F has a Unique fixed point in X.

Proof :-Let a_0 be an arbitrary point in X. We construct a sequence $\{a_n\}$ in X as

$a_{n+1} = Fa_n$, $a_1 = Fa_0$, $a_2 = Fa_1, \ldots\ldots, a_{n+1} = Fa_n$ i.e. $a_n = F^n a_0$

Given F satisfies (3.1)

Consider,

$$d(a_n, a_{n+1}) = d(Fa_{n-1}, Fa_n)$$
$$\le \beta(d(a_{n-1}, a_n), d(Fa_{n-1}, a_{n-1}), d(Fa_n, a_n))$$
$$\le \beta(d(a_{n-1}, a_n), d(a_n, a_{n-1}), d(a_{n+1}, a_n))$$
$$\le kd(a_{n-1}, a_n) \quad (3.2)$$

Similarly,

$d(a_{n-1}, a_n) \le kd(a_{n-2}, a_{n-1})$

$\therefore$ (3.2) gives

$d(a_n, a_{n+1}) \le k^2 d(a_{n-2}, a_{n-1})$

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$$d(a_n, a_{n+1}) \le k^n d(a_0, a_1) \quad (3.3)$$

for $0 \le k < 1$

Letting $n \to \infty$ we have $\{a_n\}$ is a Cauchy sequence in X. And as X is complete.

$\therefore \{a_n\}$ converges to a point in X. Let $\{a_n\}$ converges to $u \in X$

$\therefore$ for a=u and b=a_n

inequality (3.1) gives

$$d(Fa, a_{n+1}) = d(Fu, Fa_n)$$
$$\le \beta(d(u, a_n), d(u, Fu), d(a_n, Fa_n))$$
$$= \beta(d(u, a_n), d(u, Fu), d(a_n, a_{n+1}))$$

Taking limit as $n \to \infty$ and since given β is continuous we have

$d(Fu, u) \le \beta(d(u, u), d(u, Fu), d(u, u))$

$\therefore d(Fu, u) \le k.0 = 0$

Thus d(Fu,u)=0

this gives Fu=u

$\therefore u$ is a fixed point of F.

Uniqueness: - Now if possible suppose x be another fixed point of F

$\therefore Fx = x$

Now we put a=x and b=u in inequality (3.1), we have

Consider,

$$d(x, u) = d(Tx, u)$$
$$\le \beta(d(x, u), d(Fx, x), d(Fu, u))$$
$$\le \beta(d(x, u), d(w, w), d(u, u))$$
$$\le \beta(d(x, u), 0, 0)$$

$\therefore d(x, u) \le k.0$

$\therefore d(x, u) = 0$

This gives x=u.

Now we prove another result of common fixed point theorem for sequence of mappings of new type of contraction.

Theorem 3.2 Let (X,d) be a complete Metric space. And if $\beta \in E$ and the sequence $\{F_n\}_{n=1}^{\infty}$ of maps on X into itself is s.t.

$$d(F_i a, F_j b) \leq \beta(d(a,b), d(F_i a, a), d(F_j b, b)) \qquad (3.4)$$

for all a,b $\in X$ .Then sequence $\{F_n\}_{n=1}^{\infty}$ has a unique common fixed point in X.

Proof: -We construct a sequence $\{a_n\}$ of points of X s.t. for some fixed a_0 in X. For each n in N

We define $a_n = F_n a_{n-1}$, as $\beta \in E$, from inequality (3.4)

Consider,

$$d(a_1, a_2) = d(F_1 a_0, F_2 a_1)$$
$$\leq \beta(d(a_0, a_1), d(a_0, F_1 a_0), d(a_1, F_2 a_1)$$
$$= \beta(d(a_0, a_1), d(a_0, a_1), d(a_1, a_2)$$
$$\leq kd(a_0, a_1) \qquad (a)$$

for $0 \leq k < 1$.

simmilarly for $x_2, x_3 \in X$

$$d(a_2, a_3) = d(F_2 a_1, F_3 a_2)$$
$$\leq \beta(d(a_1, a_2), d(a_1, F_2 a_1), d(a_2, F_3 a_2))$$
$$= \beta(d(a_1, a_2), d(a_1, a_2), d(a_2, a_3)$$
$$\leq kd(a_1, a_2) \qquad (b)$$

from (a) and (b) we have

$$\therefore d(a_2, a_3) \leq k^2 d(a_0, a_1)$$

In general ,We have

$$d(a_n, a_{n+1}) \leq k^n d(a_0, a_1)$$

For $0 \leq k < 1$. $\therefore \{a_n\}$ Is a Cauchy Sequence in X. Since X is complete. $\therefore$ it converges to $u \in X$

consider,

$$d(u, F_n u) \leq d(u, a_{m+1}) + d(a_{m+1}, F_n u)$$
$$= d(u, a_{m+1}) + d(F_{m+1} a_m, F_n u)$$
$$\leq d(u, a_{m+1}) + \beta(d(a_m, u), d(F_{m+1} a_m, a_m), d(F_n u, u)) \qquad \text{(since by (3.4))}$$
$$\leq d(u, a_{m+1}) + \beta(d(a_m, u), d(a_{m+1}, a_m), d(F_n u, u))$$

For all $m, n \in N$. Letting $m \to \infty$ then above relation gives

$$d(u, F_n u) \leq d(u, u) + \beta(d(u, u), d(u, u), d(F_n u, u))$$
$$\leq \beta(0, 0, d(F_n u, u))$$
$$\leq 0$$
$$\therefore d(u, F_n u) = o$$
$$\therefore F_n u = u, \text{ for all n in N}$$

This gives u is the common fixed point of sequence of maps$\{F_n\}$.

To prove Uniqueness of fixed point, If possible suppose w in X be another fixed point of F_n

$$\therefore F_n w = w$$

Now consider

$$d(u,w) = d(F_i u, F_j w)$$

$$\leq \beta(d(u,w), d(F_i u, u), d(w, F_j b) \quad (\text{by } (3.4))$$

$$= \beta(d(u,w), d(u,u), d(w,w)$$

$$= \beta(d(u,w), 0, 0)$$

$$\leq k.0$$

$\therefore\ d(u,w) = 0$ This gives u=w. i.e. the common fixed point of $\{F_n\}$ is unique.

Conclusion: - Thus we have defined a new type of contraction and proved one result of fixed point and one result of common fixed point of sequence of mappings.

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