# Integral Transform and the Solution of Fractional Kinetic Equation Involving Some Special Functions 

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#### Abstract

The aim of the present paper is to establish the solution of advanced generalized fractional order kinetic equation and a main theorem based upon the multivariable I-function, Mittag -Leffler function, generalized $M$-series, generalized $\boldsymbol{K}_{4}$ - function, and generalized Mittag-Leffler function, Riemann-Liouville operator. The solution of the generalized fractional kinetic equation involving the multivariable I-function is obtained with help of the Laplace and Sumudu transform. Due to its simple formulation and consequent special and useful properties, the Sumudu and Laplace transform has already shown much promise. It is revealed herein and elsewhere that is can help to solve intricate problems in mathematical physics, especially in astrophysical problems. The results derived by using certain Corollaries used in this paper are interesting, computable and very general in nature.


Keywords and Phrases: Special functions, Fractional Kinetic equation, Mittag-Leffler function, RiemannLiouville operator, Laplace transform, Sumudu transform.

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## I. Introduction and Preliminaries

The fractional calculus has many important developments. Fractional calculus is a field of applied Mathematics that deals with derivatives and integrals of fractional order. Recently, a remarkable interest has been developed in the study of the solution of fractional kinetic equations due to their importance in astrophysics and mathematical physics. Due to the importance of kinetic equation in mathematical physics many authors have generalized the standard kinetic equation time to time. The kinetic equations of fractional order have been successfully used to determine certain phenomena governing diffusion in porous media, reaction and relaxation processes in complex systems etc.

In the recent paper of Haubold and Mathai [10] have derived the fractional kinetic equation and thermonuclear function in terms of well known Mittag-Leffler function. The Sun which is a big star is assumed to be in thermal equilibrium and hydrostatic equilibrium. To describe a model, we consider it is a spherical symmetric, self-gravitating non-rotating. The features of its area, mass, luminosity, diameter, effective surface temperature, central temperature and density. The assumptions of thermal equilibrium and hydrostatic equilibrium imply that there is no time dependence in the equations describing the internal structure of the star like sun (Kourganoff 1976 ,Perdang 1976, Clayton 1983). Energy in such stars being produced by the process of chemical reactions. For details we refer to [10].

As extensions of the work of Saxena et al. [18] have generalized the standard kinetic equation with generalized Mittag- Leffler functions. Further, Chaurasia and Kumar [22] generalized and studied the kinetic equation with generalized M-series of Sharma [13], generalized $K_{4}$-function of Faraz and Salim [2, 27]. For more result one can refer to the work of Saichev and Zaslavsky [4], Sexcena et al [16, 18], Zaslavsky [9] and Sexena and Kalla [17].

Haubold and Mathai [10] have established a functional differential equation between rate of change of reaction, the destruction rate and the production rate as follows

$$
\begin{equation*}
\frac{d \mathcal{N}}{d t}=-d\left(\mathcal{N}_{t}\right)+p\left(\mathcal{N}_{t}\right) \tag{1}
\end{equation*}
$$

where $\mathcal{N}=\mathcal{N}(t)$ the rate of reaction, $d=d(\mathcal{N})$ the rate of destruction, $p=p(\mathcal{N})$ the rate of production and denotes the function defined by $\mathcal{N}_{t}\left(t^{*}\right)=\mathcal{N}\left(t-t^{*}\right), t^{*}>0$.

They have studied a the special case of (1), for spatial fluctuations or in homogeneities in the quantity $\mathcal{N}(t)$ are neglected, namely the equation

$$
\begin{equation*}
\frac{d \mathcal{N}_{i}}{d t}=-c_{i} \mathcal{N}_{i}(t) \tag{2}
\end{equation*}
$$

together with the initial condition that $\mathcal{N}_{i}(t=0)=\mathcal{N}_{0}$ is the number of density of species i at time $t=$ $0, c_{i}>0$. Dropping the index i and integrate the standard kinetic equation (2) we obtain

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0}=-c_{0} D_{t}^{-1} \mathcal{N}(t) \tag{3}
\end{equation*}
$$

Replacing the Riemann integral operator $D_{t}^{-1}$ by the fractional Riemann-Liouville operator $D_{t}^{-v}$ [19] in equation (3), we obtain

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0}=-c_{0} D_{t}^{-v} \mathcal{N}(t) \tag{4}
\end{equation*}
$$

Haubold and Mathai in [10] found the solution of (4) as follows

$$
\begin{equation*}
\mathcal{N}(t)=\mathcal{N}_{0} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{\Gamma(v n+1)}(c t)^{v k} \tag{5}
\end{equation*}
$$

Also, Sexena, Mathai and Haubold [18] studied the generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions which is the extension of the work of Haubold and Mathai [10].

Over the set of function,

$$
\begin{equation*}
\mathbb{A}=\left\{f(t)\left|\exists M, \tau_{1}, \tau_{2}>0,|f(t)|<M e^{|t| / \tau_{j}}, \text { if } t \in(-1)^{j} \times[0, \infty)\right\}\right. \tag{6}
\end{equation*}
$$

the Sumudu transform is defined by

$$
\begin{equation*}
\mathfrak{G}(u)=\mathbb{S}[f(t)]=\int_{0}^{\infty} f(u t) e^{-t} d t, u \epsilon\left(-\tau_{1}, \tau_{2}\right) \tag{7}
\end{equation*}
$$

For more detail and properties of Sumudu transform (see in [14, 17, 18, 23]).The Riemann-Liouville fractional integral of order $\vartheta$ is defined by $[1,12]$

$$
\begin{equation*}
{ }_{0} D_{t}^{-\vartheta} N(x, t)=\frac{1}{\Gamma(\vartheta)} \int_{0}^{t}(t-u)^{\vartheta-1} N(x, u) d u, \quad \operatorname{Re}(\vartheta)>0 \tag{8}
\end{equation*}
$$

The Sumudu transform of the Riemann-Liouville fractional integral is defined as [12, 1]

$$
\begin{equation*}
\mathbb{S}\left\{{ }_{0} D_{t}^{-\vartheta} f(t) ; u\right\}=u^{\vartheta} \bar{f}(u) \tag{9}
\end{equation*}
$$

We also use the following interesting result

$$
\begin{equation*}
\mathbb{S}^{-1}\left[u^{\gamma-1}\left(1-\omega u^{\beta}\right)^{-\delta}\right]=t^{\gamma-1} E_{\beta \gamma}^{\delta}\left(\omega t^{\beta}\right) \tag{10}
\end{equation*}
$$

The Laplace transform of the function $\mathrm{N}(\mathrm{x}, \mathrm{t})$ with respect to t is

$$
\begin{equation*}
\mathcal{L}[N(x, t)]=\int_{0}^{\infty} e^{-s t} N(x, t) d t=N^{*}(x, s), \quad x \in \Re, \Re(s)>0 \tag{11}
\end{equation*}
$$

And its inverse transform with respect to $s$ is given by

$$
\begin{equation*}
\mathcal{L}^{-1}\left[N^{*}(x, s)\right]=\frac{1}{2 \pi i} \int_{\gamma-i \infty}^{\gamma+i \infty} e^{s t} N^{*}(x, s) d s=N(x, t) \tag{12}
\end{equation*}
$$

$\gamma$ being a fixed real number.
In 1903, the Swedish mathematician Gosta Mittag-Leffler introduced the function $\mathrm{E}_{\alpha}(\mathrm{z})$ [15] is defined as

$$
\begin{equation*}
E_{\alpha}(z)=\sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n \alpha+1)}, \alpha \in \mathbb{C}, \mathfrak{R}(\alpha)>0 \tag{13}
\end{equation*}
$$

Where z is a complex variable and $\Gamma($.$) is a gamma function \alpha>0$.The Mittag-Leffler function is the direct generalization of the exponential function to which it reduces for $\alpha=1$. For $0<\alpha<1$, it interpolates between the pure exponential and hypergeometric function $\frac{1}{1-z}$. Mittag-Leffler function naturally occurs as the solution of the fractional order differential equations.

Wiman [6] studied the generalization of $\mathrm{E}_{\alpha}(\mathrm{z})$, that is given by

$$
\begin{equation*}
E_{\alpha, \beta}(z)=\sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n \alpha+\beta)}, \alpha, \beta \in \mathbb{C}, \mathfrak{R}(\alpha)>0, \quad \mathfrak{R}(\beta)>0 \tag{14}
\end{equation*}
$$

which is known as Wiman's function.
Prabhakar [21] investigated the function $E_{\alpha, \beta}^{\gamma}(z)$ as

$$
\begin{equation*}
E_{\alpha, \beta}^{\gamma}(z)=\sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{\Gamma(n \alpha+\beta)} \frac{z^{n}}{n!} \tag{15}
\end{equation*}
$$

## II. The Solution of Fractional Kinetic Equation in Terms of Multivariable I-Function :

In this section, we solve fraction kinetic equation to generate a solution in terms of I-function and Mittag-Leffler, using the Laplace and Sumudu transform and their inverse.
The multivariable I-function is defined [24] as

$$
\begin{align*}
& \mathrm{I}\left[z_{1, \ldots}, \ldots, z_{r}\right]:= \\
&\left.\left.I_{\left\{p_{\left.i, q_{i}\right\}_{2, r}:}:\left\{\left(n_{i}\right\}_{2, r}:\left\{\left(m^{\left.\left.(i), q^{(i)}\right)\right\}}\right\}^{1, r}\right)\right\}^{1, r}\right.} \begin{array}{c}
z_{1} \\
\vdots \\
z_{r}
\end{array} \right\rvert\, \begin{array}{ccc}
\mathcal{A} & : & \mathcal{B} \\
\mathcal{C} & : & \mathcal{D}
\end{array}\right]  \tag{16}\\
&=\frac{1}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right) z_{i}^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r}
\end{align*}
$$

where $\omega=\sqrt{-1}$,

$$
\begin{align*}
\psi\left(\xi_{1}, \ldots, \xi_{r}\right)= & \frac{\prod_{k=2}^{r}\left[\prod_{j=1}^{n_{k}} \Gamma\left(1-a_{k j}+\sum_{i=1}^{k} \alpha_{k j}^{(i)} \xi_{i}\right)\right]}{\prod_{k=2}^{r}\left[\prod_{j=n_{k}+1}^{p_{k}} \Gamma\left(a_{k j}-\sum_{i=1}^{k} \alpha_{k j}^{(i)} \xi_{i}\right)\right]} \\
& \times \frac{1}{\prod_{k=2}^{r}\left[\prod_{j=1}^{q_{k}} \Gamma\left(1-b_{k j}+\sum_{i=1}^{k} \beta_{k j}^{(i)} \xi_{i}\right)\right]}  \tag{17}\\
\phi_{i}\left(\xi_{i}\right)= & \frac{\left[\prod_{k=1}^{m^{(i)}} \Gamma\left(b_{k}^{(i)}-\beta_{k}^{(i)} \xi_{i}\right)\right]}{\left[\prod_{j=n^{(i)+1}}^{p_{i}^{(i)}} \Gamma\left(a_{j}^{(i)}-\alpha_{j}^{(i)} \xi_{i}\right)\right]} \frac{\left[\prod_{j=1}^{n^{(i)}} \Gamma\left(1-a_{j}^{(i)}+\alpha_{j}^{(i)} \xi_{i}\right)\right]}{\left[\prod_{j=m^{(i)+1}}^{q^{(i)}} \Gamma\left(1-b_{k}^{(i)}+\beta_{k}^{(i)} \xi_{i}\right)\right]} \tag{18}
\end{align*}
$$

$\forall i \in\{1, \ldots, r\}$. Also,

$$
\begin{aligned}
& \left\{0, n_{i}\right\}_{2, r} \quad:=0, n_{2}: \ldots: 0, n_{2}, \\
& \left\{p_{i}, q_{i}\right\}_{2, r} \quad:=p_{2}, q_{2}: \ldots: p_{r}, q_{r} \text {, } \\
& \left\{\left(m^{(i)}, n^{(i)}\right)\right\}^{1, r}:=\left(m^{(1)}, n^{(1)}\right) ; \ldots ;\left(m^{(r)}, n^{(r)}\right), \\
& \left\{\left(p^{(i)}, q^{(i)}\right)\right\}^{1, r}:=\left(p^{(1)}, q^{(1)}\right) ; \ldots ;\left(p^{(r)}, q^{(r)}\right) \text {, } \\
& \mathcal{A}:=:\left\{\left(a_{i j} ; \alpha_{i j}^{(1)}, \ldots, \alpha_{i j}^{(i)}\right)_{1, p_{i}}^{2, r}\right\}:=\left(a_{2 j} ; \alpha_{2 j}^{(1)}, \alpha_{2 j}^{(2)}\right)_{1, p_{2}} ; \ldots ;\left(a_{r j} ; \alpha_{r j}^{(1)}, \ldots, \alpha_{r j}^{(r)}\right)_{1, p_{r}} \text {, } \\
& \mathcal{B}:=:\left\{\left(a_{j}^{(i)}, \alpha_{j}^{(i)}\right)_{1, p^{(i)}}^{1, r}\right\} \quad:=\left(a_{j}^{(1)}, \alpha_{j}^{(1)}\right)_{1, p^{(1)}} ; \ldots ;\left(a_{j}^{(r)}, \ldots, \alpha_{j}^{(r)}\right)_{1, p^{(r)}} \text {, } \\
& \mathcal{C}:=:\left\{\left(b_{i j} ; \beta_{i j}^{(1)}, \ldots, \beta_{i j}^{(i)}\right)_{1, q_{i}}^{2, r}\right\}:=\left(b_{2 j} ; \beta_{2 j}^{(1)}, \beta_{2 j}^{(2)}\right)_{1, q_{2}} ; \ldots ;\left(b_{r j} ; \beta_{r j}^{(1)}, \ldots, \beta_{r j}^{(r)}\right)_{1, q_{r}} \text {, } \\
& \mathcal{D}:=:\left\{\left(b_{j}^{(i)}, \beta_{j}^{(i)}\right)_{1, q^{(i)}}^{1, r}\right\} \quad:=\left(b_{j}^{(1)}, \beta_{j}^{(1)}\right)_{1, q^{(1)}} ; \ldots ;\left(b_{j}^{(r)}, \ldots, \beta_{j}^{(r)}\right)_{1, q^{(r)}} \text {, }
\end{aligned}
$$

Such that $n_{i}, p_{i}, q_{i}, m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)}$ are non-negative integers and all $a_{i j}, b_{i j}, \alpha_{i j}, \beta_{i j}, a_{j}^{(i)}, b_{j}^{(i)}, \alpha_{j}^{(i)}, \beta_{j}^{(i)}$ are complex numbers and the empty product denotes unity.
The contour integral (16) convergence, if

$$
\begin{equation*}
\left|\arg z_{i}\right|<\frac{1}{2} U_{i} \pi, U_{i}>0, i=1, \ldots, r, \tag{19}
\end{equation*}
$$

where

$$
\begin{array}{r}
U_{i}=\sum_{j=1}^{n^{(i)}} \alpha_{j}^{(i)}-\sum_{j=n^{(i)}+1}^{p^{(i)}} \alpha_{j}^{(i)}+\sum_{j=1}^{m^{(i)}} \beta_{j}^{(i)}-\sum_{j=m^{(i)}+1}^{q^{(i)}} \beta_{j}^{(i)}+\left(\sum_{j=1}^{n_{2}} \alpha_{2 j}^{(i)}-\sum_{j=n_{2}+1}^{p_{2}} \alpha_{2 j}^{(i)}\right) \\
+\cdots+\left(\sum_{j=1}^{n_{r}} \alpha_{r j}^{(i)}-\sum_{j=n_{r}+1}^{p_{r}} \alpha_{r j}^{(i)}\right)-\left(\sum_{j=1}^{q_{2}} \beta_{2 j}^{(i)}+\cdots+\sum_{j=1}^{q_{r}} \beta_{r j}^{(i)}\right) \tag{20}
\end{array}
$$

And $I\left[z_{1, \ldots}, z_{r}\right]=O\left(\left|z_{1}\right|^{\alpha_{1}}, \ldots,\left|z_{r}\right|^{\alpha_{r}}\right), \max \left\{\left|z_{1}\right|, \ldots,\left|z_{r}\right|\right\} \rightarrow 0$,
where $\alpha_{i}=\min _{1 \leq j \leq m}{ }^{(i)} \mathbb{R}\left(\frac{b_{j}^{(i)}}{\beta_{j}^{(i)}}\right), \beta_{i}=\max _{1 \leq j \leq n}(i) \mathbb{R}\left(\frac{\left(\alpha_{j}^{(i)}-1\right)}{\alpha_{j}^{(i)}}\right), i=1, \ldots, r$.
For the condition of convergence and analyticity of multivariable I-function we refer [24, 25].
The Laplace transform of the caputo fractional derivative (see, e.g., Podlubny [11])

$$
\begin{equation*}
\mathcal{L}\left[{ }_{0} D_{t}^{\alpha} N(x, t)\right]=s^{\alpha} N^{*}(x, s)-\left.\sum_{r=1}^{n} s^{r-1}{ }_{0} D_{t}^{\alpha-r} N(x, t)\right|_{t=0}, \quad n-1<\Re(\alpha) \leq n, n \in N \tag{21}
\end{equation*}
$$

We also use the following result obtain by Mathai and Saxena [15] as

$$
\begin{equation*}
\mathcal{L}^{-1}\left\{s^{-\beta}\left(1-a s^{-\alpha}\right)^{-\gamma} ; x\right\}=x^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_{k}\left(a x^{\alpha}\right)^{k}}{\Gamma(\alpha k+\beta) k!}=x^{\beta-1} E_{\alpha, \beta}^{\gamma}\left(a x^{\alpha}\right) \tag{22}
\end{equation*}
$$

where, $\beta, \gamma, a \in \mathbb{C}, \Re(\alpha)>0, \Re(\beta)>0, \Re(\gamma)>0$ and $\left|a s^{-\alpha}\right|<1$.
Remark: If we put $\gamma=1$, then equation (22) reduces to

$$
\begin{equation*}
\mathcal{L}^{-1}\left\{s^{-\beta}\left(1-\mathrm{as}^{-\alpha}\right)^{-1} ; x\right\}=x^{\beta-1} E_{\alpha, \beta}\left(a x^{\alpha}\right) \tag{23}
\end{equation*}
$$

If further we put $\beta=1$, then equation (23) reduces to

$$
\begin{equation*}
\mathcal{L}^{-1}\left\{s^{-1}\left(1-a s^{-\alpha}\right)^{-1} ; x\right\}=E_{\alpha}\left(a x^{\alpha}\right) \tag{24}
\end{equation*}
$$

Theorem 1. Let $\vartheta>0, c>0, w>0, \rho>0, \operatorname{Re}(p)>|w|^{\vartheta / \alpha}, c \neq w$ then for the solution of the generalized fractional kinetic equation

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0} t^{\rho-1} I\left[w^{\vartheta} t^{\vartheta} \eta_{1}, \ldots, w^{\vartheta} t^{\vartheta} \eta_{r}\right]=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) \tag{25}
\end{equation*}
$$

Then holds the result

$$
\mathcal{N}(t)=\mathcal{N}_{0} \sum_{K=0}^{\infty}(-1)^{K}(c t)^{\vartheta K} . I_{\left\{p_{i}+1, q_{i}+1\right\}_{2, r}:\left\{\left(p^{(i)}, q^{(i)}\right)\right\}^{1, r}}^{\left\{0, n_{i}+1\right\}_{2, r}:\left\{\left(m^{(i)}, n^{(i)}\right) 1, r\right.}\left[\begin{array}{cc|c:c}
w^{\vartheta} t^{\vartheta} \eta_{1} & \mathcal{A} & : & \mathcal{B}  \tag{26}\\
\vdots & \mathcal{C} & : & \mathcal{D} \\
w^{\vartheta} t^{\vartheta} \eta_{r} & \mathcal{C} & : & 0
\end{array}\right]
$$

where
$\mathcal{A} \equiv\left\{(1-\rho ; \vartheta \ldots \vartheta),\left(a_{2 j} ; \alpha_{2 j}^{(1)}, \alpha_{2 j}^{(2)}\right)_{1, p_{2}} ; \ldots ;\left(a_{r j} ; \alpha_{r j}^{(1)}, \ldots, \alpha_{r j}^{(r)}\right)_{1, p_{r}}\right\}$,
$\mathcal{B} \equiv\left\{\left(a_{j}^{(i)}, \alpha_{j}^{(i)}\right)_{1, p^{(1)}} ; \ldots ;\left(a_{j}^{(r)}, \ldots, \alpha_{j}^{(r)}\right)_{1, p^{(r)}}\right\}$,
$\mathcal{C} \equiv\left\{\left(b_{2 j} ; \beta_{2 j}^{(1)}, \beta_{2 j}^{(2)}\right)_{1, q_{2}} ; \ldots ;\left(b_{r j} ; \beta_{r j}^{(1)}, \ldots, \beta_{r j}^{(r)}\right)_{1, q_{r}},(1-\rho-\vartheta K ; \vartheta \ldots \vartheta)\right\}$,
$\mathcal{D} \equiv\left\{\left(b_{j}^{(i)}, \beta_{j}^{(i)}\right)_{1, q^{(1)}} ; \ldots ;\left(b_{j}^{(r)}, \ldots, \beta_{j}^{(r)}\right)_{1, q^{(r)}}\right\}$,
Proof: To obtain (26), express the multivariable I-function in terms of Mellin-Barnes type of contour integral by (16), we get

$$
\mathcal{N}(t)-\frac{\mathcal{N}_{0} t^{\rho-1}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} t^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r}=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t)
$$

Applying the Laplace transform both sides, we have

$$
\begin{gathered}
\mathcal{L}\{\mathcal{N}(t)\}=\frac{\mathcal{N}_{0}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r} \\
\times \mathcal{L}\left\{t^{\vartheta \sum_{i=1}^{r} \xi_{i}+\rho-1}\right\}-c^{\vartheta} \mathcal{L}\left\{{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t)\right\}
\end{gathered}
$$

Using the result (21), we get

$$
\begin{gathered}
\mathcal{N}(p)=\frac{\mathcal{N}_{0}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r} . \\
\times \Gamma\left(\rho+\vartheta \sum_{i=1}^{r} \xi_{i}\right) p^{-\rho-\vartheta \sum_{i=1}^{r} \xi_{i}-c^{\vartheta} p^{-\vartheta} \mathcal{N}(p)}
\end{gathered}
$$

Or

$$
\begin{aligned}
\mathcal{N}(p)\left(1+c^{\vartheta} p^{-\vartheta}\right)=\frac{\mathcal{N}_{0}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots & \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r} \\
& \times \Gamma\left(\rho+\vartheta \sum_{i=1}^{r} \xi_{i}\right) p^{-\rho-\vartheta \sum_{i=1}^{r} \xi_{i}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}(p)= & \frac{\mathcal{N}_{0}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r} \\
& \times \Gamma\left(\rho+\vartheta \sum_{i=1}^{r} \xi_{i}\right) p^{-\rho-\vartheta \Sigma_{i=1}^{r} \xi_{i}}\left(1+c^{\vartheta} p^{-\vartheta}\right)^{-1}
\end{aligned}
$$

Now taking inverse Laplace transform both the sides and making use the result of (24), we get the desired result (26).

Now we will use Sumudu transform to get the result of theorem 2.
Theorem 2. Let $\vartheta>0, c>0, w>0, \rho>0, \operatorname{Re}(u)>|w|^{\vartheta / \alpha}, c \neq w$ then for the solution of the generalized fraction kinetic equation

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0} t^{\rho-1} I\left[w^{\vartheta} t^{\vartheta} \eta_{1}, \ldots, w^{\vartheta} t^{\vartheta} \eta_{r}\right]=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) \tag{27}
\end{equation*}
$$

Then holds the result

$$
\mathcal{N}(t)=\mathcal{N}_{0} \sum_{K=0}^{\infty}(-1)^{K}(c t)^{\vartheta K} . I_{\left\{p_{i}+1, q_{i}+1\right\}_{2, r}:\left\{\left(p^{\left.\left.(i), q^{(i)}\right)\right\}^{1, r}}\right.\right.}^{\left\{0, n_{i}+1\right\}_{2, r}:\left\{\left(m^{(i)}, n^{(i)}\right){ }^{1, r}\right.}\left[\begin{array}{cc}
w^{\vartheta} t^{\vartheta} \eta_{1} & \mathcal{A}  \tag{28}\\
\vdots & : \\
w^{\vartheta} t^{\vartheta} \eta_{r} & \mathcal{C} \\
\mathcal{B} & : \\
\mathcal{D}
\end{array}\right]
$$

where
$\mathcal{A} \equiv\left\{(1-\rho ; \vartheta \ldots \vartheta),\left(a_{2 j} ; \alpha_{2 j}^{(1)}, \alpha_{2 j}^{(2)}\right)_{1, p_{2}} ; \ldots ;\left(a_{r j} ; \alpha_{r j}^{(1)}, \ldots, \alpha_{r j}^{(r)}\right)_{1, p_{r}}\right\}$,
$\mathcal{B} \equiv\left\{\left(a_{j}^{(i)}, \alpha_{j}^{(i)}\right)_{1, p^{(1)}} ; \ldots ;\left(a_{j}^{(r)}, \ldots, \alpha_{j}^{(r)}\right)_{1, p^{(r)}}\right\}$,
$\mathcal{C} \equiv\left\{\left(b_{2 j} ; \beta_{2 j}^{(1)}, \beta_{2 j}^{(2)}\right)_{1, q_{2}} ; \ldots ;\left(b_{r j} ; \beta_{r j}^{(1)}, \ldots, \beta_{r j}^{(r)}\right)_{1, q_{r}},(1-\rho-\vartheta K ; \vartheta \ldots \vartheta)\right\}$,
$\mathcal{D} \equiv\left\{\left(b_{j}^{(i)}, \beta_{j}^{(i)}\right)_{1, q^{(1)}} ; \ldots ;\left(b_{j}^{(r)}, \ldots, \beta_{j}^{(r)}\right)_{1, q^{(r)}}\right\}$,
Proof. To obtain (27), express the multivariable I-function in terms of Mellin-Barnes type of contour integral by (16), we get

$$
\mathcal{N}(t)-\frac{\mathcal{N}_{0} t^{\rho-1}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} t^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r}=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t)
$$

Applying the Sumudu transform both sides, we have

$$
\begin{gathered}
\mathbb{S}\{\mathcal{N}(t)\}=\frac{\mathcal{N}_{0}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r} . \mathbb{S}\left\{t^{\vartheta} \sum_{i=1}^{r} \xi_{i}+\rho-1\right. \\
-c^{\vartheta} \mathbb{S}\left\{{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t)\right\}
\end{gathered}
$$

Using the result (9), we get

$$
\begin{gathered}
\overline{\mathcal{N}}(u)=\frac{\mathcal{N}_{0}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r} \\
\times \Gamma\left(\rho+\vartheta \sum_{i=1}^{r} \xi_{i}\right) p^{-\rho-\vartheta \sum_{i=1}^{r} \xi_{i}}-c^{\vartheta} u^{\vartheta} \overline{\mathcal{N}}(u)
\end{gathered}
$$

Or

$$
\begin{aligned}
\overline{\mathcal{N}}(u)\left(1+c^{\vartheta} u^{\vartheta}\right)=\frac{\mathcal{N}_{0}}{(2 \pi \omega)^{r}} & \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r} \\
& \times \Gamma\left(\rho+\vartheta \sum_{i=1}^{r} \xi_{i}\right) u^{\rho+\vartheta \sum_{i=1}^{r} \xi_{i}-1}
\end{aligned}
$$

$$
\left.\begin{array}{rl}
\mathcal{N}(p)= & \frac{\mathcal{N}_{0}}{(2 \pi \omega)^{r}} \int_{\ell_{1}} \ldots \int_{\ell_{r}} \psi\left(\xi_{1}, \ldots, \xi_{r}\right) \prod_{i=1}^{r}\left\{\phi_{i}\left(\xi_{i}\right)\left(w^{\vartheta} \eta_{i}\right)^{\xi_{i}}\right\} d \xi_{1} \ldots d \xi_{r} \\
& \times \Gamma\left(\rho+\vartheta \sum_{i=1}^{r} \xi_{i}\right) p^{\rho+\vartheta} \Sigma_{i=1}^{r} \xi_{i}-1 \\
\hline
\end{array} 1+c^{\vartheta} u^{\vartheta}\right)^{-1} .
$$

Now taking inverse Laplace transform both the sides and making use the result of (10), we get the desired result (28).

Corollary 2.1 When $n_{i}=0, p_{i}=0, q_{i}=0, i=2, \cdots, r-1$ (the empty product denotes unity) the (26) reduces to the multivariable H -function.

Corollary 2.2 If we take $n_{i}=0, p_{i}=0, q_{i}=0$ and $m=n=p=q=0, r=1$ in (26) reduces to the Hfunction of single variable.

## III.The Solution of Fractional Kinetic Equation in Terms of Generalized M-Series by Using SUMUDU TransForm

A new generalization of M-series was introduced and developed by Salim et al. [27] as

$$
\begin{equation*}
\underset{\mathbb{M}}{\vartheta, \mu}(\eta)=\sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k m}, \ldots,\left(a_{p}\right)_{k m}}{\left(b_{1}\right)_{k n}, \ldots,\left(b_{q}\right)_{k n}} \frac{\eta^{k}}{\Gamma(\vartheta k+\mu)} \tag{29}
\end{equation*}
$$

where $\eta, \vartheta, \mu \in \mathbb{C}, \mathfrak{R}(\vartheta)>0$ and $\mathrm{m}, \mathrm{n}$ are non-negative real numbers. The series in (29) is absolutely convergent for all values of $\eta$ provided that $p m<q n+\Re(\vartheta)$, moreover if $p m=q n+\Re(\vartheta)$ the series convergent for $|\eta|<\delta=\vartheta^{\vartheta}$.

$$
\vartheta, \mu
$$

Some special cases of the generalization $\mathbb{M}$-series $\quad \mathbb{M} \quad(\eta)$ are the following:

$$
p, q ; m, n
$$

(I) If we set $m=n=1$ in (29), we get generalization of M-series introduced by Sharma and Jain [23], and defined as

$$
\begin{equation*}
\underset{\mathbb{M}}{\vartheta, \mu}(\eta)=\stackrel{\vartheta, \mu}{\underset{\sim}{\mathbb{M}}(\eta)}=\sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k}, \ldots,\left(a_{p}\right)_{k}}{\left(b_{1}\right)_{k}, \ldots,\left(b_{q}\right)_{k}} \frac{\eta^{k}}{\Gamma(\vartheta k+\mu)} \tag{30}
\end{equation*}
$$

(II) If we take $p=q=1$ in (29), the get the following generalized Mittag-Leffler function by means of power series, derived by Salim and Faraz [20]:

$$
\begin{equation*}
\underset{\mathbb{M}}{\vartheta, \mu}(\eta)=\mathbb{E}_{\vartheta, \mu, n}^{a_{1}, b_{1}, m}(\eta)=\sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k m}}{\left(b_{1}\right)_{k n}} \frac{\eta^{k}}{\Gamma(\vartheta k+\mu)} \tag{31}
\end{equation*}
$$

where $\vartheta, \mu, a_{1}, b_{1} \in \mathbb{C} ; \min \left(\Re(\vartheta), \Re(\mu), \mathfrak{R}\left(a_{1}\right), \mathfrak{R}\left(b_{1}\right)\right)>0$.
(III) If we consider $=\mu=1$ and $p=q=1$, then (29) reduces in to generalized hypergeometric function ${ }_{p} F_{q}$ [27] as

$$
\begin{equation*}
\underset{p, q ; 1,1}{1,1}(\eta)=\sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k}, \ldots,\left(a_{p}\right)_{k}}{\left(b_{1}\right)_{k}, \ldots,\left(b_{q}\right)_{k}}={ }_{p} F_{q}\left[\left(a_{1}\right)_{1}^{p} ;\left(b_{1}\right)_{1}^{q} ; \eta\right] \tag{32}
\end{equation*}
$$

where $a_{i}, b_{j} \in \mathbb{C} ; i=1,2, \ldots, p ; j=1,2, \ldots, q$ and $b_{j} \neq 0,-1,-2, \ldots$ and $(a)_{n}$ is the Pochhammer symbols.
Remarks 3.1 Throughout this section, we need the following well known relation

$$
\begin{equation*}
\sum_{r=0}^{\infty} \frac{(a)_{r}(x)^{r}}{r!}=(1-x)^{-a} \tag{33}
\end{equation*}
$$

Theorem 3. If $\vartheta>0, c>0, \mu>0, \mathrm{w} \neq c$, then for the solution of the generalized fractional kinetic equation

$$
\begin{equation*}
\underset{p, q ; m, n}{\mathcal{N}(t)-\mathcal{N}_{0} t^{\mu-1} \underset{\mathbb{M}}{\vartheta, \mu}\left(-w^{\vartheta} t^{\vartheta}\right)=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) .} \tag{34}
\end{equation*}
$$

Then holds the result

$$
\begin{equation*}
\mathcal{N}(t)=\mathcal{N}_{0} t^{\mu-1} \sum_{r=0}^{\infty}(-1)^{r}(c t)^{\vartheta r} \underset{\substack{\vartheta \\ p, q ; m, n}}{\substack{\vartheta, \mu+w^{\prime} \\ \hline}}\left(-w^{\vartheta} t^{\vartheta}\right) \tag{35}
\end{equation*}
$$

Proof. Applying the Sumudu transform both sides of eq. (34), by using the definition (29), we get
solving for $\overline{\mathcal{N}}(\mathrm{u})$

$$
\overline{\mathcal{N}}(u)-N_{0} \sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k m}, \ldots,\left(a_{p}\right)_{k m}}{\left(b_{1}\right)_{k n}, \ldots,\left(b_{q}\right)_{k n}} \frac{\left(-w^{\vartheta}\right)^{k}}{\Gamma(\vartheta k+\mu)} \mathbb{S}\left\{t^{\vartheta k+\mu-1}\right\}=-c^{\vartheta} u^{\vartheta} \overline{\mathcal{N}}(u)
$$

$$
\overline{\mathcal{N}}(u)=\left(1+c^{\vartheta} u^{\vartheta}\right)^{-1} N_{0} \sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k m}, \ldots,\left(a_{p}\right)_{k m}}{\left(b_{1}\right)_{k n}, \ldots,\left(b_{q}\right)_{k n}}\left(-w^{\vartheta}\right)^{k} u^{\vartheta k+\mu-1}
$$

using the result (33),we get

$$
\overline{\mathcal{N}}(u)=\mathcal{N}_{0} \sum_{r=0}^{\infty}(-1)^{r}\left(c^{\vartheta}\right)^{r} \sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k m}, \ldots,\left(a_{p}\right)_{k m}\left(-w^{\vartheta}\right)^{k} u^{\vartheta k+\mu+\theta r-1}}{\left(b_{1}\right)_{k n}, \ldots,\left(b_{q}\right)_{k n}}
$$

Now, taking inverse Sumudu transform on both sides for the last equation, we have

$$
\mathcal{N}(t)=\mathcal{N}_{0} t^{\mu-1} \sum_{r=0}^{\infty}(-1)^{r}\left(c^{\vartheta} t^{\vartheta}\right)^{r} \sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k m}, \ldots,\left(a_{p}\right)_{k m}}{\left(b_{1}\right)_{k n}, \ldots,\left(b_{q}\right)_{k n}} \frac{\left(w^{\vartheta} t^{\vartheta}\right)^{k}}{\Gamma(\vartheta k+(\mu+\vartheta r))}
$$

Or

$$
\mathcal{N}(t)=\mathcal{N}_{0} t^{\mu-1} \sum_{r=0}^{\infty}(-1)^{r}(c t)^{\vartheta r} \underset{\substack{\vartheta \\ p, q ; m, n}}{\substack{\vartheta \\ \hline}}\left(-w^{\vartheta} t^{\vartheta}\right)
$$

This completes the proof of theorem 3 .
When $m=n=1$ in (34), then we arrive at the following result recently obtained by Chourasia and Kumar [22]
Corollary 3.1 Let $\vartheta>0, c>0, w>0, \mu>0, \Re(u)>|w|^{9 / \alpha}, c \neq w$ then for the solution of the generalized fraction kinetic equation

$$
\begin{equation*}
\underset{\mathcal{N}(t)-N_{0} t^{\mu-1} \underset{\mathbb{M}}{\vartheta}\left(a_{1}, . ., a_{p} ; b_{1}, . ., b_{p} ;-w^{\vartheta} t^{\vartheta}\right)=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t)}{p, q} \tag{36}
\end{equation*}
$$

Then holds the result

$$
\begin{equation*}
\mathcal{N}(t)=\mathcal{N}_{0} t^{\mu-1} \sum_{r=0}^{\infty}(-1)^{r}\left(c^{\vartheta} t^{\vartheta}\right)^{r}{ }_{p}^{\vartheta} \mathbb{M}_{q}, \mu+\vartheta r \quad\left(a_{1}, . ., a_{p} ; b_{1}, . ., b_{p} ;-w^{\vartheta} t^{\vartheta}\right) \tag{37}
\end{equation*}
$$

If we set $p=q=1$ the result in (34) reduces to the following result.
Corollary 3.2 Let $\vartheta>0, c>0, w>0, \mu>0$ and $c \neq w$ then for the solution of the generalized fraction kinetic equation

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0} t^{\mu-1} \mathbb{E}_{\vartheta, \mu, n}^{a_{1}, b_{1}, m}\left(-w^{\vartheta} t^{\vartheta}\right)=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) \tag{38}
\end{equation*}
$$

Then holds the formula

$$
\begin{equation*}
\mathcal{N}(t)=\mathcal{N}_{0} t^{\mu-1} \sum_{r=0}^{\infty}(-1)^{r}\left(c^{\vartheta} t^{\vartheta}\right)^{r} \mathbb{E}_{\vartheta, \mu+\vartheta r, n}^{a_{1}, b_{1}, m}\left(-w^{\vartheta} t^{\vartheta}\right) \tag{39}
\end{equation*}
$$

Further taking $p=q=1, m=n=1, b_{1}=1$ and $\mathrm{c}=\mathrm{w}$ in (38), then we obtain the interesting result given by Mathai et.al. [3].
Corollary 3.3 Let $\vartheta>0, c>0, w>0, \mu>0$, then for the solution of the generalized fraction kinetic equation

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0} t^{\mu-1} \mathbb{E}_{\vartheta, \mu}^{a_{1}}\left(-w^{\vartheta} t^{\vartheta}\right)=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) \tag{40}
\end{equation*}
$$

Then the following integral formula holds true:

$$
\begin{equation*}
\mathcal{N}(t)=\mathcal{N}_{0} t^{\mu-1} \mathbb{E}_{\vartheta, \mu}^{a_{1}+1,}\left(-w^{\vartheta} t^{\vartheta}\right) \tag{41}
\end{equation*}
$$

If we set $p=q=0$ in (36), it becomes in to the known result given by Mathai et.al. [3].
Corollary 3.4 Let $\vartheta>0, c>0, w>0, \mu>0$ and $\mathrm{c} \neq \mathrm{w}$, then the solution of equation

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0} t^{\mu-1} \mathbb{E}_{\vartheta, \mu}\left(-w^{\vartheta} t^{\vartheta}\right)=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) \tag{42}
\end{equation*}
$$

is given as

$$
\begin{equation*}
\mathcal{N}(t)=\mathcal{N}_{0} \frac{t^{\mu-\vartheta-1}}{c^{\vartheta}-w^{\vartheta}}\left[\mathbb{E}_{\vartheta, \mu-1}\left(-w^{\vartheta} t^{\vartheta}\right)-\mathbb{E}_{\vartheta, \mu-\vartheta}\left(-w^{\vartheta} t^{\vartheta}\right)\right] \tag{43}
\end{equation*}
$$

If we can also obtain results concerning fractional kinetic equation by putting $c=w, p=q=0$ in (36), then we arrive another result obtained by Mathai et.al. [3].
Corollary 3.5 Let $\vartheta>0, c>0, w>0, \mu>0$, then the solution of equation

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0} t^{\mu-1} \mathbb{E}_{\vartheta, \mu}\left(-w^{\vartheta} t^{\vartheta}\right)=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) \tag{44}
\end{equation*}
$$

is given as follows

$$
\begin{equation*}
\mathcal{N}(t)=\frac{\mathcal{N}_{0}}{\vartheta} t^{\mu-1}\left[\mathbb{E}_{\vartheta, \mu-1}\left(-w^{\vartheta} t^{\vartheta}\right)+(1+\vartheta-\mu)-\mathbb{E}_{\vartheta, \mu}\left(-w^{\vartheta} t^{\vartheta}\right)\right] \tag{45}
\end{equation*}
$$

## IV.The Solution of Fractional Kinetic Equation in Terms of Generalized $\mathbb{K}_{4}$-Series by Using Sumudu Transform

Recently, Salim and Faraz [5] have introduced the generalized $\mathbb{K}_{4}$-function defined as

$$
\begin{equation*}
\mathbb{K}_{4(m, n)}^{(\vartheta, \mu, \gamma) ;(a, c) ;(p, q)}(\eta)=\sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k m}, \ldots,\left(a_{p}\right)_{k m}}{\left(b_{1}\right)_{k n}, \ldots,\left(b_{q}\right)_{k n}} \frac{(\gamma)_{k}(a)^{k}(\eta-c)^{(k+\gamma) \vartheta-\mu-1}}{k!\Gamma((k+\gamma) \vartheta-\mu)} \tag{46}
\end{equation*}
$$

where $\mathfrak{R}(\vartheta \gamma-\mu)>0$.
The series (2.1) is defined when non of the parameters $b_{j}{ }^{\prime} s$ is a negative integer or zero. If any numerator parameter $a_{i}$ is a negative integer or zero, then the series terminate to a polynomial of $\eta$.
From the ratio test it is evident that the series is convergent for all $\eta$, if $p m<q n+\mathfrak{R}(\vartheta)$, also when $p m=$ $q n+\Re(\vartheta)$,it is convergent in some cases, let $\xi=\sum_{i=1}^{p m} a_{i}-\sum_{j=1}^{q n} b_{j}$. It can be shown that when $p m=q n+$ $\mathfrak{R}(\vartheta)$, the series is absolutely convergent for $|\eta|=1$.If $\mathfrak{R}(\xi)<0$, conditionally convergent for $\eta=-1$ if $0 \leq \Re(\xi)<1$ and divergent for $|\eta|=1$, if $\mathfrak{R}(\xi) \geq 1$.

Now, we state further relation with other special functions.
(I) On setting $m=n=1$ in (46), it reduces to generalized $\mathbb{K}_{4}$-function defined by Sharma [29]

$$
\begin{equation*}
\mathbb{K}_{4}^{(\vartheta, \mu, \gamma) ;(a, c) ;(p, q)}\left(a_{1}, \ldots, a_{p} ; b_{1}, \ldots, b_{q} ; \eta\right)=\sum_{r=0}^{\infty} \frac{\left(a_{1}\right)_{r}, \ldots,\left(a_{p}\right)_{r}}{\left(b_{1}\right)_{r}, \ldots,\left(b_{q}\right)_{r}} \frac{(\gamma)_{r}(a)^{r}(\eta-c)^{(r+\gamma) \vartheta-\mu-1}}{r!\Gamma((r+\gamma) \vartheta-\mu)} \tag{47}
\end{equation*}
$$

(II) If we set $\mu=\vartheta-\mu, \gamma=1, a=1$ and $c=0$ in (50), then we obtain the following relation

$$
\begin{align*}
& \mathbb{K}_{4(m, n)}^{(\vartheta, \vartheta-\mu, 1) ;(1,0) ;(p, q)}\left(a_{1}, \ldots, a_{p} ; b_{1}, \ldots, b_{q} ; \eta\right)=\eta^{\mu-1} \sum_{r=0}^{\infty} \frac{\left(a_{1}\right)_{r m}, \ldots,\left(a_{p}\right)_{r m}}{\left(b_{1}\right)_{r m}, \ldots,\left(b_{q}\right)_{r m}} \frac{\eta^{r \vartheta}}{\Gamma(r \vartheta+\mu)} \\
&= \eta^{\mu-1} \underset{\substack{\text { M }}}{\mathbb{M}}\left(a_{1}, \ldots, a_{p} ; b_{1}, \ldots, b_{q} ; \eta^{\vartheta}\right)  \tag{48}\\
& p, q ; m, n
\end{align*}
$$

(III) If we take $p=q=1$ and $a_{i}, b_{j}$ in (47), then we arrive at the following relation:

$$
\begin{equation*}
\mathbb{K}_{4}^{(\vartheta, \mu, \gamma) ;(a, c) ;(1,1)}(1 ; 1 ; \eta)=\mathcal{G}_{\vartheta, \mu, \gamma}(a, c, \eta)=\sum_{r=0}^{\infty} \frac{(\gamma)_{r} a^{r}}{r!} \frac{(\eta-c)^{(r+\gamma) \vartheta-\mu-1}}{\Gamma((r+\gamma) \vartheta-\mu)} \tag{49}
\end{equation*}
$$

where $\mathcal{G}_{(\vartheta, \mu, \gamma)}(a, c, \eta)$ is the function $\mathcal{G}$-function (but not the Meijer's $\mathcal{G}$-function) defined by Lorenzo and Hartley [29].
(IV) Further, if we put $\gamma=1$ in (49), then $\mathbb{K}_{4}$ function readily yields the following relationship with $\mathcal{R}$-function

$$
\begin{equation*}
\mathbb{K}_{4}^{(\vartheta, \mu, 1) ;(a, c) ;(1,1)}(1 ; 1 ; \eta)=\mathcal{R}_{\vartheta, \mu}(a, c, \eta)=\sum_{r=0}^{\infty} a^{r} \frac{(\eta-c)^{(r+1) \vartheta-\mu-1}}{\Gamma((r+1) \vartheta-\mu)} \tag{50}
\end{equation*}
$$

where $\eta>c \geq 0, \vartheta \geq 0, \mathfrak{R}(\vartheta-\mu)>0$ and $\mathcal{R}_{\vartheta, \mu}(a, c, \eta)$ is the $\mathcal{R}$-function defined by Lorenzo and Hartley [28]. (V) If we put $c=\mu=0$ in (50), we get

$$
\begin{equation*}
\mathbb{K}_{4}^{(\vartheta, 0,1) ;(a, 0) ;(1,1)}(1 ; 1 ; \eta)=\mathcal{F}_{\vartheta}(a, \eta)=\sum_{r=0}^{\infty} a^{r} \frac{(\eta-c)^{(r+1) \vartheta-1}}{\Gamma((r+1) \vartheta)} \tag{51}
\end{equation*}
$$

Where $\mathcal{F}_{\vartheta}(a, \eta)$ is the $\mathcal{F}$-function defined by Robotnov and Hartley, for example see [30].
Remark 4.1: The following formula is needed for investigation of next theorem.

$$
\begin{equation*}
\sum_{r=1}^{n}\binom{n}{r}(a)^{r}=(1+x)^{n} \tag{52}
\end{equation*}
$$

Theorem 4. The general fractional kinetic equation

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0} \mathbb{K}_{4(m, n)}^{(\vartheta, \mu, \delta) ;\left(-c^{\vartheta}, 0\right) ;(p, q)}(t)=-\sum_{r=1}^{n}\binom{n}{r} c^{r \vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) \tag{53}
\end{equation*}
$$

## holds the formula

$$
\begin{equation*}
N(t)=N_{0} \mathbb{K}_{4(m, n)}^{(\vartheta, \mu+n \vartheta, \delta+n) ;\left(-c^{\vartheta}, 0\right) ;(p, q)}(t) \tag{54}
\end{equation*}
$$

Proof. Beginning with eq. (53), applying Sumudu transform both the sides and using result (46), we get

$$
\overline{\mathcal{N}}(\mathrm{u})-\mathcal{N}_{0} \sum_{\mathrm{k}=0}^{\infty} \frac{\left(\mathrm{a}_{1}\right)_{\mathrm{km}}, \ldots,\left(\mathrm{a}_{\mathrm{p}}\right)_{\mathrm{km}}}{\left(\mathrm{~b}_{1}\right)_{\mathrm{kn}}, \ldots,\left(\mathrm{~b}_{\mathrm{q}}\right)_{\mathrm{kn}}} \frac{(\delta)_{\mathrm{k}}\left(-\mathrm{c}^{\vartheta}\right)^{\mathrm{k}} \mathbb{S}\left\{\mathrm{t}^{(\mathrm{k}+\delta) \vartheta-\mu-1}\right\}}{\mathrm{k}!\Gamma((\mathrm{k}+\delta) \vartheta-\mu)}=-\sum_{\mathrm{r}=1}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{r}} \mathrm{c}^{\mathrm{r} \vartheta} \mathrm{u}^{\mathrm{r} \vartheta} \overline{\mathcal{N}}(\mathrm{u})
$$

solving for $\overline{\mathcal{N}}(\mathrm{u})$, it gives

Or

$$
\overline{\mathcal{N}}(\mathrm{u})\left(1+\sum_{\mathrm{r}=1}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{r}}\left(\mathrm{c}^{\vartheta} \mathrm{u}^{\vartheta}\right)^{\mathrm{r}}\right)=\mathcal{N}_{0} \mathrm{u}^{\delta \vartheta-\mu-1} \sum_{\mathrm{k}=0}^{\infty} \frac{\left(\mathrm{a}_{1}\right)_{\mathrm{km}}, \ldots,\left(\mathrm{a}_{\mathrm{p}}\right)_{\mathrm{km}}}{\left(\mathrm{~b}_{1}\right)_{\mathrm{kn}}, \ldots,\left(\mathrm{~b}_{\mathrm{q}}\right)_{\mathrm{kn}}} \frac{(\delta)_{\mathrm{k}}\left(-\mathrm{c}^{\vartheta} \mathrm{u}^{\vartheta}\right)^{\mathrm{k}}}{\mathrm{k}!}
$$

$$
\overline{\mathcal{N}}(\mathrm{u})\left(\sum_{\mathrm{r}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{r}}\left(\mathrm{c}^{\vartheta} \mathrm{u}^{\vartheta}\right)^{\mathrm{r}}\right)=\mathcal{N}_{0} \mathrm{u}^{\delta \vartheta-\mu-1} \sum_{\mathrm{k}=0}^{\infty} \frac{\left(\mathrm{a}_{1}\right)_{\mathrm{km}}, \ldots,\left(\mathrm{a}_{\mathrm{p}}\right)_{\mathrm{km}}}{\left(\mathrm{~b}_{1}\right)_{\mathrm{kn}}, \ldots,\left(\mathrm{~b}_{\mathrm{q}}\right)_{\mathrm{kn}}} \frac{(\delta)_{\mathrm{k}}\left(-\mathrm{c}^{\vartheta} \mathrm{u}^{\vartheta}\right)^{\mathrm{k}}}{\mathrm{k}!}
$$

using the formulae (33) and (52), we have

$$
\overline{\mathcal{N}}(\mathrm{u})=\mathcal{N}_{0} \mathrm{u}^{\delta \vartheta-\mu-1}\left(1+\mathrm{c}^{\vartheta} \mathrm{u}^{\vartheta}\right)^{-1} \sum_{\mathrm{k}=0}^{\infty} \frac{\left(\mathrm{a}_{1}\right)_{\mathrm{km}}, \ldots,\left(\mathrm{a}_{\mathrm{p}}\right)_{\mathrm{km}}}{\left(\mathrm{~b}_{1}\right)_{\mathrm{kn}}, \ldots,\left(\mathrm{~b}_{\mathrm{q}}\right)_{\mathrm{kn}}}
$$

Or

$$
\overline{\mathcal{N}}(\mathrm{u})=\mathcal{N}_{0} \sum_{\mathrm{k}=0}^{\infty} \frac{\left(\mathrm{a}_{1}\right)_{\mathrm{km}}, \ldots,\left(\mathrm{a}_{\mathrm{p}}\right)_{\mathrm{km}}}{\left(\mathrm{~b}_{1}\right)_{\mathrm{kn}}, \ldots,\left(\mathrm{~b}_{\mathrm{q}}\right)_{\mathrm{kn}}} \frac{(\mathrm{n}+\delta)_{\mathrm{k}}\left(-\mathrm{c}^{\vartheta}\right)^{\mathrm{k}}}{\mathrm{k}!} \mathrm{u}^{(\delta+\mathrm{k}) \vartheta-\mu-1}
$$

Now, taking inverse Sumudu transform on both sides for the last equation, we obtain the desired result (54).
Theorem 5.The general fractional kinetic equation

$$
\begin{equation*}
\mathcal{N}(t)-\mathcal{N}_{0} \mathbb{K}_{4(m, n)}^{(\alpha, \beta, \gamma) ;(a, 0) ;(p, q)}(t)=-c^{\vartheta}{ }_{0} D_{t}^{-\vartheta} \mathcal{N}(t) \tag{55}
\end{equation*}
$$

There holds the result

$$
\begin{equation*}
\mathcal{N}(t)=\mathcal{N}_{0} \sum_{r=0}^{\infty}(-1)^{r}\left(-c^{\vartheta}\right)^{r} \mathbb{K}_{4(m, n)}^{(\alpha, \beta-\vartheta r, \gamma) ;(a, 0) ;(p, q)}(t) \tag{56}
\end{equation*}
$$

Proof. If we apply Sumudu transform and using (53) and (9), (55) becomes

$$
\overline{\mathcal{N}}(\mathrm{u})-\mathrm{N}_{0} \sum_{\mathrm{k}=0}^{\infty} \frac{\left(\mathrm{a}_{1}\right)_{\mathrm{km}}, \ldots,\left(\mathrm{a}_{\mathrm{p}}\right)_{\mathrm{km}}}{\left(\mathrm{~b}_{1}\right)_{\mathrm{kn}}, \ldots,\left(\mathrm{~b}_{\mathrm{q}}\right)_{\mathrm{kn}}} \frac{(\gamma)_{\mathrm{k}}(\mathrm{a})^{\mathrm{k}} \mathbb{S}\left\{\mathrm{t}^{\mathrm{k}+\gamma) \alpha-\beta-1}\right\}}{\mathrm{k}!\Gamma((\mathrm{k}+\gamma) \alpha-\beta)}=-\mathrm{c}^{\vartheta} \mathrm{u}^{\vartheta} \overline{\mathcal{N}}(\mathrm{u})
$$

solving for $\overline{\mathcal{N}}(\mathrm{u})$, it gives

$$
\overline{\mathrm{N}}(\mathrm{u})=\mathcal{N}_{0}\left(1+\mathrm{c}^{\vartheta} \mathrm{u}^{\vartheta}\right)^{-1} \sum_{\mathrm{k}=0}^{\infty} \frac{\left(\mathrm{a}_{1}\right)_{\mathrm{km}}, \ldots,\left(\mathrm{a}_{\mathrm{p}}\right)_{\mathrm{km}}}{\left(\mathrm{~b}_{1}\right)_{\mathrm{kn}}, \ldots,\left(\mathrm{~b}_{\mathrm{q}}\right)_{\mathrm{kn}}} \frac{(\gamma)_{\mathrm{k}}(\mathrm{a})^{\mathrm{k}}}{\mathrm{k}!} \mathrm{u}^{(\mathrm{k}+\gamma) \alpha-\beta-1}
$$

making use of (33), we get

$$
\overline{\mathcal{N}}(\mathrm{u})=\mathcal{N}_{0} \sum_{\mathrm{r}=0}^{\infty}(-1)^{\mathrm{r}}\left(-\mathrm{c}^{\vartheta}\right)^{\mathrm{r}} \frac{(\gamma)_{\mathrm{k}}(\mathrm{a})^{\mathrm{k}}}{\mathrm{k}!} \mathrm{u}^{(\mathrm{k}+\gamma) \alpha-\beta+\vartheta \mathrm{r}-1}
$$

Now, taking inverse Sumudu transform on both sides for the last equation, we arrive at the required result (56).
The result in Theorems 4 and 5 can be easily specialized to yield the corresponding kinetic equations involving $\mathrm{K}_{4}$-function, $\mathcal{G}$-function, $\mathcal{R}$-function and $\mathcal{F}$-function.

## V. CONCLUSION

The generalized fractional kinetic equation taken in the present paper involves various special functions. It is not difficult to obtain several further analogues fractional kinetic equations and their solution as those exhibited here by theorems 2 and 3 and its Corollaries. Moreover, in view of close relationships of the generalized Hfunction, generalized M -series and generalized $\mathrm{K}_{4}$-function with other special functions, it does not seem difficult to construct various known and new fractional kinetic equations. The Mittag-Leffler function is a
special function having an essential role in the solution of fractional order integral and differential equations. Recently, this function is frequently used in modeling phenomena of fractional order appearing in the physics, biology, and engineering and applied sciences. The results calculated are suitable for various numerical computations like dynamical properties of the particle reaction rate, statistical mechanics associated with the particle distribution.

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