Notes on Depth of B and Height of B of Fuzzy Graphs

C.Subramani[1], P.Pandiammal[2] & B.Vasudevan[3]

1. Department of Mathematics,G.T.N Arts College,G.T.N nagar Dindigul-624005, Tamilnadu, India.

2. Department of Mathematics,G.T.N Arts College,G.T.N nagar Dindigul-624005, Tamilnadu, India.

3.Department of Mathematics, Yadava College , Madurai - 625014, Tamilnadu, India.

ABSTRACT: *In this paper, depth of B and height of B are introduced. We observe some of their properties.*

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KEY WORDS: *Fuzzy subset, Fuzzy relation, Strong fuzzy relation, Fuzzy graph, Fuzzy loop, Fuzzy pseudo graph, Degree of fuzzy vertex, Total Degree of fuzzy vertex, order of the fuzzy graph, size of the fuzzy graph, Fuzzy regular graph, Fuzzy totally regular graph, Fuzzy complete graph, Depth of B, height of B.*

INTRODUCTION: In 1965, Zadeh [10] introduced the notion of fuzzy set as a method of presenting uncertainty. Since complete information in science and technology is not always available. Thus we need mathematical models to handle various types of systems containing elements of uncertainty. After that Rosenfeld[8] introduced fuzzy graphs. Yeh and Bang[9] also introduced fuzzy graphs independently. Fuzzy graphs are useful to represent relationships which deal with uncertainty and it differs greatly from classical graph. It has numerous applications to problems in computer science, electrical engineering system analysis, operations research, economics, networking routing, transportation, etc. Nagoor Gani . A and Ratha . K [5] introduced fuzzy regular graphs, total degree and totally regular fuzzy graphs. Ramakrishnan P.V and Lakshmi . T [6,7] introduced depth of μ, height of μ. In this paper, some results on depth of B and height of B are studied.

1. PRELIMINARIES: **1.1**

Definition: Let X be any nonempty set. A mapping M: X$\rightarrow$ [0,1] is called a fuzzy subset of X.

1.2 Example: A fuzzy subset A = { (a, 0.3), (b, 0.4), (c, 0.6) } of a set X = { a, b, c }. **1.3**

Definition: Let A and B be any two fuzzy subset of X. We define the following relations and operations:

(i) $A \subseteq B$ if and only if $A(x) \leq B(x)$ for all x in X. (ii) A = B if and only if A(x) = B(x) for all x in X. (iii) $A\cap B$ if and only if $(A\cap B)(x) = \min \{ A(x), B(x) \}$ for all x in X. (iv) $A\cup B$ if and only if $(A\cup B)(x) = \max \{ A(x), B(x) \}$ for all x in X. (v) $A^C = 1- A = \{ (x, 1- A(x)) \ / \ x\in X \}$.

1.4 Definition: Let M be a fuzzy subset in a set S, the **strongest fuzzy relation** on S, that is a fuzzy relation V with respect to M given by V(x,y) = min { M(x), M(y) } for all x and y in S.

1.5 Definition: Let V be any nonempty set, E be any set and f: E$\rightarrow$ V$\times$V be any function. Then A is a fuzzy subset of V, S is a fuzzy relation on V with respect to A and B is a fuzzy subset of E such that $B(e) \leq \underset{e\in f^{-1}(x,y)}{S(x,y)}$.

Then the ordered triple F = (A, B, f) is called a **fuzzy graph**, where the elements of A are called **fuzzy points** or **fuzzy vertices** and the elements of B are called **fuzzy lines** or fuzzy edges of the fuzzy graph F. If f(e) = (x, y), then the fuzzy points (x, A(x)), (y, A(y)) are called **fuzzy adjacent points** and fuzzy points (x, A(x)), fuzzy line (e, B(e)) are called **incident** with each other. If two district fuzzy lines (e_1, B(e_1)) and (e_2, B(e_2)) are incident with a common fuzzy point, then they are called **fuzzy adjacent lines.**

1.6 Definition: A fuzzy line joining a fuzzy point to itself is called a **fuzzy loop**.

1.7 Definition: Let F = (A, B, f) be a fuzzy graph. If more than one fuzzy line joining two fuzzy vertices is allowed, then the fuzzy graph F is called a **fuzzy pseudo graph**.

1.8 Definition: F = (A, B, f) is called a **fuzzy simple graph** if it has neither fuzzy multiple lines nor fuzzy loops.

1.9 Example: F = (A, B, f), where V = { v_1, v_2, v_3, v_4, v_5 }, E = (a, b, c, d, e, h, g } and f : E$\rightarrow$ V$\times$V is defined by f(a) = (v_1, v_2) , f(b) = (v_2, v_2), f(c) = (v_2, v_3), f(d) = (v_3, v_4), f(e) = (v_3, v_4), f(h) = (v_4, v_5), f(g) = (v_1, v_5). A fuzzy subset A = { (v_1, 0.3), (v_2, 0.5), (v_3, 0.6), (v_4, 0.7), (v_5, 0.9) } of V. A fuzzy relation S = { ((v_1, v_1), 0.3), ((v_1, v_2), 0.3), ((v_1, v_3), 0.3), ((v_1, v_4), 0.3), ((v_1, v_5), 0.3), ((v_2, v_1), 0.3), ((v_2, v_2), 0.5), ((v_2, v_3), 0.5), ((v_2, v_4), 0.5), ((v_2, v_5), 0.5), ((v_3, v_1), 0.3), ((v_3, v_2), 0.5), ((v_3, v_3), 0.6), ((v_3, v_4), 0.6), ((v_3, v_5), 0.6), ((v_4, v_1), 0.3), ((v_4, v_2), 0.5), ((v_4, v_3), 0.6), ((v_4, v_4), 0.7), ((v_4, v_5), 0.7), ((v_5, v_1), 0.3), ((v_5, v_2), 0.5), ((v_5, v_3), 0.6), ((v_5, v_4), 0.7), ((v_5, v_5), 0.9) } on V with respect to A and a fuzzy subset B = { (a, 0.2), (b, 0.4), (c, 0.4), (d, 0.4), (e, 0.5), (h, 0.6), (g, 0.2) } of E.

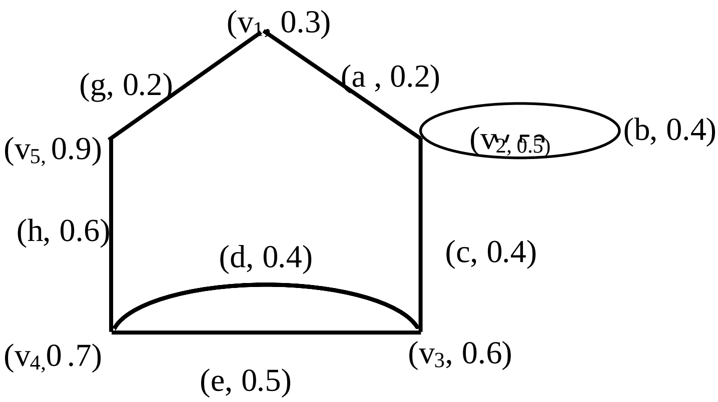

Fig 1.1

In figure 1.1, (i) (v_1, 0.3) is a fuzzy point. (ii) (a, 0.2) is a fuzzy edge. (iii) (v_1, 0.3) and (v_2, 0.5) are fuzzy adjacent points. (iv) (a, 0.2) join with (v_1, 0.3) and (v_2, 0.5) and therefore it is incident with (v_1, 0.3) and (v_2, 0.5). (v) (a, 0.2) and (g, 0.2) are fuzzy adjacent lines. (vi) (b, 0.4) is a fuzzy loop. (vii) (d, 0.4) and (e, 0.5) are fuzzy multiple edges. (viii) It is not a fuzzy simple graph. (ix) It is a fuzzy pseudo graph.

1.10 Definition: The fuzzy graph H = (C, D, f) is called a **fuzzy subgraph** of F = (A, B, f) if C $\subseteq$ A and D $\subseteq$ B.

1.11 Definition: Let F = (A, B, f) be a fuzzy graph. Then the **degree of a fuzzy vertex** is difined by d(v) = $\sum_{e\in f-1(u,v)} B(e)$+2 $\sum_{e\in f-1(v,v)} B(e)$.

1.12 Definition: Let F = (A, B, f) be a fuzzy graph. The **total degree of fuzzy vertex v** is defined by d_T(v) = = d(v) + A(v) for all v in V.

1.13 Defination: The **minimum degree** of the fuzzy graph F = (A, B, f) is δ(F) = $\wedge$ { d(v) / v$\in$V } and the **maximum degree** of F is Δ(F) = $\vee$ {d(v) / v$\in$V }.

1.14 Definition: Let F = (A, B, f) be a fuzzy graph. Then the **order of fuzzy graph** F is defined to be o(F) = $\sum_{v \in V} A(v)$.

1.15 Definition: Let F = (A, B, f) be a fuzzy graph. Then the **size of the fuzzy graph F** is defined to be S(F) = $\sum_{e \in E} B(e)$.

1.16 Definition: A fuzzy graph F = (A, B, f) is called **fuzzy k- regular graph** if d(v) = k for all v in V.

1.17 Definition: A fuzzy graph F is **fuzzy k-totally regular graph** if each vertex of F has the same total degree k.

1.18 Theorem[1]: The sum of the degree of all fuzzy vertices in a fuzzy graph is equal to twice the sum of the membership value of all fuzzy edges. i.e., $\sum_{v \in V} d(v) = 2S(F)$.

2. DEPTH OF B AND HEIGHT OF B

2.1 Definition: Let $F = (A, B, f)$ be a fuzzy graph. Then the depth of B is defined by $D(B) = \min\{B(e)/\ e \in E\}$.

2.2 Definition: Let $F = (A, B, f)$ be a fuzzy graph. Then the height of B is defined by $H(B) = \max\{B(e)/\ e \in E\}$.

2.3 Example:

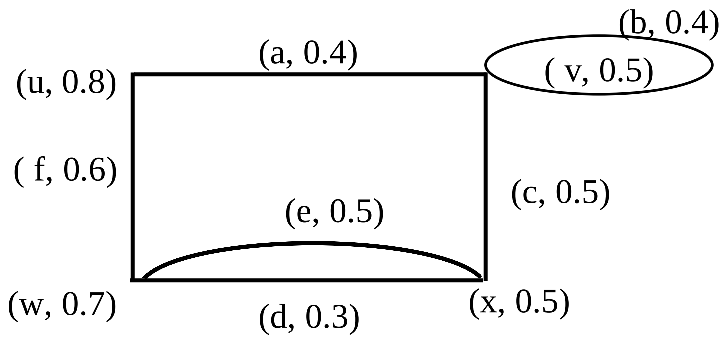

Fig. 2.1 Fuzzy graph F

Here D(B) = 0.3, H(B) = 0.6,

2.4 Remark: Clearly $D(B) \leq H(B)$ and $D(B) \leq B(e) \leq H(B)$

2.5 Theorem: Let $F = (A, B, f)$ be any fuzzy graph with respect to set V and E where $|V| = p$ and $|E| = q$. Then $D(B) \leq \frac{S(F)}{q} \leq H(B)$.

Proof: Suppose $F = (A, B, f)$ is any fuzzy graph with p-fuzzy vertices.

Obviously, $D(B) \leq B(e) \leq H(B) \Rightarrow \sum_{e \in E} D(B) \leq \sum_{e \in E} B(e) \leq \sum_{e \in E} H(B)$

$\Rightarrow qD(B) \leq S(F) \leq qH(B) \Rightarrow D(B) \leq \frac{S(F)}{q} \leq H(B)$.

2.6 Theorem: Let$F = (A, B, f)$ be any fuzzy simple graph with p-fuzzy vertices. Then $\frac{2S(F)}{p(p-1)} \leq H(B)$.

Proof: By 2.5 Theorem,$\frac{S(F)}{q} \leq H(B) \Rightarrow S(F) \leq qH(B) \Rightarrow \frac{2S(F)}{p(p-1)} \leq H(B)$.

2.7 Theorem: Let $F = (A, B, f)$ be a fuzzy complete graph with p-fuzzy vertices. Then $D(B) \leq \frac{2S(F)}{p(p-1)} \leq H(B)$.

Proof: By 2.5 Theorem, $D(B) \leq \frac{S(F)}{q} \leq H(B) \Rightarrow qD(B) \leq S(F) \leq qH(B)$

Since$F$ is fuzzy complete graph, $\frac{p(p-1)}{2} D(B) \leq S(F) \leq \frac{p(p-1)}{2} H(B)$

Which implies that $D(B) \leq \frac{2S(F)}{p(p-1)} \leq H(B)$.

2.8 Theorem: Let $F = (A, B, f)$ be a fuzzy complete graph with p-fuzzy vertices and A be k-constant function. Then $D(B) = \frac{2S(F)}{p(p-1)} = H(B)$.

Proof: Assume that F is a fuzzy complete graph with p-fuzzy vertices and $A(v) = k$ for all v in V.That is $B(e) = \underset{e \in f^{-1}(x,y)}{S}(x, y)$ for all x and y in V. Then $B(e) = A(x) \cap A(y)$=k for all x and y in V, so $D(B) = B(e) = H(B)$

$\Rightarrow \sum_{e \in E} D(B) = \sum_{e \in E} B(e) = \sum_{e \in E} H(B) \Rightarrow qD(B) = S(F) = qH(B)$

which implies $\frac{p(p-1)}{2} D(B) = S(F) = \frac{p(p-1)}{2} H(B)$. Hence $D(B) = \frac{2S(F)}{p(p-1)} = H(B)$.

2.9 Corollary: Let $F = (A, B, f)$ be a fuzzy complete graph with p-fuzzy vertices and A be a k-constant function. Then $\sum_{v \in V} d(v) = p(p-1)H(B) = p(p-1)D(B)$

2.10 Theorem: If F is a fuzzy k- regular graph with p-fuzzy vertices. Then $H(B) \geq \frac{k}{p-1}$.

Proof: suppose F is a fuzzy k- regular graph with p-fuzzy vertices. Here $d(v) = k$ for all v in V, $\sum_{v \in V} d(v) = \sum_{v \in V} k = pk$ .We get $2S(F) = pk$ implies that $S(F) = \frac{pk}{2}$. By 2.6 Theorem, $\frac{pk}{2} \leq \frac{p(p-1)}{2} H(B) \Rightarrow \frac{k}{p-1} \leq H(B)$ which implies that $H(B) \geq \frac{k}{p-1}$.

2.11 Theorem: Let $F = (A, B, f)$ be a fuzzy complete graph with p-fuzzy vertices and A be k-constant function. Then $H(B) = k = D(B)$.

Proof: Assume that F is a fuzzy complete graph with p-fuzzy vertices and $A(v) = k$ for all v in V.That is $B(e) = \underset{e \in f^{-1}(x,y)}{S}(x, y)$ for all x and y in V. Then $B(e) = A(x) \cap A(y)$= k for all x and y in V.Therefore $d(v) = (p-1)k$ for all v in V.Which implies that $\sum_{v \in V} d(v) = \sum_{v \in V} (p-1)k = p(p-1)k$.

By 2.9 Corollary, $\sum_{v \in V} d(v) = p(p-1)H(B) = p(p-1)D(B)$.Hence $H(B) = k = D(B)$.

2.12 Theorem: Let $F = (A, B, f)$ be any fuzzy simple graph with p-fuzzy vertices . Then $\delta(F) \leq (p-1)H(B)$.

Proof: For any fuzzy graph, $\delta(F) \leq \frac{2S(F)}{p}$. By 1.21 Theorem, $\frac{2S(F)}{p} \leq (p-1)H(B)$ which implies that $\delta(F) \leq (p-1)H(B)$

2.13 Theorem: Let $F = (A, B, f)$ be a fuzzy complete graph with p-fuzzy vertices and A be k-constant function. Then $\delta(F) = \Delta(F) = (p-1)H(B) = (p-1)D(B)$.

Proof: By 2.11 Theorem, $d(v) = (p-1)k$ for all v in V and $H(B) = D(B) = k$

also $\delta(F) = \Delta(F) = (p-1)k$ implies that $\frac{\delta(F)}{p-1} = \frac{\Delta(F)}{p-1} = k \Rightarrow H(B) = D(B) = \frac{\delta(F)}{p-1} = \frac{\Delta(F)}{p-1}$ implies that $\delta(F) = \Delta(F) = (p-1)H(B) = (p-1)D(B)$.

2.14 Theorem: If $F = (A, B, f)$ is a fuzzy c-totally regular graph with p-fuzzy vertices. Then $O(F) \geq p[c - (p-1)H(B)]$.

Proof: For any fuzzy graph ,$S(F) = \frac{pc - O(F)}{2}$. By 2.6 Theorem, $S(F) \leq \frac{p(p-1)}{2}H(B) \Rightarrow \frac{pc-O(F)}{2} \leq \frac{p(p-1)}{2}H(B) \Rightarrow pc - p(p-1)H(B) \leq O(F)$.

Hence $O(F) \geq p[c - (p-1)H(B)]$

2.15 Theorem: If $F = (A, B, f)$ is both fuzzy k-regular graph and fuzzy c-totally regular graph with p-fuzzy vertices. Then $H(B) \geq \frac{k}{p-1}$.

Proof: By 2.14 Theorem, $O(F) \geq p[c - (p-1)H(B)]$

For any fuzzy graph, $O(F) = p(c-k) \Rightarrow p[c - (p-1)H(B)] \leq p(c-k)$

$\Rightarrow c - (c-k) \leq (p-1)H(B) \Rightarrow k \leq (p-1)H(B) \Rightarrow H(B) \geq \frac{k}{p-1}$.

2.16 Theorem: If $F = (A, B, f)$ is a fuzzy complete graph with p-fuzzy vertices and A is a k-constant function. Then $O(F) = pH(B) = pD(B)$.

2.17 Theorem: Let $F = (A, B, f)$ be any fuzzy graph with respect to set V and E where $|V| = p$ and $|E| = q$. Then $q\,D(B) \leq \frac{\sum_{v \in V} d(v)}{2} \leq qH(B)$.

2.18 Theorem: Let$F = (A, B, f)$ be any fuzzy simple graph with p-fuzzy vertices. Then $\sum_{v \in V} d(v) \leq p(p-1)H(B)$.

2.19 Theorem: Let $F = (A, B, f)$ be a fuzzy complete graph with p-fuzzy vertices. Then $p(p-1)D(B) \leq \sum_{v \in V} d(v) \leq p(p-1)H(B)$.

2.20 Theorem: Let $F = (A, B, f)$ be a fuzzy complete graph with p-fuzzy vertices and A be k-constant function. Then $p(p-1)D(B) = \sum_{v \in V} d(v) = p(p-1)H(B)$.

Proof: By 2.8 Theorem, $D(B) = \frac{2S(F)}{p(p-1)} = H(B)$. Since F is a fuzzy complete graph with p-fuzzy vertices and by 1.18 Theorem, so $p(p-1)\,D(B) = \sum_{v \in V} d(v) = p(p-1)H(B)$.

2.21 Theorem: Let $F = (A, B, f)$ be a fuzzy complete graph with p-fuzzy vertices and A be k-constant function. Then $\sum_{v \in V} d_T(v) = p^2 H(B) = p^2 D(B)$.

Proof: By 2.20 Theorem, $\sum_{v \in V} d(v) = p(p-1)\, H(B) = p(p-1)\, D(B)$.

$$\Rightarrow \sum_{v \in V} d_T(v) = \sum_{v \in V} d(v) + \sum_{v \in V} A(v) \text{, since } A \text{ is } k\text{-constant function,}$$

$$= p(p-1)\, H(B) + pH(B) = p^2 H(B)$$

Similarly $\sum_{v \in V} d_T(v) = p^2 D(B)$.

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