Bounds on Co-Secure Domination in Graphs

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Abstract:*Let G = (V, E) be a graph and let S⊆V. The set S is a co-secure dominating set (CSDS) of a graph G if S is a dominating set, and for each u ∈S there exists a vertex v ∈ V \ S such that uv ∈E(G) and (S \ {u})∪ {v} is adominating set. The minimum cardinality of a co-secure dominating set inG is the co-secure domination number γ_{cs}(G). We determine the cosecuredomination number of some families of standard graphs and obtain sharpbounds. A set S ⊆V is a secure dominating set of a graph G = (V, E),if for each u ∈V \ S there exists a vertex v ∈ S such that uv ∈ E and (S \ {v}) ∪{u} is a dominating set. The minimum cardinality of a securedominating set in G is the secure domination number γ_s (G). We presentfew bounds on these parameter and certain graphs for which equality ofboth parameter holds.*

Keywords-*Co-secure domination, Securedomination, Jahangir graphs, Friendship graphs, Helm graph.*

I. Introduction

Let *G = (V, E)* be a finite, connected undirected graph with neither loops normultiple edges. The degree of a vertex v, *deg (v)* is the number of edges incidentto it . The open neighborhood of a vertex *v* ∈ *V* is denoted by *N (v) = {u ∈ V:uv ∈ E (G)}* and its closed neighborhood is *N [v] = N (v) ∪{v}*. For a set*S ⊆V,N (S) =* $\cup_{v\in S}$*N (v)* and *N [S] = N (S)* ∪ *S* . Given *S ⊆V* and *v* ∈ *S* , a vertex *u* ∈ *V \S* is an *S*-external private neighbor (abbreviated as *S-epn*) of *v* if *N (u)* ∩ *S = {v}*. The set of all *S*-epns of *v* is denoted by *EPN(v, S)*. The join of two graphs *G* and *H* is denoted by *G + H* and it consists of *G* ∪ *H* and all edges joining every vertex of G with every vertex of *H*. The Friendship graphF_k, is a graph on *2k + 1* vertices consisting of *k* copies of K_3 with a common central vertex of degree *2k* and all the other vertices of degree two. The Jahangir graphs $J_{n,m}$ [2] for *m*≥ 3, is a graph on *nm + 1* vertices consisting of a cycle C_{nm} with one additional vertex which is adjacent to *m* vertices ofC_{nm}, at distance *n* to each other on C_{nm}. The helm graph, H_n is a graph obtained from wheel graph W_n by attaching a pendant edge to each rim vertex. It contains three types of vertices, the vertex of degree *n* called apex, *n* pendant vertices and *n* vertices of degree four.

Let *D* be a subset of *V*. If each vertex of *V \ D* is adjacent to at least one vertex of *D*, then *D* is called a dominating set in *G*. The domination number of a graph G denoted as γ(G) is the minimum cardinality of a dominating set in *G*. A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property. This paper deals with the secure and co-secure version of the problem in which the configuration of guards induces a dominating set before and after the attack has been defended. The concept of secure domination is motivated by the following situation. Given a graph *G = (V, E)* we wish to place one guard at each vertex of a subset *S* of *V* in such a way that *S* is a dominating set of *G* and if a guard at *v* moves along an edge to protect an unguarded vertex *u*, then the new configuration of guards also forms a dominating set. In other words, for each *u* ∈ *V \ S* there exists a vertex *v* ∈*S* such that*v* is adjacent to *u*and*(S \ {v})*∪ *{u}* is a dominating set of *G*. In this case we say that *u* is *S*-defended by *v*or *v S*-defends *u*. A dominating set *S* in which every vertex in *V \ S* is*S*-defended by a vertex in *S* is called a secure dominating set (SDS) of *G*. The secure domination numberγ_s*(G)* is the minimum cardinality of a secure dominating set of *G*.

This concept was introduced and the secure domination numbers of a path and a cycle are determined by Cockayne et.al in [3]. The upper bound for secure domination of a connected graph other than $C_5$with minimum degree at least two was determined by Burger et.al in [1]. Co-secure domination numbers of path and a cycle are determined by Cockayne et.al in [4]. Zepeng et.al in [5] have found upper and lower bounds of secure domination number for any tree of order greater than three and have also characterized trees with secure domination number equal to $\frac{n+2}{3}$.

II. Preliminary results and observations

In domination, we study the problem of using guards to defend the vertices of G against an attacker. At most one guard is located at each vertex. A guard can protect the vertex at which it's located and can move to a neighboring vertex to defend an attack there. At most one guard is allowed to move in order to defend an attack (other papers have studied the model in which multiple guards can move simultaneously when an attack occurs). Several variations of this graph protection problem considering various other parameters have been studied, including Roman domination, weak Roman domination, k −secure sets,etc. The term Roman domination stems from the problem's ancient origins in Emperor Constantine's efforts to defend the Roman Empire from attackers. A set $S \subseteq V$ is a co-secure dominating set (CSDS) of a graph G if S is a dominating set, and for each $u \in S$ there exists a vertex $v \in V \setminus S$ such that $uv \in E(G)$ and $(S \setminus \{u\}) \cup \{v\}$ is adominating set. The minimum cardinality of a co-secure dominating set in$G$ is the co-secure domination number $\gamma_{cs}(G)$. A set $S \subseteq V$ is a secure dominating set (SDS) of a graph $G = (V, E)$ if for each $u \in V \setminus S$ there exists a vertex $v \in S$ such that $uv \in E(G)$ and $(S \setminus \{v\}) \cup \{u\}$is a dominating set. The minimum cardinality of a secure dominating set in G is the secure domination number $\gamma_s(G)$ of G. The secure domination number represents the minimum number of guards required to protect the entire location complex in a bid to save on the total guard remuneration or deployment cost and is often realised in the context of surveillance applications, military strategy analysis or the deployment of security guards by private security firms, etc.

Notice that if G has isolated vertices, then no such CSDS exists. Also given an isolate-free graph G with components G_i ,$i \in\{1, 2,\ldots k\}$, $\gamma_{cs}(G)=\sum_{i=1}^{k} \gamma_{cs}(G_i)$. Therefore, the study of co-secure domination may be restricted to connected, non-trivial graphs. Observe that the entire vertex set, V can never be a CSDS.

The following theorems given in [4] will be useful to prove many results in this paper.

Proposition 2.1. For a complete graph G, $\gamma_s(G) = \gamma_{cs}(G) = 1$.
Proposition 2.2. For a star G with n vertices, $\gamma_s(G) = \gamma_{cs}(G) = n-1$.
Proposition 2.3. For any graph G with $\delta(G) \geq 2$, $\gamma_{cs}(G) \leq \gamma_s(G)$.
Proposition 2.4. For any complete bipartite graph $K_{m,n}$ with m, n ≥ 4 , $\gamma_s(K_{m,n}) = \gamma_{cs}(K_{m,n}) = 4$ and $\gamma_s(\overline{K_{m,n}}) = \gamma_{cs}(\overline{K_{m,n}}) = 2$.

III.BOUNDS ON CO-SECURE DOMINATION

In this paper we will be discussing on the bounds of cosecure domination num- ber of certain classes of graphssay, Friendship graph F_k, Jahangir graph$J_{2,m}$ and Helm graphH_n. We also discuss on family of trees $\mathbb{T}$ for which $\gamma_s = \gamma_{cs}$.

Theorem 3.1. If G and H are two connected, non complete graphs, then $\gamma_{cs}(G+H)=2$.

Proof. Given that G and H are two non-complete connected graphs. In $G + H$ every vertex of G is adjacent to every vertex of H and vice-versa.

Let $S = \{u, v : u\in V(G) , v \in V(H)\}$ be a dominating set of G. Since G is connected, we can find atleast one vertex say, p adjacent to u in G such that$(S - \{u\}) \cup\{p\}$ is a dominating set.
Similarly, since H is connected we have, a vertex q adjacent to v in H such that $(S - \{v\}) \cup\{q\}$is also a dominating sets. Thus S forms a cosecure dominating set of $G + H$.

Thus, $\gamma_{cs}(G+H) = | S | = 2.$

Theorem 3.2. If G is a graph with atleast two vertices of degree n -2, then $\gamma_{cs}(G) = 2$.

Proof. Let $S = \{u, v \}$be a cosecure dominating set (CSDS) where $deg(u) = deg(v) = n - 2$.

Case (i):u and v are not adjacent.

Since $d(u) = d(v) = n -2$ and u is not adjacent to v, both u and v are adjacent to every vertex in $V - S$. Thus, u and v dominates every vertex in $V-S$. (Fig.1)

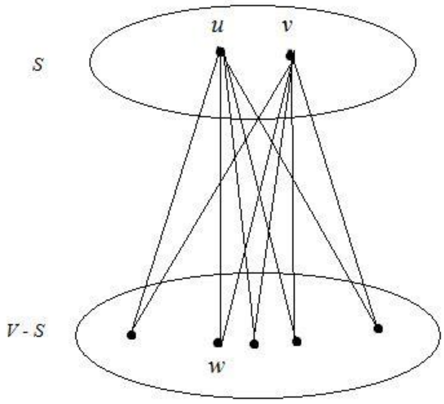

Fig 1:Vertices in *S* are not adjacent.

Moreover, there is s a vertex, $w \in V - S$ such that $(S \setminus \{u\}) \cup \{w\}$ and $(S \setminus \{v\}) \cup \{w\}$ are both dominating sets. Thus *S* forms a CSD set and hence $\gamma_{cs}(G) = 2$.

Case 2:*u* and *v* are adjacent.

Since *u* is adjacent to *v* and $d(u) = d(v) = n- 2$, both *u* and *v* are not adjacent to some vertex in $V \setminus S$. Now we have two cases i.e, either *u* and *v*are not adjacent to a common vertex in $V \setminus S$ or *u* is not adjacent to a vertex that *v* is adjacent to and vice versa.

Subcase 2.1: There exists atleast one vertex in $V \setminus S$, say *x* which is not adjacent to both *u* and *v* as given in Fig 2. But since *G* is a connected graph we have a vertex say, *w* such that, *w* is adjacent to *x*. Then $S' = \{v,x\}$ or $\{u,x\}$forms aCSDS since $(S' \setminus \{v\}) \cup \{u\}$ and $(S' \setminus \{x\}) \cup \{w\}$ are both dominating sets.

Thus, $\gamma_{cs}(G) = |S'| = 2$.

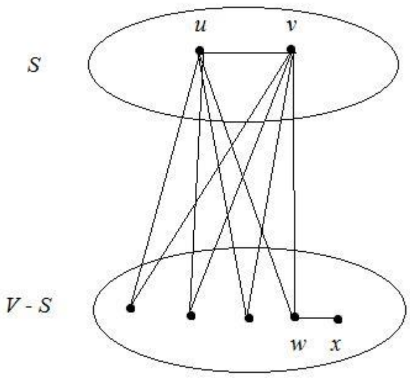

Fig 2:Vertices in *S* are adjacent
such that $u \leftrightarrow x$ and $v \leftrightarrow x$

Subcase 2.2: There exists atmost two vertex in $V \setminus S$ say a and b such that u is not adjacent to a and v is not adjacent to b. (Fig 3)

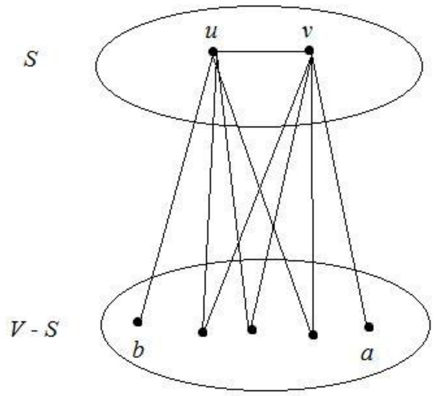

Fig 3:Vertices in S are adjacent such that u ↮a and v ↮ b

Clearly, a is anS-external private neighbour of v and b is an S-external private neighbour of u. Also (S \{u}) ∪{b} and (S \{v}) ∪{a} are both dominating sets.

Hence from above cases (1) and (2) we get,
γ_{cs} (G)= | S | = 2.

Theorem 3.3. If F_k is a friendship graph with 2k+1 vertices then γ_{cs} $(F_k) = \frac{n-1}{2}$.

Proof. Let F_k be a friendship graph with $n = 2k + 1$ vertices such that each F_k consists of k number of K_3. Let v be the label of the center vertex and $u_1, u_2, \dots, u_{2k}$ be the label of the vertices in the clockwise direction with $deg(u_i) = 2$.

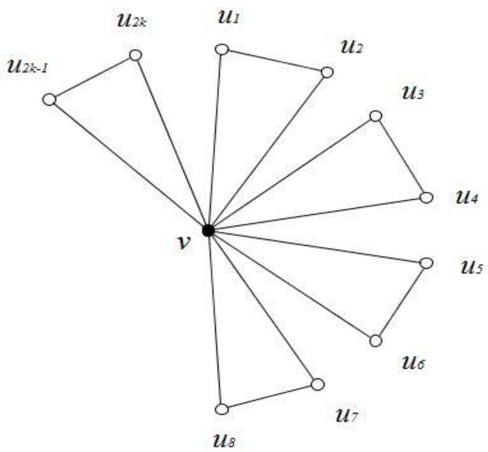

Fig 4:Friendship graph, F_k

Select one vertex from each K_3. other than the central vertex forming a CSDS with k vertices.i.e, let $D = \{u_1, u_2, \dots, u_{2k-1}\}$ or $D' = \{u_2, u_4, \dots, u_{2k}\}$ be a dominating set of F_k. Also, $(D \setminus \{u_i\}) \cup \{u_{i+1}\}$ forms a dominating setfor all $i \in \{1, 3, \dots, 2k-1\}$and $(D' \{u_i\}) \cup \{u_{i-1}\}$ forms a dominating set for all $i \in \{2, 4, \dots, 2k\}$.
Thus, γ_{cs} $(F_k) \leq k = \frac{n-1}{2}$.

Now we need to show that *D* forms the minimum cosecure dominating set. i.e, there exist no co-secure dominating set with less than *n* vertices.
Let us assume that gcs (F_k) < k. But atleast one vertex from each K_3must be there in the CSDS, which is a contradiction.Let v be the central vertex and if possible, S' =$\{v, u_1, u_3, \dots, u_{2k-5}\}$ be a CSDS. But there exist no vertex in v \ S' whose replacement with v results in a dominating set. Thus there exist atleast one K_3 which is not defended by S'. Thus D forms the minimum co-secure dominating set.

Hence, γ_{cs} (F_k) =$\frac{n-1}{2}$.

Remark: Since K_3 is a complete graph both secure and co-secure domination number remains the same. Also, each $F_k$consists of*k*number of K_3.
Thus we have, γ_s (F_k) = γ_{cs} (F_k) =$\frac{n-1}{2}$.

Theorem 3.4. For a Jahangir graph $J_{2,m}$,γ_{cs} $(J_{2,m})$ = *m.*
Proof. Suppose that *m* is even*m = 2k*, for some positive integer *k*. Let v_{2m+1}be the label of the center vertex and $v_1, v_2, \dots, v_{2m}$ be the label of the vertices that incident clockwise on cycle C_{2m} so that deg(v_1) = *3*. It is easy to verify that the set of vertices S_0 =$\{v_1, v_3, \dots, v_{2m-1}\}$ forms aco-secure dominating set for $J_{2,m}$.
Therefore γ_{cs} $(J_{2,m})$ ≤ | S_0 | = m.

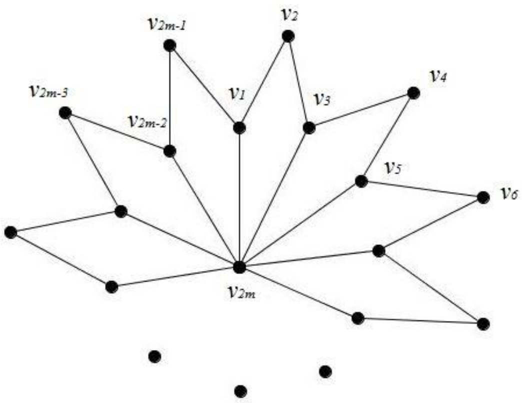

Fig 5: Jahangir graph, $J_{2,m}$

It suffices to prove that $\gamma_{cs}(J_{2,m}) \nless m$. On the contrary assume that $D \subseteq V(J_{2,m})$ be a co-secure dominating set of$J_{2,m}$with|*D* |<*m*.
We consider three cases:

Case 1. Suppose if *D* = $\{v_{2m+1}, v_2, v_4, \dots, v_{2m-4}\}$.It is clear that vertex v_{2m+1}dominates all odd index vertices, i.e.$v_1, v_3, \dots, v_{2m-1}$(Fig.5).
Therefore *D* – { v_{2m+1} }must dominate all *m* even index vertices, i.e. $v_2, v_4, \dots, v_{2m}$.Since there is at least two even index vertices on $V(J_{2,m})$ (say, v_{2m-1}, v_{2m}) that any vertices of *D* can not dominate, we have a contradiction.

Case 2. Suppose if *D* = $\{v_{2m+1}, v_1, v_2, \dots, v_{2m-5}\}$. Clearly v_{2m+1}dominates all odd indexed vertices (see Fig.5) but there exist exactly one even index vertex say, v_{2m} which is not dominated by *D* which is a contradiction. Thus, $\gamma_{cs}(J_{2,m})$ > *m -1.*

Case 3. Let $v_{2m+1} \notin D$ and all *m -1* vertices of cycleC_{2m} are in *D.* Without loss of generality, let $v_1 \in D$. Then *D* contains $\{v_1, v_3, \dots, v_{2m-3}\}$ vertices. But v_{2m-1} is not dominated by *D* which is a contradiction.
Thus, $\gamma_{cs}(J_{2,m})$ >*m -1.*

From the above three cases we have, $\gamma_{cs}(J_{2,m})$ >*m -1*. But $\gamma_{cs}(J_{2,m}$)≤| S_0 |=*m.*

Hence we have,$\gamma_{cs}(J_{2,m})$ = *m.*

Similarly we can prove the result when *m* is odd.

Thus, $\gamma_{cs}(J_{2,m}) = m$ for all m.

Theorem 3.5. For helm graph H_n, $\gamma_{cs}(H_n) = n$

Proof. H_n be a helm graph and let $u_1, u_2, \dots, u_n$ be the pendant vertices , $v_1, v_2, \dots, v_n$ be the rim vertices and v be the apex vertex as given in Figure 6. So, V (H_n) $=\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, v\}$. Since u_i's are pendant vertices, either u_i or $N(u_i)$ must belong to the cosecure dominating set (CSDS).

Suppose u_i's belongs to the CSDS, say $S_0 = \{u_1, u_2, \dots, u_n\}$ such that $|S_0| = n$. But the apex vertex v is not dominated by S_0. Thus, v must also be included in S_0 and hence $|S_0| = n + 1$.

If not suppose that v_i's belongs to the CSDS, say S such that $|S| = |v_i|$ = n. Clearly S dominates every vertex in H_n and $(S \setminus \{v_i\}) \cup \{u_i\}$ forms a dominating set of H_n. Clearly, both S and S_0 forms a CSDS, but the minimum cosecure number $\gamma_{cs}(H_n)$ is n.

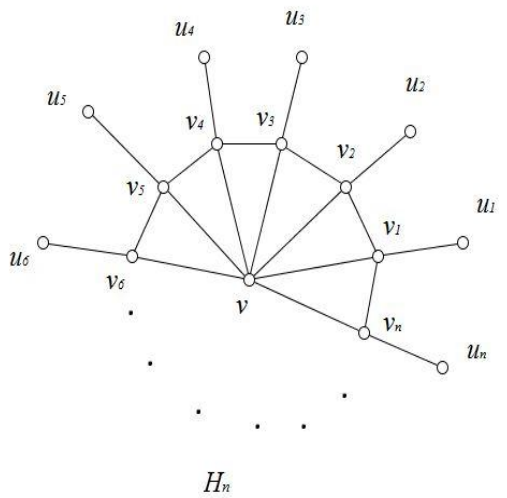

Fig 6: Helm graph, H_n

Hence we have $\gamma_{cs}(H_n)$ | S | = n. Now we need to show that S forms the minimum cosecure dominating set. i.e, there exist no cosecure dominating set with less than n vertices.
On the contrary assume that there exist a co secure dominating set $D = \{v_1, v_2, \dots, v_n\} \subseteq V(H_n)$ with $|D| < n$.

From the given labeling, it is clear that every vertex v_i in D dominates u_i. Since v_n does not belong to D, there exist atleast one vertex say, u_n on $V(H_n)$ which is not dominated by D. This contradicts our assumption that D forms a CSDS of H_n. Hence, $\gamma_{cs}(H_n) \nless n$.

Therefore,we have $\gamma_{cs}(H_n) = n$.

Theorem 3.6. If $T \in \mathbb{T}$, then $\gamma_{cs}(T) = \frac{n+2}{3}$

Proof. The family $\mathbb{T}$ of trees $T = T_k$ is obtained as follows. For any $k \geq 1$, T_k can be constructed recursively from T_{k+1} by adding a copy $P_3^k = u_k v_k w_k$ of P_3 and joining an edge between x and u_k.

Since x is adjacent to each u_i's the vertex x must belongs to every CSDS. Since w_i's are pendant vertices, either w_i or N(u_i) = v_i must belong to the co-secure dominating set (CSDS) for $i \in \{1, 2, \dots, k\}$
Suppose that v_i's belongs to the CSDS, say $S = \{x, v_1, v_2, \dots, v_k\}$ such that $|S| = |v_i| = k + 1$. Clearly S dominates every vertex in T_k and $(S \setminus \{v_i\}) \cup \{w_i\}$ forms a dominating set of T_k. Also, $(S \setminus \{x\}) \cup \{u_i\}$ forms a dominating set of T_k. Clearly, $S = \{x, v_1 v_2, \dots, v_k\}$ forms a CSDS of T_k.

Thus we have $\gamma_{cs}(T_k) \leq k + 1$.

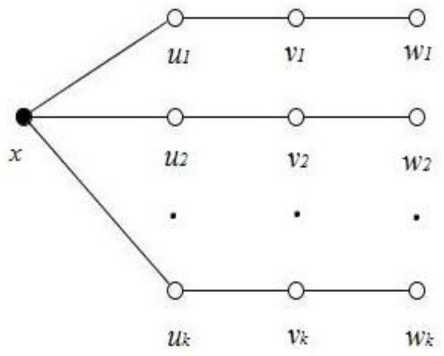

Fig 7: A family $\mathbb{T}$ of trees $T = T_k$.

Now we need to show that S forms the minimum co-secure dominating set.
i.e, there exist no co-secure dominating set with less than $k + 1$ vertices.On the contrary assume that there exist a co-secure dominating set S' = $\{x, v_k, \dots v_{k-1}\} \subseteq V(T_k)$ with $|S'| < k + 1$.
It is clear that x in S' dominates u_i (Figure 7). Since v_k does not belong to S'there exist atleast two vertex say, v_k and w_k on $V(T_k)$ which is not dominated by S'. This contradicts our assumption that S' forms a CSDS of T_k.
Hence, $\gamma_{cs}(T_k) \nless k + 1$.

Thus we have, $\gamma_{cs}(T_k) = k + 1 = \frac{n+2}{3}$

Remark: Also, $\gamma_s(T_k) = \frac{n+2}{3}$is already found in [5].

Thus we have, $\gamma_s(T) = \gamma_{cs}(T)$ for all $T \in \mathbb{T}$.

IV. Conclusions

In this paper we have studied the cosecure domination number of some families of graphs namely, Friendship graph F_k, Jahangir graph $J_{2,m}$ and Helm graph H_n. We also discussed on family of trees $\mathbb{T}$for which $\gamma_{cs} = \gamma_s$. Further works can be done in this area by finding co-secure domination numbers for other classes of graphs and by characterizing graphs G such that $\gamma_{cs} = \gamma_s$. The concepts ofco-secure and secure domination has various applications in communication networks and security system. The secure domination number represents the minimum number of guards required to protect the entire system in a bid to save on the total guard renumeration or deployment cost whereas co-secure domination set gives the provision for a substitute guard for each guards in the system and thereby safeguarding the system in the absence of a guard.

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