Changing and Unchanging of Minus Domination in Graphs

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Abstract— *A function $f:V(G)\rightarrow\{-1,0,1\}$ defined on the vertex set of a graph $G=(V,E)$ is said to be a minus dominating function if the sum of its function values over every closed neighbourhood is at least one. That is for every $v\in V$, $f(N[v])\geq 1$, where $N(v)$ consists of v and every vertex adjacent to v. The weight of a minus dominating function is $f(V)=\sum f(v)$, over all vertices $v\in V$. The minus domination number of a graph G, denoted by $\gamma^-(G)$ is equal to the minimum weight of a minus domination function of G. In this paper, we study the change in minus domination number after adding an edge to paths and $C_n\circ K_1, n\geq 3$. We also investigate the bounds for minus domination number of Jahangir graph and the line graph of sunlet graphs.*

Keywords — *Jahangir graph, sunlet graph, line graph, corona graph.*

I. Introduction

For a graph $G=(V,E)$ with vertex set V and edge set E, the open neighbourhood of $v\in V$ is $N(v)=\{u\in V: uv\in E\}$ and closed neighbourhood of v is $N[v]=\{v\}\cup N(v)$. For the set S of vertices, we define open neighbourhood $N(S)=\bigcup_{v\in S}N(v)$, and the closed neighbourhood $N[S]=N(S)\cup S$. The line graph of a graph G, written $L(G)$ is the graph whose vertices are the edges of G, with $ef\in E(L(G))$ when $e=uv$ and $f=vw$ in G. Let G and H be two graphs. The corona of G and H denoted as $G\circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the ith copy of H to the ith vertex of G, where $1\leq i\leq |V(G)|$. The corona of C_n with $K_1, N\geq 3$ is called as sunlet graph.

A set $S\subseteq V$ for graph $G=(V,E)$ is said to be a dominating set if each $v\in V$ is either in S or adjacent to a vertex of S. The domination number of G denoted by $\gamma(G)$, equals minimum cardinality of a dominating set. So far, different kinds of domination like roman domination, total domination, restrained domination, etc have been studied. By a minus domination function, we mean a three-valued function $f:V(G)\rightarrow\{-1,0,1\}$ such that $f(N[v])\geq 1\ \forall\ v\in V$. We use the following notation which we shall frequently use in the proofs that follow. For a given minus domination function f on a graph G, let $P_f=\{v\in V: f(v)=1\}$ and $M_f=\{v\in V: f(v)=-1\}$. Then, weight of the minus domination function f is equal to $|P_f|-|M_f|$. By a Jahangir graph $J_{n,m}$, we mean a graph on $nm+1$ vertices i.e., a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at a distance n to each other on C_{nm}. The domination in Jahangir graphs $J_{n,m}$ where $n=2$ has been studied by D. A. Mojdeh and A. N. Ghameshlou in [2] and also, it has been established that $\gamma(J_{2,m})=\left\lceil\frac{m}{2}\right\rceil+1$. In [1], J. Dunbar and et. al, studied the properties of minus domination number and they classified the graphs according to their minus domination number. Also, they have shown that the line graph of the smallest sunlet graph $C_3\circ K_1$ which is nothing but the $Hajös\ graph$ is said to be the smallest graph with minus domination number equal to zero. The domination in corona and join of graphs are studied by C. E. Go and S.R. Canoy in [4] and it has been shown that $\gamma(G \circ H) = n$ where G is a graph of order n and H is any graph. In [3], some properties of minus total domination number are studied and lower bounds for minus total domination number of trees, complete graphs, complete multipartite graphs are obtained. According to H. M. Xing et al. [3], the necessary and sufficient condition for a minus total dominating function f to be minimal is every vertex $v \in V$, with $f(v) \geq 0$ f, there exists a vertex $u \in N(v)$ with $f(N(u)) = 1$.

II. MAIN RESULTS

In this paper, we obtain the bounds for minus domination number of Jahangir graphs and find a result for the minus domination number of the line graph of any general sunlet graph. We study the changes in minus domination number after adding an edge to paths and show that minus domination number is reduced by 1 whenever we add an edge between any two pendent vertices of the graph $G \circ H$, where $G = C_n$ and $H = K_1$.

2.1 Bounds for the minus domination number of Jahangir graphs and line graph of sunlet graphs:

In this section, we find the bounds for the minus domination number of Jahangir graphs $J_{n,m}$ where $n = 2,3,4$. And we also prove that the minus domination number of line graph of any sunlet graph is equal to zero.

Proposition 2.1.1: For the Jahangir graph $J_{2,m}$, $\gamma^{-}(J_{2,m}) \leq \left\lceil \frac{m}{3} \right\rceil + 1$.

Proof:

We know that $J_{2,m}$ has $2m+1$ vertices. Let $V(J_{2,m}) = \{v_1, v_2, \ldots, v_{2m+1}\}$.

Let $v_2, v_4, \ldots, v_{2m}$ be the vertices on the cycle C_{2m} that are adjacent to the central vertex v_{2m+1}. We know that any positive integer can be written in the form $3k$ or $3k+1$ or $3k+2$ for some positive integer k. Since m is a positive integer greater than or equal to 3, we prove the bound in three cases:

Case (i): let $m = 3k$

Define $f : V \rightarrow \{-1, 0, 1\}$ as follows:

$$f(v_i) = \begin{cases} -1, i \cong 1 (\mathrm{mod}\ 6) \\ 1, i \cong 0, 2 (\mathrm{mod}\ 6) \\ 0,\ otherwise \end{cases}$$

And fix $f(v_{2m+1}) = 1$.

Thus f assigns the label -1 to every pair of vertices at a distance 6 from each other. (i.e., v_1, v_7, v_{13},... get the label −1. We can see fig (1) that all these vertices are of degree 2).

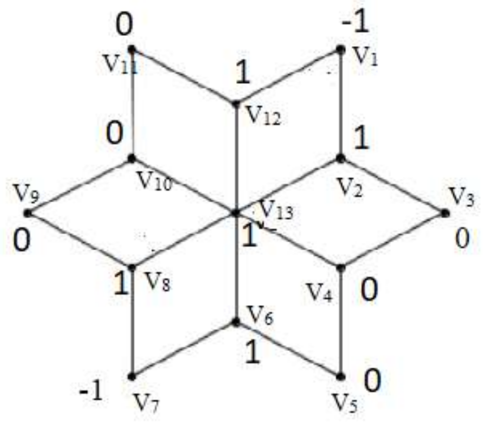

Figure 1: Jahangir graph where n=2, m=6

f is clearly a minus dominating function since the following conditions hold:

(i) f assigns the label 1 to both the neighbours of v whenever $f(v) = -1$. Hence the condition $f(N[v]) \geq 1$ is maintained.

(ii) Since all the vertices with label 1 are adjacent to the central vertex and the central vertex also has the label 1, $f(N[v_{2m+1}]) = |P_f| \geq 1$.

(iii) Since all the vertices that are adjacent to vertices with label -1 are also adjacent to the central vertex and they have the label 1, their closed neighbourhood sum is maintained as 1.

Since each pair of vertices at a distance 6 have the label -1, the total number of vertices on the cycle C_{2m} with the label -1 is equal to $\frac{m}{3}$.

In fig (1), an example where $m = 6$ is shown. Since each vertex with the label -1 has two neighbours with the label 1, the number of vertices on the cycle C_{2m} with the label 1 is equal to $2\frac{m}{3}$. Then weight of the function f is

$w(f) = |P_f| - |M_f| = 2\frac{m}{3} + 1 - \frac{m}{3}$ where +1 represents the label on the central vertex.

(However, $|M_f| = 0$ if $m = 3$.)

$$w(f) = \frac{m}{3} + 1.$$

We have $\gamma^-(J_{2,m}) \leq w(f) = \frac{m}{3} + 1$.

In this case since m is a multiple of 3, $\left\lceil \frac{m}{3} \right\rceil = \frac{m}{3}$.

Therefore, $\gamma^-(J_{2,m}) \leq \left\lceil \frac{m}{3} + 1 \right\rceil$.

case(ii): Let $m = 3k + 1$

This implies that there are $6k+2$ vertices on the cycle C_{2m} since the outer cycle of any Jahangir graph $J_{2,m}$ has $2m$ vertices.

Label the first $6k$ vertices according to the function defined in case(i). Now the last two vertices i.e., v_{2m-1} and v_{2m} are left unlabelled. Since, v_{2m} is adjacent to v_1 and v_1 has the label -1, label v_{2m} with 1. But v_{2m-1} is adjacent to v_{2m-2} which has the label 1. Therefore label v_{2m-1} with 0.

Since $m=3k+1$, it follows that $m \cong 1 \bmod 3$.

$$\Rightarrow \left\lceil \frac{m}{3} \right\rceil = \frac{m-1}{3}+1 \qquad (1)$$

It is clear that each pair (except the last) of vertices at a distance 6 from each other have the label '-1'. Therefore, the number of vertices with the label -1 is equal to $\frac{m-1}{3}$ since $k=\frac{m-1}{3}$. And every vertex with label -1 has exactly two neighbours with label 1. Therefore the number of vertices on the outer cycle with label 1 is equal to $2\frac{m-1}{3}+1$, where $+1$ is the label on v_{2m}. Thus, total number of vertices with label 1 is equal to $2\frac{m-1}{3}+2$ since central vertex also has the label 1. One such example is shown in fig (2) where $m=7$:

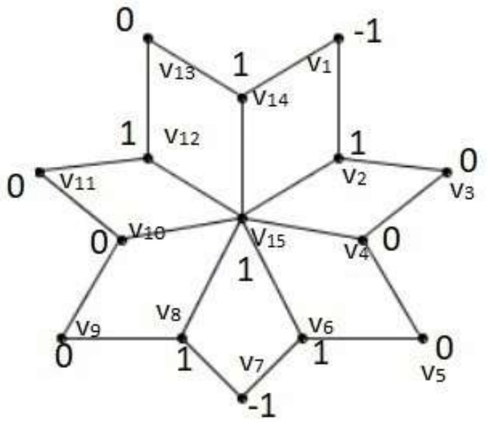

Figure 2: Jahangir graph where n=2, m=7.

We have

$$w(f)=|P_f|-|M_f|$$
$$=2\frac{m-1}{3}+2-\frac{m-1}{3}$$
$$=\frac{m-1}{3}+1+1$$
$$=\left\lceil \frac{m-1}{3} \right\rceil+1, \; from\,(1).$$

Thus, we have $\gamma^{-}(J_{2,m}) \le w(f) = \left\lceil \frac{m}{3} \right\rceil + 1$.

case(iii): Let $m = 3k + 2$. Thus, there are $6k + 4$ vertices on the outer cycle C_{2m}. Label the first 6k vertices on the cycle C_{2m} using the rule defined in case(i). Now there are 4 unlabelled vertices. They are $v_{2m-3}, v_{2m-2}, v_{2m-1}$ and v_{2m}. Label v_{2m} with 1 since $v_{2m} \leftrightarrow v_1$ and $f(v_1) = -1$.

Now $f(v_{2m-1}) = 0$.

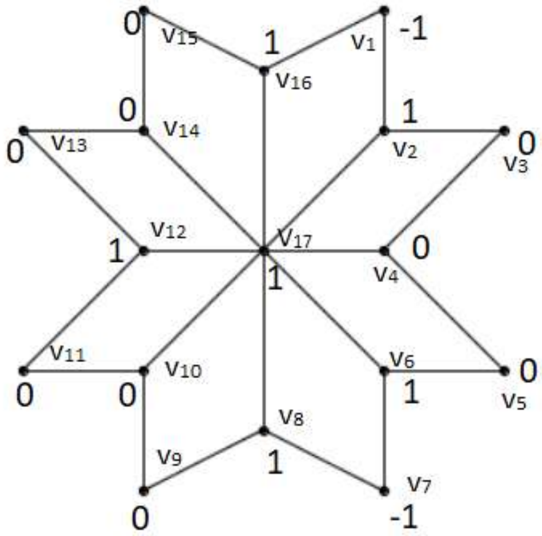

Figure 3: Jahangir graph where n=2, m=8

Label v_{2m-3} with 0 since $v_{2m-3} \leftrightarrow v_{2m-4}$ and $f(v_{2m-4}) = 1$ [since $2m - 4 \cong 0 \bmod 6$].

Now since v_{2m-2} is adjacent to the central vertex, it can be labelled with 0 .

Since $m = 3k + 2$, it follows that $m \cong 2 \bmod 3$.

$$\Rightarrow \left\lceil \frac{m}{3} \right\rceil = \frac{m-2}{3} + 1 \qquad (2)$$

Since $k = \frac{m-2}{3}$, there are $\frac{m-2}{3}$ vertices on the cycle C_{2m} with the label -1. One example is demonstrated in fig (3) where $m = 8$:

There are $\frac{m-2}{3} + 2$ vertices with the label +1 including v_{2m} and the central vertex.

We have

$$w(f) = |P_f| - |M_f|$$
$$= 2\frac{m-2}{3} + 2 - \frac{m-2}{3}$$
$$= \frac{m-2}{3} + 1 + 1$$
$$= \left\lceil \frac{m}{3} \right\rceil + 1, \text{from } (2).$$

Thus, we have $\gamma^{-}(J_{2,m}) \le w(f) = \left\lceil \frac{m}{3} \right\rceil + 1$.

Proposition 2.1.2. For the Jahangir graph $J_{3,m}, \gamma^{-}(J_{3,m}) \le m$.

Proof:

We know that $J_{3,m}$ has $3m+1$ vertices where v_{3m+1} is the central vertex. Define $f : V \to \{-1, 0, 1\}$ as follows:

$$f(v) = \begin{cases} 1, v \leftrightarrow v_{3m+1} \\ 0, otherwise \end{cases}$$

Then f is clearly a minus domination function since

(i) Since every vertex adjacent to the central vertex v_{3m+1} has the label 1, $f(N[v_{2m+1}]) = m \geq 3 \geq 1$.

(ii) By the definition of f, the distance between every pair of vertices is at most three. Thus, the closed neighborhood sum for every vertex on the outer cycle C_{3m} is maintained as 1. The labelling of $J_{3,3}$ is demonstrated in fig (4):

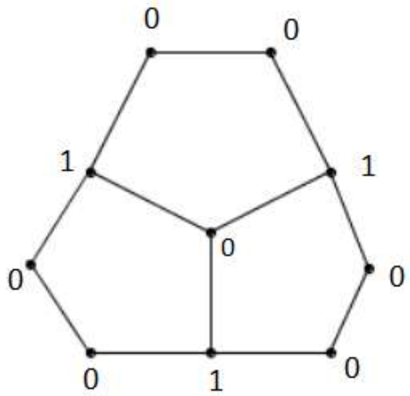

Figure 4: Jahangir graph n=m=3.

The weight of the function f is equal to

$w(f) = 1 + 1 + \ldots + 1 (m \ times)$

$\Rightarrow w(f) = m.$

$But \ \gamma^{-}(G) \leq w(f) = m.$

$Thus, \gamma^{-}(G) \leq m.$

Proposition 2.1.3. For the Jahangir graph $J_{4,m}, \gamma^{-}(J_{4,m}) \leq m+1$.

Proof: We know that $J_{4,m}$ has $4m+1$ vertices, where v_{4m+1} is the central vertex.

Define $f : V \to \{-1, 0, 1\}$ as follows:

$$f(v) = \begin{cases} 1, \quad v \ is \ the \ central \ vertex \ or \\ \ if \ d(v) = 2 \ and \ v \ is \ not \ adjacent \ to \ any \ vertex \ of \ degree \ 3. \\ 0, otherwise \end{cases}$$

The function defined above is a minus domination function since:

(i) The central vertex has the label 1 and all its neighbors have the label 0.

$f(N[v_{4m+1}]) = 1$.

(ii) The distance between any pair of vertices with label 1 is at most three. Therefore, closed neighborhood sum of all the vertices on the cycle C_{4m} is maintained as 1.

The weight of the function f is equal to

$w(f) = 1 + 1 + \ldots + 1$ (m+1) times, since central vertex also has got the label 1. The labelling of $J_{4,3}$ is shown in fig (5):

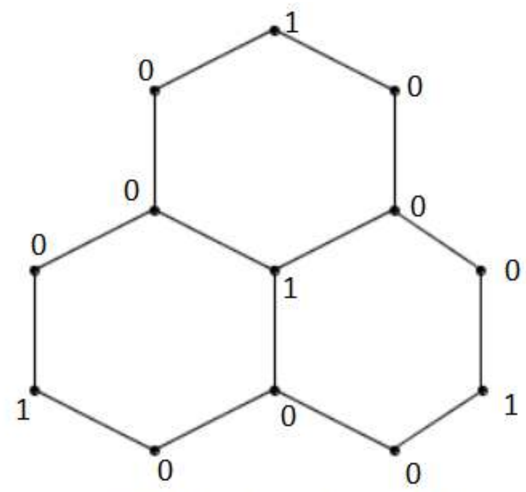

Figure 5: Jahangir graph where n=m=4.

$$w(f) = m + 1$$

Therefore, $But\ \gamma^{-}(J_{4,m}) \leq w(f) = m + 1..$

$$\therefore \gamma^{-}(J_{4,m}) \leq m + 1.$$

Theorem 2.1.4. Let $G = L(C_n \circ K_1)$. Then, $\gamma^{-}(G) = 0$.

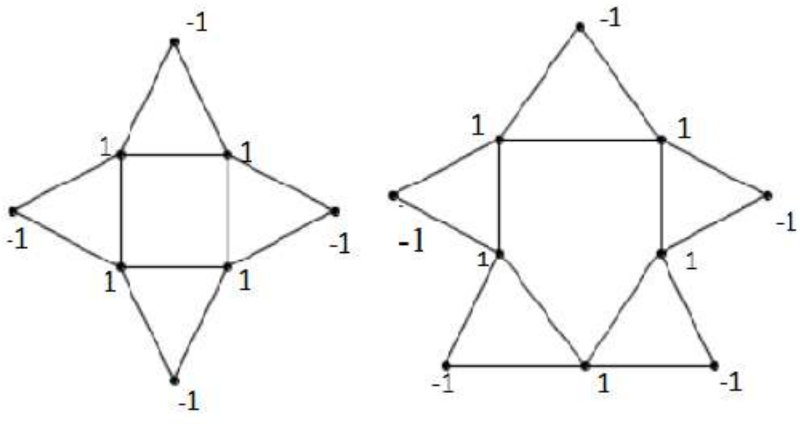

Figure 6: Line graph of sunlet graph

Proof:

Clearly, $L(C_n \circ K_1)$ is a graph with $2n$ vertices with degree of each vertex being equal to 2 or 4.

Define a function as follows:

$$f(v) = \begin{cases} -1, & \deg(v) = 2 \\ +1, & \deg(v) = 4 \end{cases}$$

The labelling of $L(C_4 \circ K_1)$ and $L(C_5 \circ K_1)$ are shown in fig (6):

Then the number of vertices with label 1 will be equal to n, since all the vertices on the cycle have degree 4. And the number of vertices with degree 2 is also equal to n since all the other vertices have degree equal to 2. [Because, $|V(G)| = 2n$]. Clearly, f is a minus dominating function.

Then, the weight of the function f is

$w(f) = 1(n) + (-1)(n) = 0$

$But, \gamma^{-}(G) \le w(f) = 0$ Now

$\Rightarrow \gamma^{-}(G) \le 0.$ (3)

we prove that this function attains the smallest possible weight for the graph G. Let v be any vertex with degree 2. Let $N(v) = \{u, w\}$. Then by above definition, $f(v) = -1$. Then 0 cannot be assigned to u or w as $f(N[v]) \le 0$, which is not allowed. For f to have negative weight, either there must be $n+1$ vertices with the label -1 So that $w(f) = (-1)(n+1) + 1(n-1) = -2 \le 0$ or at least one vertex with degree 4 must have the label 0 [So that $w(f) = -1(n) + 1(n-1) + 0 = -1 < 0$]. But we see that both of these situations cannot arise as the condition $f(N[V]) \ge 1$ has to be maintained. Therefore, there exists no minus dominating function with negative weight.

Thus,

$\gamma^{-}(G) \ge 0$ (4) From (3) and (4), $\gamma^{-}(G) = 0$.

2.2 Change in the minus domination number on the addition of an edge to paths and corona of C_n with K_1:

In this section, we study the change in minus domination number of paths and corona of C_n and K_1 after the addition of an edge.

Theorem 2.2.1. Let $G = C_n \circ K_1$. Then,

(a) $\gamma^{-}(G + v_i v_j) = \gamma^{-}(G) - 1$, where $v_i v_j$ is an edge between any two pendent vertices.

(b) $\gamma^{-}(G + u_i u_j) = \gamma^{-}(G)$, where $u_i u_j$ is an edge between any two vertices on the cycle.

Proof of (a):

Let $V = V_1 \cup V_2$, where

$V_1 = \{v : v \text{ is a pendent vertex}\}$

$V_2 = \{u : u \text{ is a vertex on the cycle } C_n\}$

Before adding the edge, define a function $f : V \rightarrow \{-1, 0, 1\}$ as follows:

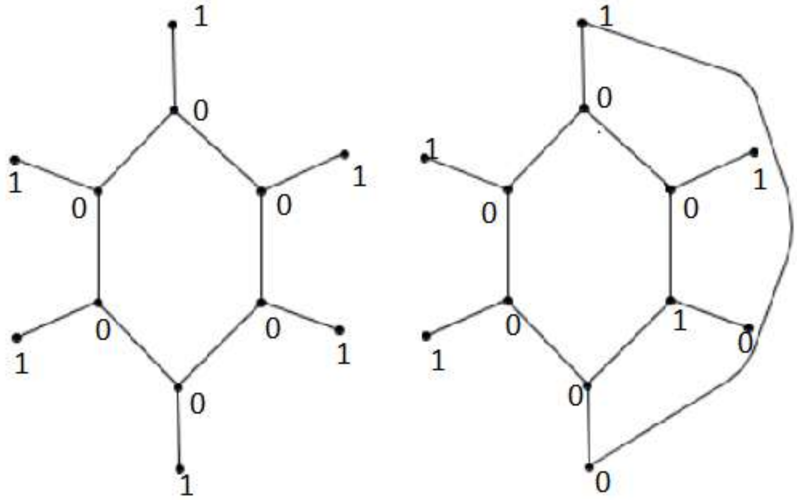

Figure 7: Corona of graphs before and after adding an edge

$f(v) = 1 \ \forall v \in V_1$

$f(u) = 0 \ \forall u \in V_2$.

Since no vertex is assigned the value -1 under f, f is same as a dominating function. Clearly, all the vertices on the cycle have the label 0 and all the pendent vertices have the label 1. Every dominating function is also a minus dominating function. Thus, f is a minus dominating function.

Therefore, $\gamma^-(G) = \gamma(G)$.

The weight of the function f is $w(f) = 1+1+\ldots+1\ (n\ times) = n$.

We have $\gamma(C_n \circ K_1) = n$.

Thus, f is attains the smallest possible weight.

$\Rightarrow f$ is minimum. Now add an edge between the vertices v_i and v_j $1 \le i < j \le n$. Then according to above definition $f(v_i) = 1$.

$f(u_j) = 0$ where u_j is the vertex on the cycle that is adjacent to the pendent vertex v_j. Now define $f(v_j) = 0$, $f(u_{j-1}) = 1$, $f(v_{j-1}) = 0$. Labelling of the rest of the vertices remains the same as that before adding the edge $v_i v_j$. i.e., u_{j-1} has the label 1 and v_j, v_{j-1} have the label 0. In other words, one of the vertices on the cycle has the label 1 and two of the pendent vertices have the label 0 after adding the edge.

Thus, $w(f) = n-2+1 = n-1$ However since f is minimum, we get $\gamma^-(G + v_i v_j) = n-1$ as stated.

The labelling of $C_6 \circ K_1$ before and after adding the edge is shown in fig (7):

Proof of (b): Since n vertices are required to dominate the n pendent vertices of the corona, addition of an edge within the cycle does not alter the domination number. Therefore, $\gamma^-(G + u_i u_j) = n = \gamma^-(G)$.

We observe that the addition of an edge to a path P_n changes the domination number of the path (precisely, decreases γ by 1) only when $n = 3m+1,\ m = 1,2,3\ldots$ Various cases where the domination number changes is discussed in the following propositions.

Theorem 2.2.2. Let G be a path on n vertices such that $n = 3m+1,\ m = 1,2,3\ldots$

If e is an edge added between a pendent vertex and the support vertex of the other pendent vertex, then

$\gamma^-(G+e) = \gamma^-(G) - 1$.

Proof:

Given G is a path on n vertices such that $n = 3m+1,\ m = 1,2,3\ldots$

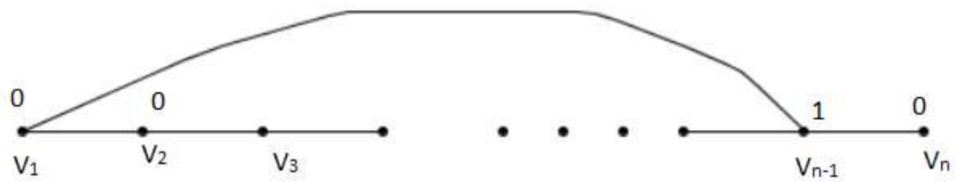

Figure 8: Addition of an edge to paths.

Let $V(P_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ in the order such that v_1 and v_n are the pendent vertices.

Let f be a minus domination function such that $w(f) = \gamma^-(G)$. Then no vertex is assigned the value -1 under f. We have,

$$\gamma^-(P_n) = \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{3m+1}{3} \right\rceil = m+1.$$

$$i.e., \gamma^-(G) = m+1 \qquad (5)$$

Now add an edge between a pendent vertex and the support vertex of the other pendent vertex. It forms a pan graph on $3m+1$ vertices consisting of a cycle of length $3m$.

The minus domination number of the cycle is: $\gamma^{-}(C_{3m}) = \left\lceil \frac{3m}{3} \right\rceil = m.$. Now, v_n being $(3m+1)^{th}$ vertex is dominated by the vertex v_{n-1}. Minimality is achieved when v_{n-1} gets the label 1 under f as shown in fig (8). Therefore, m vertices suffice to dominate the whole graph after adding the edge. Since no vertex is assigned the value -1 under f, f is a minimum domination function which assigns either 1 or 0 to each vertex of G.

For paths

$$\gamma^{-}(G+e) = m \qquad (6)$$

From (5) and (6), we have $\gamma^{-}(G+e) = \gamma^{-}(G) - 1$.

Proposition 2.2.3. let G be a path on n vertices such that $n = 3m+1,\ m = 1,2,3\ldots$ If an edge is added between two vertices u and v such that the edge uv belongs to a cycle of length $3k,\ k < n,\ k = 1,2,3\ldots$ where one of them is a pendent vertex, then $\gamma^{-}(G+uv) \le \gamma^{-}(G) - 1$.

Proof:

Let u be a pendent vertex. If v is the support vertex of the other pendent vertex, then the result holds by the previous proposition. Hence assume v is any vertex of degree 2 other than the support vertex of the other pendent vertex.

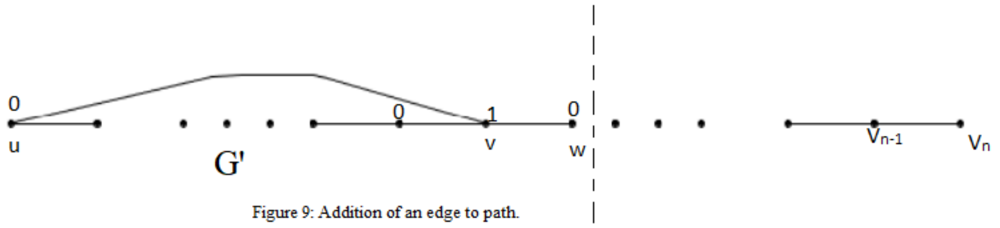

Figure 9: Addition of an edge to path.

Then, $\gamma^{-}(G+uv) \le \gamma^{-}(G') + \gamma^{-}(P_{n-(3k+1)})$ where G' is the pan graph on $3k+1$ vertices. However, $\gamma^{-}(G') = k$, from the previous proposition.

$$\gamma^{-}(G+uv) \le k + \gamma^{-}(P_{n-(3k+1)}) = k + \left\lceil \frac{n-3k-1}{3} \right\rceil, k < n$$

$$= k + \left\lceil \frac{3m+1-3k-1}{3} \right\rceil$$

$$= k - m + k = m.$$

$\therefore\ \gamma^{-}(G+uv) \le \gamma^{-}(G) - 1$.

In the following proposition we show that whenever we add an edge between any two intermediate vertices such that the new edge belongs to a cycle of length $3k$, then the minus domination number either is decreased or it is maintained as it was before but it never increases.

Proposition 2.2.4. Let G be a path on $n = 3m+1$ vertices where $m = 1,2,3\ldots$ If an edge is added between two vertices u and v such that $d(v) = d(u) = 2$ and $uv \in C_{3k}, k = 1,2,3\ldots$ then, $\gamma^{-}(G+uv) \le \gamma^{-}(G)$.

Proof: Given, $G = P_n,\ n = 3m+1,\ m = 1,2,3\ldots$.

Let $P_n' = P_n + uv$. Then $P_n' - C_{3k}$ will have two disconnected paths say P_s and P_r such that $r + s \cong 1 \bmod 3$.

There are two cases.

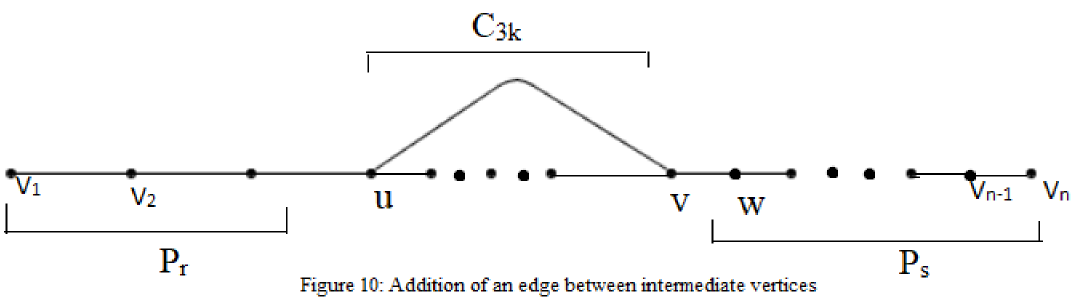

Figure 10: Addition of an edge between intermediate vertices

case(i): $r \cong 0 \bmod 3$ and $s \cong 1 \bmod 3$.

Thus, it follows that

$3 \mid r \Rightarrow r = 3p$ and

$3 \mid s-1 \Rightarrow s-1 = 3q$ where p and q are integers.

We have $\gamma^-(G+uv) \le \gamma^-(P_r) + \gamma^-(P_s) + \gamma^-(C_{3k})$.

However, $\gamma^-(C_{3k}) = k$.

Let w be the vertex adjacent to v. Then, from the proposition 2.2.2, we have $\gamma^-(G') = k$ where G' is the pan graph on $3k+1$ vertices.

$$\therefore \gamma^-(G+uv) \le \gamma^-(P_r) + \gamma^-(G') + \gamma^-(P_{s-\{w\}})$$

$$= \left\lceil \frac{r}{3} \right\rceil + k + \left\lceil \frac{s-1}{3} \right\rceil$$

$$= \left\lceil \frac{3p}{3} \right\rceil + k + \left\lceil \frac{3q+1-1}{3} \right\rceil$$

$$= p + k + q.$$

$$\therefore \gamma^-(G+uv) \le p + q + k \qquad (7)$$

However, the total number of vertices is

$n = r + 3k + s$

$3m + 1 = 3p + 3k + 3q + 1$

$m = p + k + q$

$\Rightarrow p + q = m - k$

Substituting for $p+q$ in (7), we get

$\gamma^-(G+uv) \le m \le m+1$

In other words, $\gamma^-(G+uv) < \gamma^-(G)$.

Case(ii): $r \cong 2 \bmod 3$ and $s \cong 2 \bmod 3$.

$\therefore 3 \mid r-2 \Rightarrow r-2 = 3p$.

$3 \mid s-2 \Rightarrow s-2 = 3q$

Where p and q are integers.

$r = 3p + 2, s = 3q + 2.$

$$\begin{aligned}
\gamma^-(G) &\leq \gamma^-(P_r) + \gamma^-(C_{3k}) + \gamma^-(P_s) \\
&= \left\lceil \frac{r}{3} \right\rceil + k + \left\lceil \frac{s}{3} \right\rceil \\
&= \left\lceil \frac{3p+2}{3} \right\rceil + k + \left\lceil \frac{3q+2}{3} \right\rceil \\
&= p + 1 + k + q + 1 \\
&= p + q + k + 2.
\end{aligned}$$

$$\gamma^-(G + uv) \leq p + q + k + 2. \qquad (8)$$

The total number vertices is given by:

$n = r + s + 3k$

$3m + 1 = 3p + 2 + 3q + 2 + 3k$

$3m = 3p + 3q + 3k + 1$

$\Rightarrow p + q + k = m - 1.$

Substituting $p + q + k = m - 1$ in (8), we get

$\gamma^-(G + uv) \leq m + 1 = \gamma^-(G)$

Hence,

$\gamma^-(G + uv) \leq \gamma^-(G)$.

III. CONCLUSION

In this paper, we discussed the bounds for the minus domination number of Jahangir graphs and line graph of sunlet graphs. We also discussed the change in the minus domination number after adding an edge to $C_n \circ K_1$ and paths P_n where $n = 3m + 1, m = 1, 2, 3 \ldots$ Further works can be done in this area by finding the bounds for the minus domination number of Jahangir graph $J_{n,m} n \geq 5$. The application of minus domination is that by assigning the values $-1, 0$ or 1 to the vertices of the graphs, we can model networks of people or organizations in which global decisions must be made.

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