Difference Property of an Integer Function

Xingbo Wang

Department of MechatronicsEnigneering, Foshan University
Foshan City, Guangdong Province, China

Abstract— *This article makes an investigation on the difference of an integer function. Through proving several inequalities, the article gives a bound for the first order difference of the function. The article also presents some new inequalities that are helpful for general purpose of bound estimation of integer functions.*

Keywords —*Difference, integer function, bound estimation, inequality*.

I. Introduction

The floor function of real variable x, denoted by $\lfloor x \rfloor$, is an integer function defined by an inequality $x-1<\lfloor x \rfloor \le x$. Due to such a definition in the form of inequality, mathematical deduction or mathematical modelling involved with the function always requires quite a lot of special skills related with inequalities and is of quite individuality. Every case remains a classic. A recent study came across such a function and required to estimate the lower and upper bounds of its first order difference. In solving the problem, several new inequalities are derived out and proved. This article introduces the related contents.

II. Preliminaries

This section lists notations, symbols and lemmas that are adopted in this article.

A. Definitions

Definition 1: (D1). For arbitrary real number x, the floor function of x, denoted by $\lfloor x \rfloor$, is an integer that satisfies inequality $x-1<\lfloor x \rfloor \le x$, or equivalently $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$. The fraction part $x-\lfloor x \rfloor$ is denoted by $\{x\}$.

Definition 2: (D2). Let a be a given positive constant real number and *x* be a variable on $(0,\infty)$. Define $d(x)$ and its first order difference Δ_x by

$$d(x)=\left\lfloor \frac{1}{2}\left(1+\sqrt{\frac{a}{x}}\right)\right\rfloor$$

$$\Delta_x = d(x)-d(x+1)=\left\lfloor \frac{1}{2}\left(1+\sqrt{\frac{a}{x}}\right)\right\rfloor-\left\lfloor \frac{1}{2}\left(1+\sqrt{\frac{a}{x+1}}\right)\right\rfloor$$

B. Symbols and Notations

Symbol $A \Rightarrow B$ means conclusion *B* can be derived from condition *A*; $A \Leftrightarrow B$ means *A* is equivalent to *B*. Symbol $A \,\square\, \frac{B}{2}$ means

$$A=\begin{cases}\dfrac{B}{2}, & B \text{ is even}\\[2mm] \dfrac{B-1}{2}, & B \text{ is odd}\end{cases}$$

For convenience in deduction of a formula, comments are inserted by symbols that express their related mathematical foundations. For example, the following deduction

$$\begin{aligned} A &= B\\ (\mathbf{T}) &= C\\ (\mathbf{C}) &\le D\end{aligned}$$

means that, theorem (**T**) supports the step from *B* to *C*, and corollary (**C**) supports the step from *C* to *D*.

C. Lemmas

Lemma 1: (**L**1, see in [1]) For integer *n*, real numbers *x* and *y*, the following inequalities and equalities hold

(P1) $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$

(P2) $\lfloor x \rfloor - \lfloor y \rfloor - 1 \le \lfloor x-y \rfloor \le \lfloor x \rfloor - \lfloor y \rfloor < \lfloor x \rfloor - \lfloor y \rfloor + 1$

(P8) $n\lfloor x\rfloor \le \lfloor nx\rfloor$ with $n > 0$

(P13) $x \le y \Rightarrow \lfloor x\rfloor \le \lfloor y\rfloor$

(P17) $\lfloor x\rfloor + \left\lfloor x + \frac{1}{2}\right\rfloor = \lfloor 2x\rfloor$

Lemma 2: (**L**2, see in [2]). Let α and x be a positive real numbers; then it holds

$$\alpha\lfloor x\rfloor - 1 < \lfloor \alpha x\rfloor < \alpha(\lfloor x\rfloor + 1)$$

Particularly, if α is a positive integer, say $\alpha = n$, then it yields

$$n\lfloor x\rfloor \le \lfloor nx\rfloor \le n(\lfloor x\rfloor + 1) - 1$$

Lemma 3: (**L**3, see in [2]). For arbitrary positive real numbers α, x and y with $x > y$, it holds

$$\lfloor \alpha(x-y)\rfloor + \alpha\lfloor y - x\rfloor \le 0$$

III. Main Results and Proofs

Theorem 1: (**T**1). Let θ be a positive real number and $\Theta(x) = \frac{1}{2}\sqrt{\frac{1}{x}} - \frac{\theta+1}{2}\sqrt{\frac{1}{x+1}} + \frac{\theta}{2}\sqrt{\frac{1}{x+2}}$; then $0 < \theta \le 1$ yields $\Theta(x) > 0$, and $\theta > 1$ plus $x > \frac{2}{\sqrt[3]{\theta^2} - 1}$ yields $\Theta(x) < 0$.

Proof. Let $l(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$ and $r(x) = \theta(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x+2}})$; then it yields

$$\Theta(x) = l(x) - r(x) \tag{1}$$

Note that

$$l(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}} = \frac{1}{(\sqrt{x+1} + \sqrt{x})\sqrt{x(x+1)}} \tag{2}$$

and

$$r(x) = \theta(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x+2}}) = \frac{\theta}{(\sqrt{x+1} + \sqrt{x+2})\sqrt{(x+1)(x+2)}} \tag{3}$$

It is sure that $2(x\sqrt{x}) < \frac{1}{l(x)} < 2(x+1)\sqrt{x+1}$ and $\frac{2(x+1)\sqrt{x+1}}{\theta} < \frac{1}{r(x)} < \frac{2(x+2)\sqrt{x+2}}{\theta}$; hence it holds

$$\frac{1}{\theta} < \frac{l(x)}{r(x)} < \frac{1}{\theta}(1 + \frac{2}{x})^{\frac{3}{2}}$$

or

$$\frac{r(x)}{\theta} < l(x) < \frac{r(x)}{\theta}\cdot(1 + \frac{2}{x})^{\frac{3}{2}} \tag{4}$$

By (1) it can see $\Theta(x) > 0 \Leftrightarrow l(x) > r(x)$ and $\Theta(x) < 0 \Leftrightarrow l(x) < r(x)$. Referring to (4) knows $0 < \theta \le 1$ leads to $\Theta(x) \ge 0$. The inequality (4) also shows that, when $\theta > 1$ and $x > \frac{2}{\sqrt[3]{\theta^2} - 1}$, it holds $0 < \frac{1}{\theta}\cdot(1 + \frac{2}{x})^{\frac{3}{2}} < 1$, which leads to $l(x) < r(x)$, namely, $\Theta(x) < 0$.

□

Theorem 2: (**T**2). It always holds

$$\left\lfloor \frac{\sqrt{a}}{2((x+1)\sqrt{x} + x\sqrt{x+1})}\right\rfloor \le \Delta_x \le \left\lfloor \frac{\sqrt{a}}{2((x+1)\sqrt{x} + x\sqrt{x+1})}\right\rfloor + 1 \tag{5}$$

Proof. By Lemma 1 (**P**2), it knows

$$\begin{aligned}\Delta_x &\ge \left\lfloor \frac{1}{2}(1 + \sqrt{\frac{a}{x}}) - \frac{1}{2}(1 + \sqrt{\frac{a}{x+1}})\right\rfloor \\ &= \left\lfloor \frac{1}{2}(\sqrt{\frac{a}{x}} - \sqrt{\frac{a}{x+1}})\right\rfloor \\ &= \left\lfloor \frac{\sqrt{a}}{2((x+1)\sqrt{x} + x\sqrt{x+1})}\right\rfloor\end{aligned}$$

and

$$\Delta_x \le \left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x}}\Big) - \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+1}}\Big) \right\rfloor + 1$$
$$= \left\lfloor \frac{1}{2}\Big(\sqrt{\frac{a}{x}} - \sqrt{\frac{a}{x+1}}\Big) \right\rfloor + 1$$
$$= \left\lfloor \frac{\sqrt{a}}{2((x+1)\sqrt{x} + x\sqrt{x+1})} \right\rfloor + 1$$

□

Theorem 3: (**T**3). For given real numbers α , β with $0 < \alpha \le 1$ and $\beta > 1$, there always be an x_0 such that, when $x > x_0$ it holds

$$\alpha\Delta_{x+1} < \Delta_x < \beta\Delta_{x+1} \tag{6}$$

Proof. Without loss of generality, calculate $\Delta_x - \omega\Delta_{x+1}$ for positive real number ω . Note that

$$\begin{aligned}\Delta_x - \omega\Delta_{x+1} &= \left(\left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x}}\Big)\right\rfloor - \left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+1}}\Big)\right\rfloor\right) - \omega\left(\left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+1}}\Big)\right\rfloor - \left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+2}}\Big)\right\rfloor\right)\\ &= \left(\left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x}}\Big)\right\rfloor - \left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+1}}\Big)\right\rfloor\right) + \omega\left(\left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+2}}\Big)\right\rfloor - \left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+1}}\Big)\right\rfloor\right)\end{aligned} \tag{7}$$

Then by Lemma 1 (**P**2) it yields

$$\Delta_x - \omega\Delta_{x+1} \le \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + 1 + \omega\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \omega$$
$$= \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \lfloor\omega\rfloor\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \{\omega\}\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \omega + 1$$
$$(\mathbf{P}8) \le \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \left\lfloor \frac{\lfloor\omega\rfloor}{2}\sqrt{\frac{a}{x+2}} - \frac{\lfloor\omega\rfloor}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \{\omega\}\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \omega + 1$$
$$(\mathbf{P}1) \le \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor\omega\rfloor+1}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor\omega\rfloor}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \omega + 1$$
$$= \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\omega+1}{2}\sqrt{\frac{a}{x+1}} + \frac{\omega}{2}\sqrt{\frac{a}{x+2}} + \frac{\{\omega\}}{2}\sqrt{\frac{a}{x+1}} - \frac{\{\omega\}}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \omega + 1$$
$$(\mathbf{P}1) \le \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\omega+1}{2}\sqrt{\frac{a}{x+1}} + \frac{\omega}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \left\lfloor \frac{\{\omega\}}{2}\sqrt{\frac{a}{x+1}} - \frac{\{\omega\}}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \omega + 2$$
$$(\mathbf{L}3) \le \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\omega+1}{2}\sqrt{\frac{a}{x+1}} + \frac{\omega}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \omega + 2$$

Namely,

$$\Delta_x - \omega\Delta_{x+1} \le \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\omega+1}{2}\sqrt{\frac{a}{x+1}} + \frac{\omega}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \omega + 2 \tag{8}$$

which indicates by Theorem 1 that, there must be an $x_b(a)$ such that $\Delta_x - \omega\Delta_{x+1} \le 0$ when $\omega > 1$ and $x > x_b(a)$.

On the other hand, Lemma 1 (**P**2) also leads to

$$\Delta_x - \omega\Delta_{x+1} \ge \left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x}}\Big) - \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+1}}\Big) \right\rfloor + \omega\left\lfloor \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+2}}\Big) - \frac{1}{2}\Big(1+\sqrt{\frac{a}{x+1}}\Big) \right\rfloor$$
$$= \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \omega\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor$$
$$= \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + (\omega-1)\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor$$
$$(\mathbf{P}1) \ge \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1+1}{2}\sqrt{\frac{a}{x+1}} + \frac{1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + (\omega-1)\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor - 1$$
$$= \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1+1}{2}\sqrt{\frac{a}{x+1}} + \frac{1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + (\omega-2)\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor - 1$$
$$(\mathbf{P}1) \ge \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1+2}{2}\sqrt{\frac{a}{x+1}} + \frac{1+1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + (\omega-2)\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor - 2$$
$$\ldots$$

$$(\mathbf{P1}) > \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{1+(\lfloor \omega \rfloor - 1)}{2}\sqrt{\frac{a}{x+1}} + \frac{1+(\lfloor \omega \rfloor - 2)}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + (\omega - (\lfloor \omega \rfloor - 1))\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor - (\lfloor \omega \rfloor - 1)$$

$$= \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor - 1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + (1 - \{\omega\})\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor - (\lfloor \omega \rfloor - 1)$$

$$= (1 - \{\omega\})\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor - 1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + (1 - \{\omega\})\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+2}} - \frac{1}{2}\sqrt{\frac{a}{x+1}} \right\rfloor + \{\omega\}\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor - 1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - (\lfloor \omega \rfloor - 1)$$

$$(\mathbf{P1}) \geq (1 - \{\omega\})\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor - 1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - \lfloor \omega \rfloor$$

$$= \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}(\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor - 1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - \{\omega\}\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+2}} \right\rfloor) - \lfloor \omega \rfloor$$

$$(\mathbf{P2}) \geq \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}(\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+1}} - \frac{1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - \lfloor \omega \rfloor$$

$$(\mathbf{D1}) \geq \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\lfloor \omega \rfloor + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\lfloor \omega \rfloor}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}(\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+1}} - \frac{1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - \omega$$

$$= \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\omega + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\omega}{2}\sqrt{\frac{a}{x+2}} + \frac{\{\omega\}}{2}\sqrt{\frac{a}{x+1}} - \frac{\{\omega\}}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}(\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+1}} - \frac{1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - \omega$$

$$(\mathbf{P1}) \geq \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\omega + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\omega}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \left\lfloor \frac{\{\omega\}}{2}\sqrt{\frac{a}{x+1}} - \frac{\{\omega\}}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + \{\omega\}(\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+1}} - \frac{1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - \omega$$

$$(\mathbf{L2}) \geq \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\omega + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\omega}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + 2\{\omega\}(\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+1}} - \frac{1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - \omega - 1$$

That is

$$\Delta_x - \omega\Delta_{x+1} \geq \left\lfloor \frac{1}{2}\sqrt{\frac{a}{x}} - \frac{\omega + 1}{2}\sqrt{\frac{a}{x+1}} + \frac{\omega}{2}\sqrt{\frac{a}{x+2}} \right\rfloor + 2\{\omega\}(\left\lfloor \frac{1}{2}\sqrt{\frac{a}{x+1}} - \frac{1}{2}\sqrt{\frac{a}{x+2}} \right\rfloor - \omega - 1 \tag{9}$$

Since $\frac{1}{2}\sqrt{\frac{a}{x+1}} - \frac{1}{2}\sqrt{\frac{a}{x+2}} > 0$, it knows by Theorem 1 that, there must be an $x_s(a)$ such that $\Delta_x - \omega\Delta_{x+1} \geq 0$ when $0 < \omega \leq 1$ and $x > x_s(a)$.

Consequently, it is sure that Theorem 3 holds if x_0 is taken by $x_0 = \max(x_s, x_b)$.

□

Theorem 4: (**T**4). Suppose k_1 and k_2 are positive integers with $k_1 < k_2$, a is real with $a > \max(k_1 + 1, k_2 + 1)$ and b is a constant real number. Let $s_1 = \left\lfloor \frac{1}{2}(1 + \sqrt{\frac{a}{k_1 + 1}}) \right\rfloor + 1$, $s_2 = \left\lfloor \frac{1}{2}(1 + \sqrt{\frac{a}{k_2 + 1}}) \right\rfloor + 1$, $b_1 = \left\lfloor \frac{1}{2}(1 + \sqrt{\frac{a}{k_1}}) \right\rfloor$ and $b_2 = \left\lfloor \frac{1}{2}(1 + \sqrt{\frac{a}{k_2}}) \right\rfloor$; then it holds

$$l_1 < r_1 \leq l_2 < r_2 \tag{10}$$

where $l_1 = b - 2(b_1 - 1)$, $l_2 = b - 2(b_2 - 1)$, $r_1 = b - 2(s_1 - 1)$ and $r_2 = b - 2(s_2 - 1)$.

Particularly, when $k_2 = k_1 + 1$ it holds

$$l_1 < r_1 = l_2 < r_2 \tag{11}$$

Proof. It obviously holds $s_2 < s_1$ and $b_2 < b_1$ under the given conditions.

Note that

$$b_2 - s_1 = \left\lfloor \frac{1}{2}(1 + \sqrt{\frac{a}{k_2}}) \right\rfloor - \left\lfloor \frac{1}{2}(1 + \sqrt{\frac{a}{k_1 + 1}}) \right\rfloor - 1$$

$$(\mathbf{P}2) \leq \left\lfloor \sqrt{\frac{a}{k_2}} - \sqrt{\frac{a}{k_1 + 1}}) \right\rfloor = \left\lfloor (\frac{\sqrt{k_1 + 1} - \sqrt{k_2}}{\sqrt{k_2(k_1 + 1)}})\sqrt{a} \right\rfloor$$

Since $k_2 \geq k_1 + 1$, it yields

$$b_2 - s_1 \begin{cases} = 0, k_2 = k_1 + 1 \\ < 0, k_2 > k_1 + 1 \end{cases} \tag{12}$$

Consequently it results in

$$s_2 < b_2 \leq s_1 < b_1$$

which indicates

$$b-2(s_2-1)>b-2(b_2-1)\geq b-2(s_1-1)>b-2(b_1-1)$$

Namely

$$r_2>l_2\geq r_1>l_1 \tag{13}$$

By the way, by (12) it surely holds when $k_2=k_1+1$.

$$r_2>l_2=r_1>l_1 \tag{14}$$

□

Theorem 5:(T5). For real number $a>0$, let $\Delta_0=\left\lfloor\frac{\sqrt{a}+1}{2}\right\rfloor$, $\Delta_1=\left\lfloor\frac{\sqrt{a}}{4+2\sqrt{2}}\right\rfloor$, $\Delta_2=\left\lfloor\frac{\sqrt{a}}{2(2\sqrt{3}+3\sqrt{2})}\right\rfloor$ and $\Delta_3=\left\lfloor\frac{\sqrt{a}}{2(4\sqrt{3}+3\sqrt{4})}\right\rfloor$; then $\Delta_0\geq 26$ yields $\Delta_1+\Delta_2>\frac{1}{4}\Delta_0$ and $\Delta_1+\Delta_2+\Delta_3 \square \frac{1}{2}\Delta_0$ holds for arbitrary $x>0$.

Proof. Let $X=\frac{\sqrt{a}+1}{2}$, namely, $\Delta_0=\left\lfloor\frac{\sqrt{a}+1}{2}\right\rfloor$; then

$$(\Delta_1+\Delta_2)=\left(\left\lfloor\frac{\sqrt{a}}{4+2\sqrt{2}}\right\rfloor+\left\lfloor\frac{\sqrt{a}}{4\sqrt{3}+6\sqrt{2}}\right\rfloor\right)$$

$$(\mathbf{P}1)\geq\left\lfloor\left(\frac{1}{4+2\sqrt{2}}+\frac{1}{4\sqrt{3}+6\sqrt{2}}\right)\sqrt{x}\right\rfloor-1=\left\lfloor\left(\frac{3-\sqrt{3}}{6}\right)\sqrt{a}\right\rfloor-1$$

$$(\mathbf{P}13)\geq\left\lfloor\frac{1}{6}\sqrt{a}\right\rfloor-1$$

Note that,

$$\left\lfloor\frac{1}{6}\sqrt{a}\right\rfloor-1-\frac{1}{4}\Delta_0=\left\lfloor\frac{1}{6}\sqrt{a}\right\rfloor-1-\frac{1}{4}\left\lfloor\frac{\sqrt{a}+1}{2}\right\rfloor$$

$$(\mathbf{L}2)>\left\lfloor\frac{1}{6}\sqrt{a}\right\rfloor-1-\left(\left\lfloor\frac{1}{4}\frac{\sqrt{a}+1}{2}\right\rfloor+1\right)$$

$$(\mathbf{P}2)\geq\left\lfloor\frac{4}{24}\sqrt{a}-\frac{3\sqrt{a}+3}{24}\right\rfloor-2=\left\lfloor\frac{\sqrt{a}-3}{24}\right\rfloor-2$$

That's to say, $\left\lfloor\frac{1}{6}\sqrt{a}\right\rfloor-1-\frac{1}{4}\Delta_0>0$ when $\frac{\sqrt{a}-3}{24}>2$, or $\frac{\sqrt{a}+1}{2}\geq 26$. Consequently, when $\Delta_0\geq 26$ it holds

$$(\Delta_1+\Delta_2)>\frac{1}{4}\Delta_0 \tag{15}$$

Now estimate $\Delta_1+\Delta_2+\Delta_3$. Direct calculation shows

$$\begin{aligned}2(\Delta_1+\Delta_2+\Delta_3)&=2\left(\left\lfloor\frac{\sqrt{a}}{4+2\sqrt{2}}\right\rfloor+\left\lfloor\frac{\sqrt{a}}{2(2\sqrt{3}+3\sqrt{2})}\right\rfloor+\left\lfloor\frac{\sqrt{a}}{2(4\sqrt{3}+3\sqrt{4})}\right\rfloor\right)\\(\mathbf{P}8)&\leq\left\lfloor\frac{\sqrt{a}}{2+\sqrt{2}}\right\rfloor+\left\lfloor\frac{\sqrt{a}}{2\sqrt{3}+3\sqrt{2}}\right\rfloor+\left\lfloor\frac{\sqrt{a}}{4\sqrt{3}+6}\right\rfloor\\(\mathbf{D}1)&\leq\frac{\sqrt{a}}{2+\sqrt{2}}+\frac{\sqrt{a}}{2\sqrt{3}+3\sqrt{2}}+\frac{\sqrt{ax}}{4\sqrt{3}+6}=\frac{\sqrt{a}}{2}<\frac{\sqrt{a}+1}{2}=X\end{aligned} \tag{16}$$

On the other hand, it yields

$$
\begin{aligned}
&2(\Delta_1+\Delta_2+\Delta_3) \\
&(\mathbf{P}1) \geq 2\left\lfloor \left(\frac{1}{4+2\sqrt{2}}+\frac{1}{2(2\sqrt{3}+3\sqrt{2})}+\frac{1}{2(4\sqrt{3}+3\sqrt{4})}\right)\sqrt{a}\right\rfloor = 2\left\lfloor \frac{\sqrt{a}}{4}\right\rfloor \\
&(\mathbf{L}2) \geq \left\lfloor \frac{\sqrt{a}}{2}\right\rfloor - 1 = \left\lfloor \frac{\sqrt{a}+1}{2}+\frac{1}{2}\right\rfloor - 1 \\
&(\mathbf{P}17) = \left\lfloor \sqrt{a}+1\right\rfloor - \left\lfloor \frac{\sqrt{a}+1}{2}\right\rfloor - 1 \\
&= \left\lfloor 2\times\frac{\sqrt{a}+1}{2}\right\rfloor - \left\lfloor \frac{\sqrt{a}+1}{2}\right\rfloor - 1 \\
&(\mathbf{P}8) \geq \lfloor X \rfloor - 1
\end{aligned} \tag{17}
$$

Since $\Delta_1+\Delta_2+\Delta_3$ is an integer, it yields

$$\lfloor X \rfloor - 1 \leq 2(\Delta_1+\Delta_2+\Delta_3) \leq \lfloor X \rfloor \tag{18}$$

which is

$$\Delta_1+\Delta_2+\Delta_3 \;\square\; \frac{\Delta_0}{2}$$

□

IV. Conclusions

Integer functions are a kind of special functions. Due to their discrete characteristics, study of the functions requires special skills. Since there has not been a general purpose to study the kind of functions, every case of knowing them is worthy of investigation. The function $d(x)$ studied in this article surely sets an example. Hope more excellent results come in the future.

Acknowledgment

The research work is supported by the State Key Laboratory of Mathematical Engineering and Advanced Computing under Open Project Program No.2017A01, Department of Guangdong Science and Technology under project 2015A010104011, Foshan Bureau of Science and Technology under projects 2016AG100311, Project gg040981 from Foshan University. The authors sincerely present thanks to them all.

References

[1] Xingbo WANG, *Brief Summary of Frequently-Used Properties of the Floor Function*, IOSR Journal of Mathematics, vol.13, No.5, pp46-48, 2017

[2] Xingbo WANG, *Some New Inequalities With Proofs and Comments on Applications*, Journal of Mathematics Research, vol.11, No.3, pp15-19,2018