On Some Partial Ordering for Con. Secondary-K Normal Bimatrices

Dr.N.Elumalai[#1], Mr. R.Manikandan[#2]
[#1]Associate professor,[#2]Assistant Professor,
[##], Department of Mathematics, A.V.C. College (Autonomous), Mannampandal, Tamilnadu India.

Abstract

The inequalities of con s-k normal bimatrices are introduced, Also it is shown that all these ordering are partial ordering in bimatrices. The result can be extended to study $A^n \leq^* B^n$

Keywords: *Star partial order, bimatrix ,normal bimatrix, sk-normal bimatrix ,Inequality.*
Ams classification: 15B05, 15A09,15A15,15A57

I Introduction

Let C_{nxn} be the set of mxn complex bimatrices we order it by the standard partial ordering $\leq^*$, So $A_B \leq B_B^*$ means that $A_1^*A_1 = A_1^*B_1$; $A_2^*A_2 = A_2^*B_2$. The symbol A_B^* $R(A_B)$ and $r(A_B)$ denote the conjugate transpose rank spaces and rank subtractivity of $A_B \in C_{nxn}$

II Some of definitions and results

Bimatrix 2.1 [3]

A bimatrix A_B is defined as the union of two square array of numbers A_1 and A_2 arranged into rows and columns. It is written as follows $A_B = A_1 \cup A_2$
where $A_1 \neq A_2$ with

$$A_1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & \cdots\cdots & a_{1n}^1 \\ a_{21.}^1 & a_{22.}^1 & \cdots\cdots & a_{2n.}^1 \\ \vdots & \vdots & & \vdots \\ a_{m1}^1 & a_{m2}^1 & \cdots & a_{mn}^1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots\cdots & a_{1n}^2 \\ a_{21.}^2 & a_{22.}^2 & \cdots\cdots & a_{2n.}^2 \\ \vdots & \vdots & & \vdots \\ a_{m1}^2 & a_{m2}^2 & \cdots & a_{mn}^2 \end{bmatrix}$$

`∪` the notational convenience (symbol) only.

Definition 2.2[2]

A bimatrix $A_B = A_1 \cup A_2$ is said to be normal bimatrix , if $A_B A_B^* = A_B^* A_B$

Definition :

1. The standard ordering for con s-k normal bimatrices is defined by $A_B \leq^* A_B$ ⇔ $A_B^* A_B = A_B^* B_B$

   That is $A_1^*A_1 = A_1^*B_1$,
   $A_2^*A_2 = A_2^*B_2$ and $A_B A_B^* = B_B A_B^*$
   That is $A_1A_1^* = B_1A_1^*$; $A_2A_2^* = B_2A_2^*$
2. The standard ordering for con s-k normal bimatrices is defined by

   $A_B^\# = \overline{A_B(K_B V_B A_B^* V_B K_B)}$

   $A_B \leq^* B_B$
   ⇔ $A_B^\# A_B = A_B^\# B_B$

That is $A_1^{\#}A_1= A_1^{\#}B_1$,
$A_2^{\#}A_2= A_2^{\#}B_2$ and $A_BA_B^{\#}=B_BA_B^{\#}$
That is $A_1A_1^{\#}=B_1A_1^{\#};A_2A_2^{\#}=B_2A_2^{\#}$

Example :

$A_1=\begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix}$ and $A_2=\begin{pmatrix} 2i & 2i \\ 0 & 0 \end{pmatrix}$

$B_1=\begin{pmatrix} i & i \\ i & -1 \end{pmatrix}$ and $B_2=\begin{pmatrix} 2i & 2i \\ 2i & -2i \end{pmatrix}$

$A_1=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix}$

$A_2=\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2i & 2i \\ 0 & 0 \end{pmatrix}$

$A_1^{*}A_1= A_1^{*}B_1$; $A_2^{*}A_2= A_2^{*}B_2$ $A_B\leq *B_B$

III On Some Partial Ordering For Secondary k-Normal Bimatrices

Definition 1.3

The left-star ordering for bimatrices is defined by

$A_B{}^{*}\leq B_B \Leftrightarrow A_B{}^{*}A_B= A_B{}^{*}B_B$

That is $A_1^{*}A_1= A_1^{*}B_1$, $A_2^{*}A_2= A_2^{*}B_2$

And $R(A_B)\subseteq R(B_B)$ $R(A_B)\subseteq R(B_B)$

$R(A_1)\subseteq R(B_1)$; $R(A_2)\subseteq R(B_2)$

Definition 1.4

The Right-star ordering for bimatrices is defined by

$A_B \leq {}^{*}B_B \Leftrightarrow A_BA_B{}^{*}= B_BA_B{}^{*}$

That is $A_1A_1^{*}= B_1A_1^{*}$, $A_2A_2^{*}= B_2A_2^{*}$

And $R(A_B{}^{*})\subseteq R(B_B{}^{*})$

$R(A_i)\subseteq R(B_i)$; $R(A_2^{*})\subseteq R(B_2^{*})$

Definition 1.5

The plus-order for bimatrices is defined as $A_B<B_B$ whenever $A_B{}^{\#}A_B=A_B{}^{\#}B_B$ and $A_BA_B{}^{\#}=B_BA_B{}^{\#}$ for some reflexive generalized inverse $A_B{}^{\#}$ of A_B (satisfying both $A_BA_B{}^{\#}A_B=A_B$ and $A_B{}^{\#}A_BA_B{}^{\#}=A_B{}^{\#}$)

Definition 1.6

The minus (rank subtractivity)ordering is defined for bimatrices as,

$A_B\leq {}^{-}B_B \Leftrightarrow r(B_B-A_B)=r(B_B)-r(A_B)$

That is $r(B_1-A_1)=r(B_1)-r(A_1)$and

$r(B_2-A_2)=r(B_2)-r(A_2)$

or as $A_B\leq {}^{-}B_B \Leftrightarrow A_BB_B{}^{\#}B_B=A_B$, $B_BB_B{}^{\#}A_B=A_B$ and $A_BB_B{}^{\#}A_B=A_B$

Lemma 1.10

Let $A_B,B_B\in C_{mxn}$ and let $a=r(a)<r(b)$ then $A_B\leq {}^{*}B_B$ if and only if there exist $U\in C_{mxn}$ $_V \in C_{nxn}$ satisfying $U^{*}U=l_b=V^{*}V$ for which $A_B=U\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}V^{*}$ and $B_B=U\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}V$ where D_1 and D_2 are positive definite diagonal matrices of degree a and b-a respectively. For $A_B,B_B\in CH_n$,The matrix U in (1-8) may be replaced by v but then $D_1$and D_2 represent any non singular real diagonal matrices

Lemma 1.11

Let $A_B,B_B\in C_{mxn}$ and let $a=r(A)<r(B)=b$

Then $A_B\leq {}^{-}B_B$ if only if there exist $U\in C_{mxb}$ $V\in C_{nxb}$ satisfying $U*U=l_b=V*V$ for which $A_B=U\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}V$ and $B_B=U\begin{pmatrix} D_1+RD_2s & RD_2 \\ D_2s & D_2 \end{pmatrix}V^{*}$ where $D_1$and D_2 are positive definite diagonal matrices of degree a and b-a, while $R\in C_{axb-a}$ and $S\in C_{b-axa}$ are arbitrary.For $A_B,B_B\in CH_n$ the matrices U and S in(1.9) may be replaced by V and R^{*}respectively but then D_1 and D_2 represent any non singular real diagonal matrices

Lemma 1.12

Let $A_B, B_B \in C_n H$ be star ordered as $A_B \leq^* B_B$ Then $A_B \leq_L B_B$ if and only if $\gamma(A_B) = \gamma(B_B) where \gamma(.)$ denotes the number of negative eigen values of a given bimatrix

Proof:

Case(i):

Let $r(A_B)=r(B_B)$ that is $r(A_1)=r(B_1)$ and $r(A_2)=r(B_2)$

The result is trivial

Case(ii):

Let $r(A_B)<r(B_B)$ that is $r(A_1)<r(B_1)$ and $r(A_2)<r(B_2)$

Lemma(1.1) ensures that if $A_B \leq^* B_B$ then

$$B_B - A_B = (B_1 \cup B_2) - (A_1 \cup A_2)$$

$$= (B_1 - A_1) \cup (B_2 - A_2)$$

$$= \left[U_1 \begin{pmatrix} D_{11} & 0 \\ 0 & D_{12} \end{pmatrix} V_1^* - U_1 \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix} V_1^* \right] \cup$$

$$\left[U_2 \begin{pmatrix} D_{21} & 0 \\ 0 & D_{22} \end{pmatrix} V_2^* - U_2 \begin{pmatrix} D_{21} & 0 \\ 0 & 0 \end{pmatrix} V_2^* \right]$$

$$= (U_1 \cup U_2) \begin{pmatrix} D_{11} \cup D_{12} & 0 \\ 0 & D_{12} \cup D_{22} \end{pmatrix} (V_1^* \cup V_2^*) - (U_1 \cup U_2) \begin{pmatrix} P_{11} \cup D_{21} & 0 \\ 0 & 0 \end{pmatrix} (V_1^* \cup V_2^*)$$

$$= U_B \begin{pmatrix} D_{B1} & 0 \\ 0 & D_{B2} \end{pmatrix} V_B^* - U_B \begin{pmatrix} D_{B1} & 0 \\ 0 & 0 \end{pmatrix} V_B^*$$

$$B_B - A_B = U_B \begin{pmatrix} 0 & 0 \\ 0 & D_{B2} \end{pmatrix} V_B^*$$

Hence it is seen that the order $A_B \leq^L B_B$ is equivalent to the non-negative definiteness of D_{B2} that is $\gamma(D_2) = 0$ .consequently the result follows by noting that

$$\gamma(A_B) = \gamma(D_{B1}) and \gamma(B_B) = \gamma(D_{B1}) + \gamma(D_{B2})$$

Theorem:1

Let A and B be con s-k Normal bimatrices $A_B \leq^* B_B \Rightarrow A_B^2 \leq^* B_B^2$ and $A_B B_B = B_B A_B$

Proof:

Since $A_B = A_1 \cup A_2 \in C_n \Leftrightarrow A_B A_B^{\#} = A_B^{\#} A_B$

That is $A_1 A_1^{\#} = A_1^{\#} A_1$ and $A_2 A_2^{\#} = A_2^{\#} A_2$

Now,

$$A_B^2 A_B^{\#} = A_1^2 A_1^{\#} \cup A_1^2 A_2^{\#}$$

$$= A_1 (A_1 A_1^t)^* \cup A_2 (A_2 A_2^t)^*$$

$$= A_1 \cup A_2$$

$$A_B^2 A_B^{\#} = A_B$$

Also,

$$A_B^{\#} A_B^2 = A_1^{\#} A_1^2 \cup A_2^{\#} A_1^2$$

$$= (A_1^{\#} A_1) A_1 \cup (A_2^{\#} A_2) A_2$$

$$= A_1 \cup A_2$$

$$A_B^{\#} A_B^2 = A_B$$

$$\Rightarrow A_B^2 A_B^* = A_B^* A_B^2$$

$$And (A_B^2)^* = (A_1^2 \cup A_2^2)^*$$

$$= (A_1 A_1)^* \cup (A_2 A_2)^*$$

$$= A_1^{\#} A_1^{\#} \cup A_2^{\#} A_2^{\#}$$
$$= (A_1^{\#} \cup A_2^{\#})^2$$
$$= (A_B^{\#})^2$$

Consequently, in view of(1.3)

$A_B B_B = A_1 B_1 \cup A_2 B_2$

$$= A_1^2 A_1^{\#} B_1 \cup A_2^2 A_2^{\#} B_2$$
$$= A_1^2 A_1^{\#} A_1 \cup A_2^2 A_2^{\#} A_2$$

$A_B B_B = A^2_B$

$B_B A_B = B_1 A_1 \cup B_2 A_2$

$$= B_1 A_1^{\#} A_1^2 \cup B_2 A_2^{\#} A_2^2$$

$B_B A_B = A^2_B$

$A_B B_B = B_B A_B = A^2_B$

More over,

$$(A_B^2)^{\#} B_B^2 = (A_1^2 \cup A_2^2)^{\#} (B_1^2 \cup B_2^2)$$

$$= \left(A_1^2\right)^{\#} B_1^2 \cup \left(A_2^2\right)^{\#} B_2^2$$
$$= \left(A_1^2\right)^{\#} A_1^2 \cup \left(A_2^2\right)^{\#} A_2^2$$
$$= (A_1^2 \cup A_2^2)^* (A_1^2 \cup A_2^2)$$
$$(A_B^2)^{\#} B_B^2 = (A_B^2)^{\#} A_B^2$$

And $B_B^2(A_B^2)^* = B_1^2(A_1^2)^* \cup B_2^2(A_2^2)^*$

$= B_1A_1(A_1^*)^2 \cup B_2A_2(A_2^*)^2$

$= A_1^2(A_1^2)^* \cup A_2^2(A_2^2)^*$

$B_B^2(A_B^2)^* = A_B^2(A^2_B)^*$

$$\Rightarrow A_B^2 \le B_B^2$$

Theorem 2.4

Let $A_B, B_B \in C_{mxn}$, Then $A_B \le^* B_B$ and $A_B B_B = B_B A_B \Rightarrow A_B^2 \le^* B_B^2$

Proof

On account of (1.2),it follows that if $A_B \le^* B_B$ and $A_B B_B = B_B A_B$ then,

$$(A_B^2)^* B_B^2 = (A_1^2 \cup A_2^2)^* (B_1^2 \cup B_2^2)$$
$$= [(A_1^2)^* \cup (A_2^2)^*] (B_1^2 \cup B_2^2)$$

$$= \left(A_1^2\right)^* B_1^2 \cup \left(A_2^2\right)^* B_2^2$$
$$= A_1^* (A_1^* A_1) B_1 \cup A_2^* (A_2^* A_2) B_2$$
$$= (A_1^*)^2 A_1 B_1 \cup (A_2^*)^2 A_2 B_2$$
$$= A_1^* (A_1^* B_1) A_1 \cup A_2^* (A_2^* B_2) A_2$$
$$= \left(A_1^2\right)^* A_1^2 \cup \left(A_2^2\right)^* A_2^2$$
$$(A_B^2)^* B_B^2 = (A_B^2)^* A_B^2$$

and

$$B_B^2 (B_B^2)^* = (B_1^2 \cup B_2^2)(A_1^2 \cup A_2^2)^*$$

$$= B_1^{\,2}\left(A_1^{\,2}\right)^* \cup B_2^{\,2}\left(A_2^{\,2}\right)^*$$
$$= B_1(B_1A_1^{\,*})A_1^{\,*} \cup B_2(B_2A_2^{\,*})A_2^{\,*}$$
$$= B_1(A_1A_1^{\,*})A_1^{\,*} \cup B_2(A_2A_2^{\,*})A_2^{\,*}$$
$$= A_1(B_1A_1^{\,*})A_1^{\,*} \cup A_2(B_2A_2^{\,*})A_2^{\,*}$$
$$= A_1(A_1A_1^{\,*})A_1^{\,*} \cup A_2(A_2A_2^{\,*})A_2^{\,*}$$
$$= A_1^{\,2}(A_1A_1)^* \cup A_2^{\,2}(A_2A_2)^*$$
$$= A_1^{\,2}(A_1^{\,2})^* \cup A_2^{\,2}(A_2^{\,2})^*$$
$$B_B^{\,2}(A_B^{\,2})^* = A_B^{\,2}(A_B^{\,2})^*$$

IV REFERENCES

[1].Ann Lee, Secondary symmetric , Secondary skew symmetric, Secondary orthogonal matrices period math . Hungary 7(1976),63-76

[2]. Dr.Elumalai, N. and Manikandan,R.”On S-k normal and con S-k normal bimatrices”, International journal of innovative Research and Technology,Volume3,Issue3,2016, pages(s):102-107

[3] Vasanthakandasamy, W.B,Florentinsamaranache and ilathendral.k,” Application of

Bimatrices to some fuzzy and Neutrosophic Models-2005

[4]Weddurburn,J.H.M.,Lectures on matrices,colloq.publ.Amer.Math.Soc.No.17,1934.

[5]. Horn.R.A. and Johnson.C.R..’’Matrix Analysis”. Cambridge University Press, NewYork,1985.

[6] Horn.R.A.and Johnson.C.R..”Topics in Matrix Analysis”. Cambridge University Press,Newyork,1991.

[7] Horn.R.A. and Merino. D.I..” Contragredient equivalence: a canonical form and some applications”. Linear Algebra Appl..214(1995).43-92.

[8] Ramesh. G. and Maduranthaki. P..”On Unitary Bimatrices” .*International Journal of Current Research. Vol. 6.* Issue 09 ,September 2014.pp.8395-8407.