Dominance of Rooks and Bishops in Fractal Chessboard

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Abstract — *This paper proves that a maximum of $N-2$ non- attacking Rooks and N non- attacking Bishops can be placed in a $N \times N$ Fractal Chessboard (FC) which is defined as a board that grows progressively in a consistent manner using a 2×2 chessboard (B) by adding the copy of B to its sides and corners. The Rook Domination Number of FC is $\frac{N}{2}+1$ and $\frac{N}{2}-1$ for even and odd iteration respectively. Also it is proved that $N-1$ Rooks can be separated by one Pawn for any N of a $N \times N$ Fractal Chessboard. A maximum of $2(k+1)$ non-attacking Rooks and non-attacking Bishops can be placed in a $N \times N$ Fractal Chessboard if there is no limitation on the number of pawns needs to be placed.*

Keywords — *Rook Domination Number, Rook Independence Separation Number, Fractal Chessboard, Non-attacking Rook, Non- attacking Bishop.*

I. INTRODUCTION

The study of domination number in chessboards commenced in early 1850 [4]. Cockayne and others have immensely contributed on domination numbers and its related parameters ([1], [6]). He has given the upper and lower bounds for the domination number of the Queen's graph [3]. Various other notable results have been obtained on the domination parameters for Rook's graph, Bishop's graph and Queen's graph ([2], [7]). The earlier research works on domination numbers and its related parameters primarily derived the results for a classical chessboard (8×8 Chessboard) ([1], [6]).

In this paper, the dominance of the chess pieces Rook and Bishop on Fractal Chessboard (FC) is being analyzed. It is a board that grows progressively in a consistent manner using a 2×2 chessboard (*B*) by adding the copy of *B* to its sides and corners. The Maximum Rooks Problem and The Maximum Bishops Problem are being explored.

A. *Fractal Chessboard*

Consider a 2×2 board (*B)* from which the growth of the board FC begins by adding a copy of this *B* to each of its side. If two boards intersect at a point then they share a common board at their point of intersection. The board thus obtained by this method is called as Fractal Chessboard. The Figure 1 portrays the growth of Fractal Chessboard. Here the board with darker edges is the base board *B*.

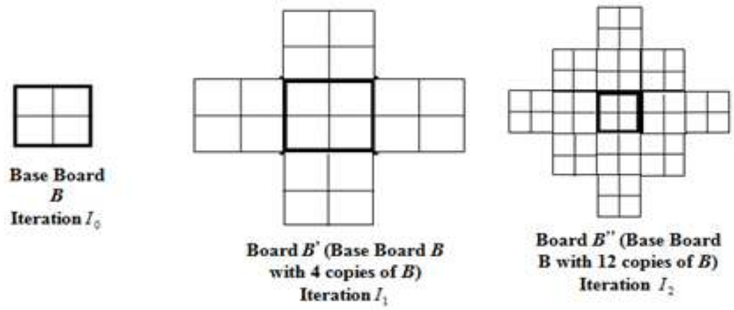

Fig. 1 A Fractal Chessboard at its first three iterations

B. Remark

In any $N \times N$ Fractal Chessboard, the value of $N = 2(2I + 1)$ where I is the iteration number.

II. DOMINANCE of ROOKS on FRACTAL CHESSBOARD

In the game of chess, a Rook moves horizontally or vertically through any number of unoccupied square and they are very powerful towards the end of a game. It is a very powerful piece to deliver checkmate. Here, the maximum and minimum numbers of non-attacking Rooks that can be placed for the Fractal Chessboard are obtained.

A. Rook Domination Number

Let F be a $N \times N$ Fractal Chessboard. The Rook Domination Number is defined as the minimum number of non- attacking Rooks required to cover the entire $N \times N$ Fractal Chessboard F. It is denoted by $\gamma(R_F)$.

1) Theorem

The maximum number of non -attacking Rooks that can be placed in a $N \times N$ Fractal Chessboard F is $N - 2$, where $N \geq 6$.

Proof

Considering each 2×2 board (B and its copies) of $N \times N$ Fractal Chessboard as a cell, we have $\frac{N}{2}$ rows in FC. By placing the Rooks, two in each 2×2 board (B and its copies) diagonally in the left corner board or right corner (moving upwards from the $\frac{N}{2}^{th}$ row), in the right corner board or left corner (moving downwards from the $\frac{N}{2}^{th}$ row) and at the left or right corner board B of the $\frac{N}{2}^{th}$ row, a maximum of two non-attacking Rooks can be placed in each row except for the last row. Thus the centre row $\frac{N}{2}$ comprises two Rooks and the remaining $\left(\frac{N}{2} - 2\right)$ rows (leaving the centre and last row) comprise two Rooks.

Hence the maximum number of non-attacking Rooks that can be placed in FC is given by,

$$2 + 2\left(\frac{N}{2} - 2\right) = N - 2$$

2) Theorem

For any $N \geq 6$, the Rook Domination Number of a $N \times N$ Fractal Chessboard $\gamma(R_F) = \begin{cases} \frac{N}{2} + 1, & for \quad even \quad iteration \quad I \\ \frac{N}{2} - 1, & for \quad odd \quad iteration \quad I \end{cases}$ where I is the iteration number.

Proof

In the Fractal Chessboard, the shortest diagonal that covers all the cells of the board is the diagonal along the base board *B*. At any iteration I, the number of 2×2 boards along the diagonal is I, for even iteration I and $I + 1$, for odd iteration I. Thus placing the Rooks along this diagonal will cover the entire FC and thus the Rook Domination Number of the Fractal Chessboard is given by,

$$2(I+1) = \frac{N}{2} + 1 \text{ for even iteration } I$$

$$2(I) = \frac{N}{2} - 1 \text{ for odd iteration } I \qquad \text{(By Remark B)}$$

$$\text{Hence } \gamma(R_F) = \begin{cases} \frac{N}{2}, & for \quad even \quad iteration \quad I \\ \frac{N}{2} - 1, & for \quad I \; odd \quad iteration \quad I \end{cases}$$

B. Rook Independence Separation Number for the Fractal Chessboard

The rules of the classic chess problem can have not more than N non- attacking Rooks on a Classical $N \times N$ Chessboard. By changing the rules in order to allow more Rooks on the board, we place a Pawn in a square between two Rooks that are in the same row, column; those Rooks no longer attack each other. This is the "Maximum Queens Problem" done by K.Zhao in 1998 for classical $N \times N$ chessboard on Queens [8]. The same work has been done on rectangular boards by Chatam in the year 2016 [9]. Here the work of Independence Separation Number [5] is done for Rooks on Fractal Chessboard.

$S_{R_F}(\gamma, m, N)$ is the minimum number of Pawns which can be placed on a $N \times N$ Fractal Chessboard F to result in a separated board on which a maximum of 'm' independent Rooks can be placed.

1) Theorem

$S_{R_F}(\gamma, N-1, N) = 1$ is true for a $N \times N$ Fractal Chessboard.

Proof

By the theorem 1), the maximum number of non- attacking Rooks that can be placed on a $N \times N$ Fractal Chessboard F is $N - 2$. Here $N - 1$ non-attacking Rooks need to be placed.

(i.e.) $N - 1 - (N - 2)$

= 1 Rook.

Placing exactly one Pawn in any one of the empty 2×2 boards, the remaining one Rook can be placed and thus $N - 1$ Rooks can be separated by *one* Pawn.

$$\text{Hence } S_{R_F}(\gamma, N-1, N) = 1$$

C. Maximum Rooks Problem

If there is no restriction in the number of Pawns, then at most $2(k+1)$ non attacking Rooks can be placed with the help of $2(k+2)-N$ Pawns where k is the number of copies on a $N \times N$ Fractal Chessboard.

1) Theorem

For any N , it is possible to place $2(k+1)$ Rooks and $2(k+2)-N$ Pawns on a $N \times N$ Fractal Chessboard with k copies so that no two Rooks attack each other.

Proof

A maximum of two non-attacking Rooks can be placed in each $N \times N$ board of FC with no Pawns. Considering the whole FC, a maximum of $2(k+1)$ non-attacking Rooks can be placed where k is the number of copies of the base board *B*, by subtracting the maximum number of non-attacking Rooks that can be placed without the help of Pawns from $2(k+1)$

$$2(k+1) - (N-2)$$
$$= 2(k+2) - N \text{ Pawns.}$$

Thus $2(k+2)-N$ Pawns are required to place $2(k+1)$ non-attacking Rooks in a $N \times N$ Fractal Chessboard.

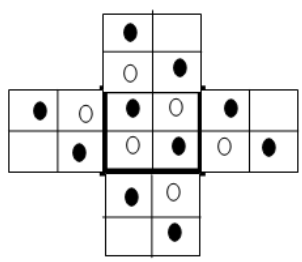

Fig. 2 The Illustration of the result of theorem 1) under section C using the Fractal Chessboard of Size $N = 6$

The above board is a Fractal Chessboard with $N = 6$ & $k = 4$. Here the filled circle represents the Rook and the unfilled circle represents the Pawn.

A maximum of 2(4+1) = 10 non-attacking Rooks are placed using 2(4+2) – 6= 12-6 = 6 Pawns.

III. DOMINANCE of BISHOPS on FRACTAL CHESSBOARD

In the game of Chess, the Bishop has no restrictions in distance for each move, but it is limited to diagonal movement. Bishops and Knights are called minor pieces and Rooks and Queens are called major pieces. Here, the maximum number of non attacking Bishops that can be placed on the Fractal Chessboard is obtained.

1) *Theorem*

The maximum number of non attacking Bishops that can be placed in a $N \times N$ Fractal Chessboard F is $N, N \geq 6$

Proof

Placing two Bishops in the alternate first row of the centre boards of our Fractal Chessboard, the maximum number of non-attacking Bishops that cover the entire Fractal Chessboard F is obtained which is given by,

$$2\left(\frac{N}{2}\right) = N \ .$$

A. Maximum Bishop Problem

If there is no restriction in the number of Pawns, then at most $2(k+1)$ non attacking Bishops can be placed with the help of $2(k-1)$ Pawns where k is the number of copies on a $N \times N$ Fractal Chessboard.

1) Theorem

For any N, it is possible to place $2(k+1)$ Bishops and $2(k-1)$ Pawns on a Fractal Chessboard with k copies and N rows or columns so that no two Bishops attack each other.

Proof

A maximum of two non attacking Bishops can be placed in each 2×2 board of our FC with or without Pawns. Considering the whole FC, placing two Pawns in each of the empty cells of 2×2 boards, a maximum of $2(k+1)$ Pawns are placed. Now by removing the extra pawns from $2(k+1)$, one from each left and right end boards of the centre row and two from the bottom end board of the centre column ,

$$2(k+1) - (1+1+2)$$

$$= 2(k-1) \text{ Pawns.}$$

Thus $2(k-1)$ Pawns are required to place $2(k+1)$ non-attacking Bishops in a $N \times N$ Fractal Chessboard.

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