Duality of "Some Famous Integral Transforms" from the polynomial Integral Transform.

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Abstract — *In this article, we derive multiplicity relation between polynomial integral transform, sumudu, Fourier and mellin integral transforms. We will obtain useful relations of polynomial integral transform and other integral transforms. Fourier integral transform and mellin integral transform can be very difficult to apply in many situations due to its complex nature. The relation between these transforms is more convenient to derive. We will show that polynomial integral transform and Laplace integral transform are equivalent to each other. We can obtain some interesting results in the forms of theorems and these results have been proven. We are taking advantage from the duality relation of mellin and Laplace integral transform for obtaining the results. It will be helpful for solving many problems of science and technology.*

Keywords — *Natural integral transform, Sumudu integral Transform, Laplace integral Transform, Fourier integral Transform, Mellin integral Transform..*

I. INTRODUCTION

INTEGRAL TRANSFORMS ARE USEFUL TOOLS FOR SOLVING DIFFERENTIAL AND INTEGRAL EQUATIONS.MOST OF THE COMPLICATED PROBLEMS IN ENGINEERING AND OTHER SCIENTIFIC DISCIPLINE LIKE, PHYSICS, CHEMISTRY AND DYNAMICS INVOLVE RATES OF CHANGE. PROBLEMS OF THESE SOLUTION ARE OFTEN EITHER DIFFICULT OR NOT FEASIBLE AT ALL FOR INITIAL CONDITION OF THE DEPENDENT VARIABLE. TECHNIQUES OF SPECIAL SUBSTITUTION HAVE BEEN ADOPTED IN FINDING SOLUTION TO DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS. SINCE A NUMBER OF INTEGRAL TRANSFORMS HAVE BEEN PROPOSED .TO SOLVE DIFFERENTIAL EQUATION AND INTEGRAL EQUATION THERE ARE VARIOUS INTEGRAL TRANSFORMS ARE FREQUENTLY USED. AMONG THESE TRANSFORM LAPLACE TRANSFORM IS MOSTLY USED. THE SUMUDU INTEGRAL TRANSFORM WAS FIRST INTRODUCED BY **WATUGALA** IN **1993**.

II . *Preliminaries related to polynomial integral transform*

2. In this section we recall basic properties of polynomial transform as mention [1]

Definition 2.1: (A polynomial integral transform): Let $f(x)$ be a function defined for $x \geq 1$ then the integral

$$B(f(x)) = B(s) = \int_1^\infty f(\ln x)\, x^{-s-1} dx, \qquad \ldots (1)$$

is the polynomial integral transform of $f(x)\ for\ x\epsilon[1,\infty)$, provided the integral converges.

Definition 2.2: (Linear property): if $\propto_1\ and\ \propto_2$ are real constants then

$$\mathrm{B}(\propto_1 f(x) + \propto_2 g(x)) = \propto_1 \mathrm{B}(f(x)) + \propto_2 B(g(x)) \qquad \ldots (2)$$

Definition 2.3: if $f(x)$ is a piecewise continuous function on $[0,\infty)$, but not of exponential order, then a polynomial integral transform.

$$B(f(x)) \to 0\ as\ s \to \infty$$

2.3.1: Shifting and changing scale property:

If $B(f(x)) = B(s), then\ B((e^{ax}f(x)) = B(s-a), for\ s > 1$

Also for a unit step function

$$H_l(x) = f(x) = \begin{cases} 0, & 0 \leq x < l \\ 1, & x \geq l \end{cases}$$

Then $B(H_l(x))f(x-l) = f(s-l)$ … (3)

Sufficient condition for the existence of polynomial integral transform.

2.4 ***: Existence of polynomial integral transform:*** If $f(x)$ be a piece wise continuous function on $[1,\infty)$ and of exponential order then the polynomial integral transform exist.

i.e. $\int_1^\infty f(x)x^{-(s+1)}\,dx = M\sum_{n=0}^{\infty}\frac{M\alpha^n}{n!(s-n)}, s > n$ …(4)

3.Connection between Polynomial Integral Transform and Other Integral Transforms

3.1Connection between Polynomial Integral Transform and Natural Integral Transform:

In this section we show that polynomial transform is theoretical dual of natural transform and the dual relation is given by the following relation:

Theorem 3.1: Let $f(t) \in B$ and let $N\{f(t)\} = R(s,u)$ and $B(f(x)) = B(s)$ Then

$$R(s,u) = \frac{1}{u}\,B(\frac{s}{u}) \quad \dots (5)$$

$$B(f(x)) = uR(us,u) \quad \dots (6)$$

Proof: we know that

Natural Transform $N\{f(t)\} = R(s,u) = \int_0^\infty f(ut)e^{-st}\,dt$

Polynomial integral transform $B(f(x)) = \int_1^\infty f(\ln x)\,x^{-s-1}dx$

Let $N\{f(t)\} = \int_0^\infty f(ut)e^{-st}dt$

Substituting $ut = \ln x \Rightarrow dt = \frac{1}{ux}dx$

Therefore

$$N\{f(t)\} = \int_1^\infty f(\ln x)\,e^{-\frac{s(\ln x)}{u}}\frac{1}{ux}dx$$

$$= \frac{1}{u}\int_1^\infty f(\ln x)\frac{1}{x}e^{-\frac{s(\ln x)}{u}}dx$$

$$= \frac{1}{u}\int_1^\infty f(\ln x)\,x^{-\left(\frac{s}{u}+1\right)}dx$$

Hence, on comparing with Eq. (2), we thus obtain

$$N\{f(t)\} = \frac{1}{u}B\left(\frac{s}{u}\right)$$

Converse

Let $B(f(x)) = \int_1^\infty f(\ln x)\,x^{-s-1}dx$

Substituting $\ln x = ut \Rightarrow dx = e^{ut}.u.dt$

Therefore

$$B(f(x)) = \int_0^\infty f(ut)e^{ut(-s-t)}e^{ut}.u.dt$$

$$= u\int_0^\infty f(ut)e^{-sut}\,dt$$

Hence, on comparing with definition of Natural Integral Transform

$$B(f(x)) = uR(us,u)$$

These prove dual Relation between Polynomial Integral Transform and Natural Integral Transform.

3.2Connection between Polynomial Integral Transform and Sumudu Integral Transform:

The dual relation between Polynomial Integral Transform and Sumudu Integral Transform can we obtained by the following theorem.

Theorem 3.2: Let $f(t)\epsilon B$ *and let* $S\{f(t)\} = G(s)$ be the sumudu transform and $B(f(x)) = B(s)$ be polynomial transform then,

$$B(f(x)) = uG(su), \quad \dots (7)$$

And $S\{f(t)\} = sB(us)$ … (8)

Proof: we know that

$$S\{f(t)\} = \int_0^1 e^{-t}\,f(ut)dt = G(s)$$

$$B(f(t)) = \int_1^\infty f(\ln x)x^{-s-1}\,dx = B(s)$$

Let $B(f(t)) = \int_1^\infty f(\ln x)x^{-s-1}\,dx$

Substituting $\ln x = ut \Rightarrow \frac{1}{x}dx = udt$

Therefore

$$B(f(t)) = \int_0^\infty f(ut)(e^{ut})^{-s}\,udt$$
$$= u\int_0^\infty e^{-sut} f(ut)\,dt$$

Hence, on comparing definition of Sumudu Integral Transform

$$B(f(x)) = uG(su)$$

Converse

Let $S\{f(t)\} = \int_0^\infty e^{-t} f(ut)dt$

Substituting $t = s\ln x \Rightarrow dt = \frac{s}{x}dx$

Therefore

$$S\{f(t)\} = \int_1^\infty e^{-s\ln x} f(us\ln x)\frac{s}{x}dx$$
$$= s\int_1^\infty x^{-s-1} f(us\ln x)dx$$

Hence, on comparing with Eq. (1) we thus obtain.

$$S\{f(t)\} = sB(us)$$

These prove dual relation.

3.3Connection between Polynomial Integral Transform and Fourier Integral Transform-

In this section we show that Polynomial Integral Transform is theoretical dual of Fourier Integral transform and the dual relation is given by the following relation:

Theorem 3.3: Let $f(t)\epsilon B$ with polynomial transform $B(s)$ and $F(s)$ is the Fourier transform of $f(t)$ respectively then their dual relation can be obtain as:

$$B(f(s)) = iF(is) \quad \ldots (9)$$

And $$\mathrm{F}\{f(t)\} = \frac{1}{i}\mathrm{B}\left(\frac{1}{i}s\right) \quad \ldots (10)$$

***Proof*:** we know that

$$\mathrm{B}(f(x)) = \int_1^\infty f(lnx)x^{-s-1}\,\mathrm{dx} = \mathrm{B}(s)$$
$$\mathrm{F}\{f(t)\} = \int_{-\infty}^\infty e^{-ist} f(t)dt = F(s)$$

Let $\mathrm{B}(f(x)) = \int_1^\infty f(lnx)x^{-s-1}\,\mathrm{dx}$

Substitute $lnx = it \Rightarrow \frac{1}{x}dx = idt$

Therefore

$$\mathrm{B}(f(x)) = \int_0^\infty f(it)x^{-s-1}ix\,dt$$
$$= i\int_0^\infty f(it)x^{-s}\,dt$$
$$= i\int_0^\infty (e^{it})^{-s} f(it)\,dt$$
$$= i\int_0^\infty e^{-its} f(it)\,dt$$

Hence, on comparing with definition of Fourier Integral Transform

$$\mathrm{B}(f(x)) = iF(is)$$

Converse

Let $\mathrm{F}\{f(t)\} = \int_{-\infty}^\infty e^{-its} f(t)\,dt$

Substitute $it = lnx \Longrightarrow \frac{1}{x}\,dx = idt$

Therefore

$$\mathrm{F}\{f(t)\} = \int_0^\infty x^{-s} f\left(\frac{1}{i}lnx\right)\frac{1}{ix}\,dx$$
$$= \frac{1}{i}\int_0^\infty x^{-s-1} f\left(\frac{1}{i}lnx\right)\,dx$$

Hence, on comparing with Eq. (1), we thus obtain

$F\{f(t)\} = \frac{1}{i}B\left(\frac{1}{i}s\right)$

Both result hold for the interval $[0,\infty)$. So we can relate to each other.

3.4Connection between Polynomial Integral Transform and Laplace Integral Transform:-

In this section we show that **Polynomial Integral Transform** is the theoretical dual of Laplace Transform and the dual relationship is given by following relation-

Theorem 3.4- Let $f(t)\epsilon B$ and $L\{f(t)\} = F(s)$and $B(f(x))$=B(s) then

$$B(f(x))=F(s) \qquad \ldots (11)$$

$$L\{f(t)\} = B(s) \qquad \ldots (12)$$

and $F(s) \Leftrightarrow G(s)$

Proof- we know that

$$B(f(x))=\int_1^\infty f(lnx)x^{-s-1}\,dx$$

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$

Let $L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$

Substitute $lnx = t \Rightarrow dt = \frac{1}{x}dx$

Therefore

$$L\{f(t)\} = \int_1^\infty f(lnx)e^{-slogx}\frac{1}{x}\,dx$$

$$= \int_1^\infty f(lnx)x^{-s-1}\,dx$$

Hence, on comparing with Eq. (2), we thus obtain

$$L\{f(t)\} = B(s)$$

Converse

Let $B(f(x)) = \int_1^\infty f(lnx)x^{-s-1}\,dx$

Substitute $lnx = t \Rightarrow dt = \frac{1}{x}dx$

Therefore

$$B(f(x)) = \int_0^\infty f(t)e^{t(-s-1)}e^t dt$$

$$= \int_0^\infty f(t)e^{-ts}dt$$

Hence, on comparing with definition of Laplace Integral Transform

$$B(f(x))=F(s)$$

3.5 The Relation between Polynomial Integral Transform and Mellin Integral Transform

In this section we show that Polynomial Integral Transform is the theoretical dual of Mellin Integral Transform and the dual relationship is given by following relation-

Theorem 3.5- Let $f(t)\epsilon B$ the Polynomial transforms $B(f(x))$ = B(s) and M $\{f(t)\}$ = M(s) is the mellin transform respectively then

$$B(f(x)) = M\{f(-lnt)\} \qquad \ldots (13)$$

And $M\{f(t)\} = B(f(e^{-x}))$ … (14)

Proof:

We have duality of Laplace Integral Transform and Polynomial Integral Transform

$$L\{f(t)\}=B(f(x)) \qquad \ldots (15)$$

But we know that relation between Laplace transform and Mellin transform is given by

$$L\{f(t)\}= M\{f(-lnt)\} \qquad \ldots (16)$$

$$M\{f(t)\} = L\{f(e^{-t})\} \qquad \ldots (17)$$

We have

$L\{f(t)\}=B(f(x))$

then from equation (17)

$B(f(x))= M\{f(-lnt)\}$

Similarly from equation (15), (16) and (17). We get

$M\{f(t)\} = L\{f(e^{-t})\} = B(f(e^{-x}))$ [by Eq. (13)]

4. Conclusion

We drive a duality relation between Polynomial Integral Transform and Natural, Sumudu, Fourier, Laplace and Mellin. In addition, it is successfully proved that Natural, Sumudu, Fourier, Laplace and Mellin Transforms can be derived from Polynomial Integral Transform by making simple substitution and converse is also valid. Just by changing parameters, Polynomial Integral Transform has the property of being converging to Laplace Integral Transform. Thus we can say that Polynomial Transform plays are important role for solving various differential and integral equations.

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