# Fixed Point Theorem in Dislocated Quasi Metric Space

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**Abstract** — *In this paper I, have proved fixed point theorem for continuous contraction mappings in dislocated quasi-metric space.*

**Keywords—** *dislocated quasi metric space, fixed point..*

## I. INTRODUCTION

Fixed point theory is a very useful tool in solving variety of problems in the control theory, economic theory, nonlinear analysis and, global analysis. The Banach contraction principle is the most famous, most simplest, and one of the most versatile elementary results in the fixed point theory. A huge amount of literature is witnessed on applications, generalizations, and extensions of this principle carried out by several authors in different directions, for example, by weakening the hypothesis, using different setups, and considering different mappings. Das [2] generalized Banach's contraction principle in metric space. Also Rhoades established a partial ordering for various definitions of contractive mappings.

The object of this note is to prove some fixed point theorems for continuous contraction mappings defined by Das and Gupta [2] and Rhoades [4], in dislocated quasi-metric spaces.

## II. PRELIMINARIES

Definition 2.1: Let *X* be a non empty set and let $d : X \times X \to [0, \infty)$ be a function, called a distance function we need the following conditions:

i.) $d(x, y) = d(y, x) = 0$, then $x = y$

ii.) $d(x, y) \le d(x, z) + d(z, y)\ \forall x, y, z \in X$.

Then *d* is called a dislocated quasi metric on *X*. If a metric *d* satisfies $d(x, y) = d(y, x)$, then it is called dislocated metric space.

Definition 2.2: A sequence $\{x_n\}$ dislocated quasi converges to *x* if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.

Definition 2.3: A sequence $\{x_n\}$ in dq- metric space $(X, d)$ is called Cauchy if $\forall \varepsilon > 0, n_0 \in N$ such that $\forall m, n \ge n_0, d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$.

Definition 2.4: dq-limits in a dq-metric space are unique.

Definition 2.5: A dq-metric space $(X, d)$ is called complete if every Cauchy sequence in it is dq-convergent.

Definition 2.6: Let $(X, d_1)$ and $(Y, d_2)$ be dq-metric spaces and let $f : X \to Y$ be a function. Then *f* is continuous to $x_0 \in X$ if for each sequence $\{x_n\}$ which is $d_1 - q$ convergent to $x_0$ the sequence $\{f(x_n)\}$ is $d_2 - q$ convergent to $\{f(x_0)\}$ in *Y*.

Definition 2.7: Let $(X, d)$ be a dq- metric space. A map $T : X \to X$ is called contraction if there exist $0 \le \lambda < 1$ such that $d(Tx, Ty) \le \lambda d(x, y) \forall x, y \in X$.

**3 Main Result:**

Theorem 3.1 Let $(X, d)$ be a complete dq- metric space and let $T : X \to X$ be a continuous mapping satisfying the following condition

$$d(Tx, Ty) \le \alpha(d(x, Tx) + d(y, Ty)) + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \forall x, y \in X, 0 \le \alpha < 1/2, 0 \le \beta < 1/2.$$

Then T has unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in *X*, defined as follows: Let $x_0 \in X, T(x_0) = x_1, T(x_1) = x_2, \ldots, T(x_n) = x_{n+1}$.

Consider,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \alpha[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)]+\beta\frac{d(x_n,Tx_n)[1+d(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)}$$

$$\leq \alpha[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+\beta\frac{d(x_n,x_{n+1})(1+d(x_{n-1},x_n))}{(1+d(x_{n-1},x_n)}$$

$$\leq \alpha[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+\beta d(x_n,x_{n+1})$$

Thus,

$$d(x_n,x_{n+1})\leq\frac{\alpha}{1-(\alpha+\beta)}d(x_{n-1},x_n)$$

$$=\lambda d(x_{n-1},x_n) \text{ where } \lambda=\frac{\alpha}{1-(\alpha+\beta)}, 0\leq\lambda<1.$$

Similarly we will show that

$d(x_{n-1},x_n)\leq\lambda d(x_{n-2},x_{n-1})$

Thus $d(x_n,x_{n+1})\leq\lambda^2 d(x_{n-2},x_{n-1})$

Continuing in this way we get

$d(x_n,x_{n+1})\leq\lambda^n d(x_0,x_1)$

Since $0\leq\lambda<1$, as $n\to\infty,\lambda^n\to 0$.

Hence $\{x_n\}$ is a dq- sequence in the complete dislocated quasi metric space $X$.

Thus $\{x_n\}$ dislocated quasi converges to some $t_0$. Since T is continuous we have, $T(t_0)=\lim_{n\to\infty}T(x_n)=\lim_{n\to\infty}x_{n+1}=t_0$.

Thus $T(t_0)=t_0$. Thus *T* has a fixed point.

Uniqueness: Let *x* be a fixed point of *T*. Then by given condition, we have

$d(x,x)=d(Tx,Tx)$

$\leq(2\alpha+\beta)d(x,x)$

$\because 0\leq(2\alpha+\beta)<1$ and $d(x,x)\geq 0$.

$\Rightarrow d(x,x)=0$, if x is a fixed point of T.

Let $x,y\in X$ be a fixed point of *T* i.e. *Tx* = *x*, *Ty* = *y*.

Then,

$d(x,y)=d(Tx,Ty)$

$$\leq\alpha(d(x,Tx)+d(y,Ty))+\beta\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}$$

and $d(x,y)\geq 0$.

$\Rightarrow d(x,y)=0$, similarly $d(y,x)=0$ and hence $x=y$. Thus fixed point of $T$ is unique.

Theorem 3.2 Let $(X,d)$ be a complete dq- metric space and let $T:X\to X$ be a continuous mapping satisfying the following condition

$d(Tx,Ty)\leq\alpha d(x,y)+\beta d(x,Ty)+\gamma d(y,Ty)+\delta[d(x,Tx)+d(y,Ty)], \alpha,\beta,\gamma,\delta$ are non negative which may depends on both $x$ and $y$, such that $\sup\{\alpha+\beta+\gamma+2\delta : x,y\in X\}<1$.

Then T has unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in *X*, defined as follows: Let $x_0\in X, T(x_0)=x_1, T(x_1)=x_2,\ldots,T(x_n)=x_{n+1}$.

Consider,

$d(x_n,x_{n+1})=d(Tx_{n-1},Tx_n)$

$$\leq\alpha d(x_{n-1},x_n)+\beta d(x_{n-1},Tx_n)+\gamma d(x_n,Tx_n)+\delta[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)]$$

$$\leq\alpha d(x_{n-1},x_n)+\beta d(x_{n-1},x_{n+1})+\gamma d(x_n,x_{n+1})+\delta[d(x_{n-1},x_n)+d(x_n,x_{n+1})]$$

Thus,

$$d(x_n, x_{n+1}) \le \frac{\alpha + \beta + \delta}{1 - (\gamma + \delta + \beta)} d(x_{n-1}, x_n)$$

$$= \lambda d(x_{n-1}, x_n) \text{ where } \lambda = \frac{\alpha + \beta + \delta}{1 - (\gamma + \beta + \delta)}, 0 \le \lambda < 1.$$

Similarly we will show that

$d(x_{n-1}, x_n) \le \lambda d(x_{n-2}, x_{n-1})$

Thus $d(x_n, x_{n+1}) \le \lambda^2 d(x_{n-2}, x_{n-1})$

Continuing in this way we get

$d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1)$

Since $0 \le \lambda < 1$, as $n \to \infty, \lambda^n \to 0$.

Hence $\{x_n\}$ is a dq- sequence in the complete dislocated quasi metric space $X$.

Thus $\{x_n\}$ dislocated quasi converges to some $t_0$. Since T is continuous we have, $T(t_0) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = t_0$.

Thus $T(t_0) = t_0$. Thus $T$ has a fixed point.

Uniqueness: Let $x$ be a fixed point of $T$. Then by given condition, we have

$d(x, x) = d(Tx, Tx)$

$\le \alpha d(x, x) + \beta d(x, x) + \gamma d(x, x) + 2\delta d(x, x)$

$\le (\alpha + \beta + \gamma + 2\delta) d(x, x)$

$\because 0 \le (\alpha + \beta + \gamma + 2\delta) < 1$ and $d(x, x) \ge 0$.

$\Rightarrow d(x, x) = 0$, if x is a fixed point of T.

Let $x, y \in X$ be a fixed point of $T$ i.e. $Tx = x$, $Ty = y$.

Then,

$d(x, y) = d(Tx, Ty)$

$\le \alpha d(x, y) + \beta d(x, Ty) + \gamma d(y, Ty) + \delta[d(x, Tx) + d(y, Ty)]$

$\le (\alpha + \beta) d(x, y)$

$\because 0 \le (\alpha + \beta) < 1$ and $d(x, y) \ge 0$.

$\Rightarrow d(x, y) = 0$, similarly $d(y, x) = 0$ and hence $x = y$. Thus fixed point of $T$ is unique.

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