Picard-Mann Hybrid Iteration Processes for Monotone Non Expansive Mappings

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Abstract

In this paper we provestrong convergence and stability results of Picard-Mann hybrid iteration scheme for a monotone nonexpansivemappings in a Uniformly Convex Banach space.

Keywords

Strong convergence, Picard –Mann hybrid iteration scheme, monotone nonexpansive mapping.

Mathematics subject classification: 54H25, 47H10

1. Introduction

Monotone nonexpansive self-mappings of a closed convex and bounded subset of a Uniformly Convex Banach space always have Lipschitz constant equal to one and have a fixed point which has been proved in 1965 by Browder[4] and Ghode[7]. Some of the authors Hussai et al.[9], Phuengrattana and Suantal [12] and Harder [8] have done research regarding rate of convergence of iteration procedures.Hussai Khan [10] in 2013, have introduced Picard-Mann hybrid iteration scheme and showed that it converges faster than all of Picard, Mann and Ishikawa iteration processes for contractions in the sense of Berinde [2]. In 1990, Rhoades [13] have proved that several iteration procedures are T-Stable for map satisfying General Contraction condition. In 2006, Rhoades and Soltuz [14] have showed the equivalence of T-stability result in Mann and Ishikawa iterations whereas Asadi [1] in 2009 discussed the T-stability of Picard's iteration procedures in cone metric space and gave an application. Many authors like Bin Dehaish and Khamsi [3], Y.song,Kumam andcho [16] and Klangpraphan and Nilsrakoo [5] have proved Mann Iteration process for monotone nonepansive mappings. In this paper, we prove the strong convergence of Picard_Mann hybrid iteration scheme for a monotone nonexpansive mapping in a uniformly convex Banach space and also proved its stability results.

2. Preliminaries

LetK be a closed convex subset of a uniformly convex Banach space (E,$\leq$).A partial order $'\leq'$ with respect to K in E is defined as follows:

$x \leq y\ (x < y)$if and only if $y - x\ \in K\ for\ all\ x, y\ \in K.$

Let F(T) =$\{p \in E : Tp = p\}$denote the set of all fixed point of mapping T.

Definition 2.1 [3]:

Let(E,$\leq$) be a Uniformly Convex BanachSpace. Let K be a nonempty subset of E. A mapping T: K→E is said to be

1. Monotone if T$x \leq Ty$ for all $x, y \in K\ with\ x \leq y$;
2. Monotone nonexpansive if T is monotone and
$$\|Tx - Ty\| \leq \|x - y\|$$
for all $x, y \in K\ with\ x \leq y.$

Definition 2.2

A Banach space E is said to be

1. Strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E\ with\ \|x\| = \|y\|$=1 and $x \neq y.$
2. Uniformly convex if for all $\epsilon\ \in [0,2], there\ exists\ \delta > 0\ such\ that$

$$\left\|\frac{x+y}{2}\right\| < 1 - \delta \text{for all } x, y \in E \ with \ \|x\| = \|y\|=1 \text{ and } \|x-y\| \geq \epsilon$$

Definition 2.3 [1]:

Picard-Mann hybrid iteration scheme is defined as follows:

Let E be a real normed linear space. For any initial point $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ where n=0 to n=∞, is defined by

x_{n+1} = Ty_n , where
$y_n = (1-\beta_n)x_n + \beta_n \mathrm{T}x_n$, n≥ 0.

$$where\ \{\beta_n\}_{n=0}^{\infty}\ is\ a\ real\ sequence\ in\ [0,1].$$

Definition2.4 [13]:

Let (X,d) be a complete metric space and E be a real Banach space. Let T: X→X be a selfmap of X. Let $F_T = \{p \in E : Tp = p\}$is the set of foxed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset$ E be the sequence generated by an iteration procedure involving T which is defined by x_{n+1} =f(T,x_n)$, n = 0,1,2$ ...where $x_0 \in X$ is the initial approximation and f is some function. Suppose$\{x_n\}_{n=0}^{\infty}$ converges to fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset$ X and set $\epsilon = d(y_{n+1}, f(T, y_n)), n = 0,1,2$ ..., then the iteration procedure is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n$=0 implies $\lim_{n\to\infty} y_n$=p

Lemma 2.5 (XU [15]:

For any real numbers q> $1\ and\ r > 0$, a Banach space E is uniformly convex if and only if there exist a continuous strictly increasing convex function g: $[0,\infty) \to [0,\infty)$ with g(0)=0 such that

$\|tx + (1-t)y\|^q \leq$t$\|x\|^q + (1-t)\|y\|^q - w(q,t)g(\|x-y\|)$

for all $x, y \in B_r(0)$= $\{x \in E , \|x\| \leq r\}$ and t $\in [0,1], where$

$w(q,t) = t^q(1-t) + t\,(1-t)^q$.

3. Main Result

Theorem 3.1

Let K be a non-empty and closed convex subset of a uniformly convex Banach space (E, ≤) and T: $K \to K$ be a monotone nonexpansive mapping with a fixed point p $\in E$. Assume that the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

x_{n+1} = Ty_n

$y_n = (1-\beta_n)x_n + \beta_n \mathrm{T}x_n$, n≥ 0.→ (1)

is monotonically non-increasing sequence and $\{\beta_n\}$ for n=0 to n=∞ is a real sequence in [0,1] satisfying the following condition $\sum_{n=0}^{n=\infty} \beta_n(1-\beta_n)$=0. Then $\{x_n\}_{n=0}^{\infty}$converges strongly to p.

Proof:

Since p$\in F(T)$, we have Tp=p and using (1), we get

$$\|x_{n+1} - \mathrm{p}\| = \|Ty_n - \mathrm{Tp}\|$$

$$\leq \|y_n - \mathrm{p}\|$$

$$\leq \|(1-\beta_n)(x_n - \mathrm{p}) + \beta_n(Tx_n - \mathrm{Tp})\|$$

In Lemma 2.5, putq=1 and $\beta_n = t$.By Lemma 2.5, since E is a Uniformly Convex Banach space there exist a continuous strictly increasing convex function g: $[0,\infty) \to [0,\infty)$ such that

$$\|x_{n+1} - p\| \leq \beta_n \|Tx_n - Tp\| + (1-\beta_n)\|x_n - p\|$$
$$-\beta_n(1-\beta_n)g(\|(x_n - Tx_n)\|)$$

$\leq \beta_n \|x_n - p\|$+$(1-\beta_n)\|x_n - p\|$

$-\beta_n(1-\beta_n)g(\|x_n - Tx_n\|)$

From the conditions of β_n in theorem, we get

$\|x_{n+1} - p\| \leq \|x_n - p\|$ as $n \to \infty$.

Since the sequence $\{\|x_n - p\|\}$ is monotonically non-increasing, it follows that

$\lim_{n\to\infty}\|x_n - p\| = 0$ and it implies $\lim_{n\to\infty} x_n = p$. Hence $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. This completes the proof.

Theorem 3.2

Let K be a non-empty and closed convex subset of a uniformly convex Banach space (E, $\leq$) and T: $K \to K$ be a monotone nonexpansive mapping with a fixed point p $\in E$. Assume that the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$x_{n+1} = Ty_n$

$y_n = (1-\beta_n)x_n + \beta_n Tx_n$, for n$\geq 0$

is monotonically non-increasing sequence. Then the Picard-Mann hybrid iteration scheme is T-stable.

Proof:

From the theorem 3.1, $\{x_n\}$ converges to p. Let

$\epsilon_n = \|x_{n+1} - Ty_n\|$, n=0,1,2,3.. where

$x_{n+1} = Ty_n$

$y_n = (1-\beta_n)x_n + \beta_n Tx_n$, n$\geq 0$

and let $\lim_{n\to\infty}\epsilon_n$=0. Then we shall prove that $\lim_{n\to\infty} x_n$=p.

$\|x_{n+1} - p\| \leq \|x_{n+1} - Ty_n\|$+$\|Ty_n - Tp\|$

$\leq \epsilon_n$ +$\|y_n - p\|$

$\leq \epsilon_n$ +$(1-\beta_n)\|x_n - p\|$+ $\beta_n\|Tx_n - Tp\|$

$\leq \epsilon_n$ +$(1-\beta_n)\|x_n - p\|$+ $\beta_n\|x_n - p\|$

$\leq \epsilon_n$ +$\|x_n - p\|$

As $\lim_{n\to\infty}\epsilon_n$=0 and the sequence $\{\|x_n - p\|\}$ is monotonically non-increasing the $\lim_{n\to\infty}\|x_n - p\|$ exist and hence $\lim_{n\to\infty} x_n = p$.

Conversely, let $\lim_{n\to\infty} x_n = p$. We show that $\lim_{n\to\infty}\epsilon_n$=0.

We know,

$\epsilon_n = \|x_{n+1} - Ty_n\|$

$\leq \|x_{n+1} - Tp\|$+ $\|Tp - Ty_n\|$

$\leq \|x_{n+1} - p\|$+ $\|y_n - p\|$

$\leq \|x_{n+1} - p\| + \|(1-\beta_n)(x_n - \mathrm{p}) + \beta_n(\mathrm{T}x_n - Tp)\|$

$\leq \|x_{n+1} - p\| + (1-\beta_n)\|(x_n - \mathrm{p})\| + \beta_n\|x_n - p\|$

$\leq \|x_{n+1} - p\| + \|x_n - p\|$

As $\lim_{n\to\infty}\|x_n - p\| = 0, \lim_{n\to\infty}\epsilon_n$=0.

Therefore Picard-Mann hybrid iterative scheme is T-stable.

We give an example to establish theorem 3.1.

Example 3.2

Let E=[0,1]. Define T:[0,1] → [0,1] by $Tx = \frac{3x}{4}$ where [0,1] has the usual metric. Then T satisfies $\|Tx - Tp\| \leq \|x - p\|$ and F(T)=[0,1].

We will show that Picard-Mann hybrid iterative scheme is T-stable.

Now, let p=0, taking $\beta_n = \frac{1}{4}, y_n = \frac{1}{n}$ for each n≥ 1. Then $\lim_{n\to\infty} y_n$=0.

We see that $\epsilon_n = |y_{n+1} - Ty_n|$

$= \left|y_{n+1} - \frac{3y_n}{4}\right|$

$= \left|y_{n+1} - \frac{(1-\beta_n)3y_n}{4} - \frac{3\beta_n}{4}\frac{(3y_n)}{4}\right|$

$= \left|\frac{1}{n+1} - \frac{9}{16n} - \frac{9}{64n}\right|$

We have,

$\lim_{n\to\infty}\epsilon_n$= 0.

Hence the Picard-Mann hybrid iterative scheme is T-stable.

References:

1. A. M. Soleimani, H.Vaezpour SM, Rhoades B.E. (2009), On T-stability of Picard's Iteration in cone metric space, Fixed point theory appl.DOl: 10.1155/2009/751090.
2.Berinde, V.Iterative Approximation of fixed points EditurraEfemeride, Baia Mare (2002).
3.B.Dehaish and Khamsi, Mann iteration process for monotone nonexpansive mappings, Fixed point theory and applications (2015) 2015:177 DOI 10.1186/s 13663-015-0416-0
4. Browder, FE: Nonexpansive nonlinear operators in a Banach Space. Proc. Natl. Acad. Sci. USA 54,1041-1044(1965).
5.Bachar, M.khamsi, MA: On common approximate fixed points of monotone nonexpansive semigroups in Banach spaces. Fixed Point Theory Appl.2015,160 (2015)
6. J.G.Flaset, E.L. Fluster and T. Suzuki, Fixed point theory for a class of generalised non-expansive mapping. J.Math.Anal.Appl.375(1) (2011), 185-195.
7. Gohde, D: ZumPrinzip der kontraktivenabbildung. Math. Nachr.30,251-258.
8. Harder, AM: Fixed pointtheory and stability results for fixed point iteration procedures. Ph.d. thesis, University of Missouri-Polla(1987).
9. Hussai.N, Chugh, R.Kumar, V.Rafiq, On the rate of convergence of Kirk-type iterative schemes. J. Appl.Math. 345(2) (2012), 186-190,
10. Khan, SH: A Picard-Mann hybrid iterative process, Fixed point theory Appl 2013 Article ID69(2013). Dol 10.1186/11687-1812-2013-69.
11.Lael.F and Heidarpour.Z, Fixed point theorems for a class of generalised nonexpansive mappings, Fixed point theory Appl, 2016:82. DOI .10,1186/S13663-016-0571-4.
12. Phuengrattana, W.Suantal, On the convergence of Mann, Ishikawa, Noor and Sp-iterations for continuous functions on an arbitrary integral. J.Comput. Appl. Math.235(9), 3006-3014 (2011).
13. Rhoades B.E. (1990), Fixed point theorems and stability results for fixed point iteration procedures, Indian J.PureAppl.Math 21:1-9.
14.Rhoades B.E, Soltuz SM (2006) ,The equivalence between the T-Stabilities of Mann and Ishikawa Iteration. J Math Anal Appl 318: 472-475.
15. Xu, HK: Inequality in Banach Spaces with Applications,Nonlinear Anal.16, 1127-1138 (1991).
16.Y.Song, PoomKumam and Y.J Cho, Fixed point theorems and iterative approximations for monotone nonexpansive mappings in orderedBanachspace, Fixed point theory and applications (2016) 2016:73DOl 10.1186/s 13663-016-0563-4