Common Fixed Point Theorems for Non-Contractive Type Mappings in Banach Space

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Abstract :

In this paper we present some fixed point and common fixed point theorems are established for non-contraction mappings in Banach Space. Our result is motivated by many authors.

Keywords :

Fixed point, Common Fixed Point, Banach spaces, non-contractive mapping.

I. INTRODUCTION

Almost differential equation and an integral equation arise from many physical problems; these problems are mostly solvable using non-linear and fixed point techniques. This technique provides the powerful tools for obtaining the solution of such equations otherwise difficult to solve by ordinary method.

While stating this however, we mention that when we solve the equation by functional analysis approach, lost the some qualitative properties. Many attempts have been made to formulate fixed point theorems in this direction and the well known Schauder's fixed point principle formulated by J. Schauder in 1930

Brouwder [1], Gohde [2],Kirk [3], Iseki [4], Sharma [5], Singh [6] and Yadav [7] have self-rule proved the common fixed point theorem for the non –expansive mappings defined on a closed , bounded and convex subset of a uniformly convex Banach space and in spaces with richer structure. Many other mathematicians gave a number of generalizations of non- expansive mappings, like Dotson [8] , Emmouele [9], Goebel [10], Goebel and Zlotkiewicz [11],Goebel, Kirk and Shimi [12], Massa and Rouk [13], Rhodes [14], are of special significance. An extensive survey about common fixed point theorem for the non-expansive mapping and related mappings can be found by Kirk [3].

Let C be closed subset of Banach space X. then the well known Banach Contraction Principle: An contraction mapping from C into itself has a unique fixed point in C. The same holds good if we assume that only some positive power of a mapping is a contraction (For example, Bryant [15]). But this result is no longer true for non expansive mappings. Many author have studied the existence of fixed points of non expansive maps defined on a closed, bounded and convex subset of a uniformly convex Banach space, and in a space with a normal structure. For the results of this kind one is referred to Browder [1], Goebel [10], kirk [16], S.K.Tiwari and Ranu modi [17] and S.Vijayvargiya et all.[18]. It is natural with a non expansive iteration. The answer in general is negative However Goebel and Zlokiewicz [11] have answered this problem in affirmative with some restriction, and thus generalizing a result of Browder [1].

A normed linear space is also linear space in set of natural number in which each vector x, there corresponds a real number denoted by $\|x\|$. It is called the norm of vector x with properties:

(1) $\|x\| \geq 0$ if $\|x\| = 0 \ if \ and \ only \ if \ x = 0$

(2) $\|x + y\| \leq \|x\| + \|y\|$

(3) $\|\alpha x\| = |\alpha|\,\|x\|$

If we consider $\|x\|$ as a real mapping defined on set of natural number N then it is easy to verify the normed function is called norm on set of natural number N. It is also very easy to verify that linear space N is metric space defining by $d(x,y) = \|x-y\|$. A banach space is a complete normed linear space. We used non- contractive mapping in banach space.

II. MAIN RESULT

2.1 Let X be Banach space. Let F be the mapping of X into itself such that

$F^2 = I$, where I is identity mapping ... …..(2.1)

$$\|F(x)-F(y)\| \le \alpha \frac{\|x-F(x)\|\|y-F(y)\|}{\|x-F(x)\|+\|x-F(y)\|} + \beta \frac{\|x-F(y)\|\|y-F(x)\|}{\|x-F(y)\|+\|y-F(x)\|}$$
$$+\gamma\|x-F(x)\| + \emptyset\|y-F(y)\| + \delta\|x-F(y)\| + \varphi\|y-F(x)\|$$
$$+\eta\|x-y\| \qquad \text{……. (2.2)}$$

For every $x, y \in X$ where $0 \le \alpha, \beta, \gamma, \emptyset, \delta, \varphi, \eta$ and $7\alpha + 8\beta + 4\gamma < 8$ and

$$\frac{\beta}{2} + \delta + \varphi + \eta < 1 \qquad \text{………. (2.3)}$$

Then F has an unique fixed point x_0 in Banach Space X.

Proof:

Let x be any point in the Banach space X, then taking $y = \frac{1}{2}(F+I)x$

$$t = F(y)$$

And $$u = 2y - t.$$

Now

$$\|t-x\| = \|F(y) - F^2(x)\|$$
$$= \|F(y) - F(F(x))\|$$
$$\le \alpha \frac{\|y-F(y)\|\|F^2(x)-F(x)\|}{\|y-F^2(x)\|+\|y-F(y)\|} + \beta \frac{\|y-F^2(x)\|\|F(y)-F(x)\|}{\|y-F^2(x)\|+\|F(y)-F(x)\|}$$
$$+\gamma\|F(x)-F^2(x)\| + \emptyset\|y-F(y)\| + \delta\|F(x)-F(y)\| + \varphi\|y-F^2(x)\|$$
$$+\eta\|F(x)-y\|$$
$$=\alpha \frac{\|y-F(y)\|\|x-F(x)\|}{\|y-x\|+\|y-F(y)\|} + \beta \frac{\|y-x\|\|F(y)-F(x)\|}{\|y-x\|+\|F(y)-F(x)\|}$$
$$+\gamma\|x-F(x)\| + \emptyset\|y-F(y)\| + \delta\|x-y\| + \varphi\|F(y)-F(x)\|$$
$$+\eta\|F(x)-y\|$$
$$= \alpha \frac{\|y-F(y)\|\|x-F(x)\|}{\|x-F(y)\|} + \beta \frac{\frac{1}{2}\|x-f(x)\|\|F(y)-x+x-F(x)\|}{\frac{1}{2}\|x-f(x)\|+\|F(y)-x+x-F(x)\|}$$
$$+\gamma\|x-F(x)\| + \emptyset\|y-F(y)\| + \delta\frac{1}{2}\|x-f(x)\| + \varphi\|F(y)-x+x-F(x)\|$$
$$+\eta\frac{1}{2}\|F(x)-x\|$$
$$\le \alpha \frac{\|y-F(y)\|\|x-F(x)\|}{\frac{1}{2}\|x-F(x)\|} + \beta \frac{\frac{1}{2}\|x-f(x)\|[\|F(y)-x\|+\|x-F(x)\|]}{\frac{1}{2}\|x-f(x)\|+\|F(y)-x\|+\|x-F(x)\|}$$
$$+\gamma\|x-F(x)\| + \emptyset\|y-F(y)\|$$
$$+\delta\frac{1}{2}\|x-f(x)\| + \varphi[\|F(y)-x\| + \|x-F(x)\|]$$
$$+\eta\frac{1}{2}\|F(x)-x\|$$
$$\le 2 \propto \|y-F(y)\| + \beta \frac{\frac{1}{2}\|x-f(x)\|[\|F(x)-x\|+\|x-F(x)\|]}{\frac{1}{2}\|x-f(x)\|+\|F(x)-x\|+\|x-F(x)\|}$$

$$+\gamma\|x-F(x)\|+\emptyset\|y-F(y)\|$$
$$+\delta\tfrac{1}{2}\|x-f(x)\|+\varphi\left[\tfrac{1}{2}\|F(x)-x\|+\|x-F(x)\|\right]$$
$$+\eta\tfrac{1}{2}\|F(x)-x\|$$
$$=(2\propto+\emptyset)\|y-F(y)\|+\left(\tfrac{3}{8}\beta+\gamma+\tfrac{1}{2}\delta+\tfrac{3}{2}\varphi+\tfrac{\eta}{2}\right)\|x-f(x)\|$$

i.e $\|t-x\|\le(2\propto+\emptyset)\|y-F(y)\|+\left(\tfrac{3}{8}\beta+\gamma+\tfrac{1}{2}\delta+\tfrac{3}{2}\varphi+\tfrac{\eta}{2}\right)\|x-f(x)\|$

Now

$$\|u-x\|=\|2y-t-x\|$$
$$=\|(F+I)x-F(y)-x\|$$
$$=\|F(x)-F(y)\|$$
$$\le\alpha\frac{\|x-F(x)\|\|y-F(y)\|}{\|x-F(x)\|+\|x-F(y)\|}+\beta\frac{\|x-F(y)\|\|y-F(x)\|}{\|x-F(y)\|+\|y-F(x)\|}$$
$$+\gamma\|x-F(x)\|+\emptyset\|y-F(y)\|+\delta\|x-F(y)\|+\varphi\|y-F(x)\|$$
$$+\eta\|x-y\|$$
$$=\alpha\frac{\|x-F(x)\|\|y-F(y)\|}{\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|}+\beta\frac{\frac{1}{4}\|x-F(x)\|\|x-F(x)\|}{\frac{1}{2}\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|}$$
$$+\gamma\|x-F(x)\|+\emptyset\|y-F(y)\|+\delta\tfrac{1}{2}\|x-F(x)\|+\varphi\|y-x+x-F(x)\|$$
$$+\eta\tfrac{1}{2}\|x-F(x)\|$$
$$\le\alpha\frac{\|x-F(x)\|\|y-F(y)\|}{\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|}+\beta\frac{\frac{1}{4}\|x-F(x)\|\|x-F(x)\|}{\frac{1}{2}\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|}$$
$$+\gamma\|x-F(x)\|+\emptyset\|y-F(y)\|+\delta\tfrac{1}{2}\|x-F(x)\|+\varphi[\|y-x\|+\|x-F(x)\|]$$
$$+\eta\tfrac{1}{2}\|x-F(x)\|$$

$$\le\alpha\frac{\|x-F(x)\|\|y-F(y)\|}{\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|}+\beta\frac{\frac{1}{4}\|x-F(x)\|\|x-F(x)\|}{\frac{1}{2}\|x-F(x)\|+\frac{1}{2}\|x-F(x)\|}$$
$$+\gamma\|x-F(x)\|+\emptyset\|y-F(y)\|+\delta\tfrac{1}{2}\|x-F(x)\|$$
$$+\varphi\left[\tfrac{1}{2}\|x-F(x)\|+\|x-F(x)\|\right]$$
$$+\eta\tfrac{1}{2}\|x-F(x)\|$$
$$=\alpha\frac{\|x-F(x)\|\|y-F(y)\|}{\frac{3}{2}\|x-F(x)\|}+\beta\frac{\frac{1}{4}\|x-F(x)\|\|x-F(x)\|}{\|x-F(x)\|}$$
$$+\gamma\|x-F(x)\|+\emptyset\|y-F(y)\|+\delta\tfrac{1}{2}\|x-F(x)\|+\varphi\tfrac{3}{2}\|x-F(x)\|$$
$$+\eta\tfrac{1}{2}\|x-F(x)\|$$
$$=(2\propto+\emptyset)\|y-F(y)\|+\left(\tfrac{3}{8}\beta+\gamma+\tfrac{1}{2}\delta+\tfrac{3}{2}\varphi+\tfrac{\eta}{2}\right)\|x-f(x)\|$$

i.e.

$$\|u-x\|\le(2\propto+\emptyset)\|y-F(y)\|+\left(\tfrac{3}{8}\beta+\gamma+\tfrac{1}{2}\delta+\tfrac{3}{2}\varphi+\tfrac{\eta}{2}\right)\|x-f(x)\|$$

… …………………………………………….2.4

Now

$$\|t-u\|\le\|t-x\|+\|x-u\|$$
$$\le(2\propto+\emptyset)\|y-F(y)\|+\left(\tfrac{3}{8}\beta+\gamma+\tfrac{1}{2}\delta+\tfrac{3}{2}\varphi+\tfrac{\eta}{2}\right)\|x-f(x)\|$$
$$+(2\propto+\emptyset)\|y-F(y)\|+\left(\tfrac{3}{8}\beta+\gamma+\tfrac{1}{2}\delta+\tfrac{3}{2}\varphi+\tfrac{\eta}{2}\right)\|x-f(x)\|$$

$$= (4 \propto +2\emptyset)\|y - F(y)\| + \left(\frac{3}{4}\beta + 2\gamma + \delta + 3\varphi + \eta\right)\|x - f(x)\|$$

i.e. $\|t-u\| \leq (4 \propto +2\emptyset)\|y - F(y)\| + \left(\frac{3}{4}\beta + 2\gamma + \delta + 3\varphi + \eta\right)\|x - f(x)\|$ …..2.5

One more time

$$\|t-u\| = \|F(y) - (2y - t)\|$$
$$= \|F(y) - 2y + t\|$$
$$= 2\|y - F(y)\|$$

Combining 2.4 and 2.5, we get

$$2\|y - F(y)\| \leq (4 \propto +2\emptyset)\|y - F(y)\| + \left(\frac{3}{4}\beta + 2\gamma + \delta + 3\varphi + \eta\right)\|x - f(x)\|$$

Therefore

$$(2 - 4 \propto -2\emptyset)\|y - F(y)\| \leq \left(\frac{3}{4}\beta + 2\gamma + \delta + 3\varphi + \eta\right)\|x - f(x)\|$$

Therefore

$$\|y - F(y)\| \leq \frac{\left(\frac{3}{4}\beta + 2\gamma + \delta + 3\varphi + \eta\right)}{(2 - 4\propto - 2\emptyset)}\|x - f(x)\|$$

Therefore

$\|y - F(y)\| \leq q\|x - f(x)\|$, where $q = \frac{\left(\frac{3}{4}\beta + 2\gamma + \delta + 3\varphi + \eta\right)}{(2 - 4\propto - 2\emptyset)} < 1$

Since $7\alpha + 8\beta + 4\gamma < 8$

Let $G = \frac{1}{2}\|F + I\|$

For all $x \in X$,

$$\|G^2(x) - G(x)\| = \|G(y) - y\|$$
$$= \left\|\frac{1}{2}\|F + I\|(y) - y\right\|$$
$$= \frac{1}{2}\|y - F(y)\|$$
$$= \frac{q}{2}\|x - F(x)\|$$

But by defining term q, we claim that $\{G^n(x)\}$ is Cauchy sequence in banach space X.

Therefore $\{G^n(x)\}$ is converges to some point say $x_0 \in X$. As completeness of X.

i.e. $\lim_{n\to\infty} G^n(x) = x_0$

which is implies that $G(x_0) = x_0$.

Therefore $F(x_0) = x_0$.

i.e. F has fixed point x_0.

Now we only to show that this fixed point is unique.

If possible suppose that y_0 is another fixed point of F.

$$\|x_0 - y_0\| = \|F(x_0) - F(y_0)\|$$
$$\leq \alpha \frac{\|x_0 - F(x_0)\|\|y_0 - F(y_0)\|}{\|x_0 - F(x_0)\| + \|x_0 - F(y_0)\|} + \beta \frac{\|x_0 - F(y_0)\|\|y_0 - F(x_0)\|}{\|x_0 - F(y_0)\| + \|y_0 - F(x_0)\|}$$
$$+\gamma\|x_0 - F(x_0)\| + \emptyset\|y_0 - F(y_0)\| + \delta\|x_0 - F(y_0)\| + \varphi\|y_0 - F(x_0)\|$$
$$+\eta\|F(x_0) - F(y_0)\|$$
$$= \alpha \frac{\|x_0 - x_0\|\|y_0 - y_0\|}{\|x_0 - x_0\| + \|x_0 - y_0\|} + \beta \frac{\|x_0 - y_0\|\|y_0 - x_0\|}{\|x_0 - y\_0\| + \|y_0 - x_0\|}$$
$$+\gamma\|x_0 - x_0\| + \emptyset\|y_0 - y_0\| + \delta\|x_0 - y_0\| + \varphi\|y_0 - x_0\|$$
$$+\eta\|x_0 - y_0\|$$
$$= \frac{\beta}{2}\|x_0 - y_0\| + (\delta + \varphi)\|x_0 - y_0\| + \eta\|x_0 - y_0\|$$
$$= \left(\frac{\beta}{2} + \delta + \varphi + \eta\right)\|x_0 - y_0\|$$

i.e. $\|x_0 - y_0\| \leq \left(\frac{\beta}{2} + \delta + \varphi + \eta\right)\|x_0 - y_0\|$

Therefore

$$\left(1 - \frac{\beta}{2} + \delta + \varphi + \eta\right)\|x_0 - y_0\| \leq 0$$

Since $\left(\frac{\beta}{2}+\delta+\varphi+\eta\right)<1$
$\therefore \|x_0-y_0\|=0$
$\therefore x_0=y_0$
$\therefore F$ has Unique fixed point x_0.
Now we generalized the theorem 2.1

2.2 Let $\boldsymbol{X}$ be banach space and $\boldsymbol{K}$ be closed and convex subset of $\boldsymbol{X}$. Let $\boldsymbol{F}$ **and** $\boldsymbol{G}$ be two mapping from $\boldsymbol{K}$ into itself such that

F and G are commute ……………………2.6

$F^2=I$ and $G^2=I$, where I is identity mapping ……………………2.7

$$\|F(x)-F(y)\|\le\alpha\frac{\|F(x)-G(x)\|\|F(y)-G(y)\|}{\|F(x)-G(x)\|+\|F(y)-G(x)\|}+\beta\frac{\|F(y)-G(x)\|\|F(x)-G(y)\|}{\|F(y)-G(x)\|+\|F(x)-G(y)\|}$$
$$+\gamma\|F(x)-G(x)\|+\emptyset\|F(y)-G(y)\|$$
$$+\delta\|F(y)-G(x)\|+\varphi\|F(x)-G(y)\|$$
$$+\eta\|G(x)-G(y)\| \qquad \ldots\ldots 2.8$$

For every $x,y\in K$, where $0\le\alpha,\beta,\gamma,\emptyset,\delta,\varphi,\eta$ and $7\alpha+8\beta+4\gamma<8$.

………………………..2.9

Then there exist at least on fixed point $x_0\in K$ such that $F(x_0)=G(x_0)=x_0$. Moreover x_0 is Unique fixed point of mapping F and G, if $\frac{\beta}{2}+\delta+\varphi+\eta<1$.

Proof:
From (2.6) and (2.7) it follows that Occasionally Weakly Compatible
$(FG)^2=I.$
Again from (2.7) and (2.8)

$$\|FGG(x)-FGG(y)\|\le\propto\frac{\|FG^2(x)-GG^2(x)\|\|FG^2(y)-GG^2(y)\|}{\|FG^2(x)-GG^2(x)\|+\|FG^2(y)-GG^2(x)\|}$$
$$+\beta\frac{\|FG^2(y)-GG^2(x)\|\|FG^2(x)-GG^2(y)\|}{\|FG^2(y)-GG^2(x)\|+\|FG^2(x)-GG^2(y)\|}$$
$$+\gamma\|FG^2(x)-GG^2(x)\|+\emptyset\|FG^2(y)-GG^2(y)\|$$
$$+\delta\|FG^2(y)-GG^2(x)\|+\varphi\|FG^2(x)-GG^2(y)\|$$
$$+\eta\|GG^2(x)-GG^2(y)\|$$
$$=\alpha\frac{\|FG.G(x)-G(x)\|\|FG.G(y)-G(y)\|}{\|FG.G(x)-G(x)\|+\|FG.G(y)-G(x)\|}$$
$$+\beta\frac{\|FG.G(y)-G(x)\|\|FG.G(x)-G(y)\|}{\|FG.G(y)-G(x)\|+\|FG.G(x)-G(y)\|}$$
$$+\gamma\|FG.G(x)-G(x)\|+\emptyset\|FG.G(y)-G(y)\|$$
$$+\delta\|FG.G(y)-G(x)\|+\varphi\|FG.G(x)-G(y)\|$$
$$+\eta\|G(x)-G(y)\|$$

Taking $G(x)=u$ and $G(y)=v$, we get

$$\|FG(u)-FG(v)\|\le\alpha\frac{\|FG(u)-u\|\|FG(v)-v\|}{\|FG(u)-u\|+\|FG(v)-u\|}++\beta\frac{\|FG(v)-u)\|\|FG(u)-v\|}{\|FG(v)-u)\|+\|FG(u)-v\|}$$
$$+\gamma\|FG(u)-u\|+\emptyset\|FG(v)-v\|$$
$$+\delta\|FG(v)-u)\|+\varphi\|FG(u)-v\|$$
$$+\eta\|u-v\|$$

Since $(FG)^2=I$.
So by theorem 2.1 FG has at least one fixed point say $x_0\in K$.
i.e. $FG(x_0)=x_0$
$\therefore FFG(x_0)=F(x_0)$ or $G(x_0)=F(x_0)$.
Now

$$\|F(x_0)-x_0\|=\|F(x_0)-F^2(x_0)\|$$
$$=\|F(x_0)-F(F(x_0))\|$$

$$\leq \alpha \frac{\|F(x_0)-G(x_0)\|\|FF(x_0)-GF(x_0)\|}{\|F(x_0)-G(x_0)\|+\|FF(x_0)-GF(y_0)\|} + \beta \frac{\|FF(x_0)-G(x_0)\|\|F(x_0)-GF(x_0)\|}{\|FF(x_0)-G(x_0)\|+\|F(x_0)-GF(x_0)\|}$$
$$+\gamma\|F(x_0)-G(x_0)\| + \emptyset\|FF(x_0)-GF(x_0)\|$$
$$+\delta\|FF(x_0)-G(x_0)\| + \varphi\|F(x_0)-GF(x_0)\|$$
$$+\eta\|G(x_0)-GF(x_0)\|$$
$$= \alpha \frac{\|F(x_0)-F(x_0)\|\|x_0-x_0\|}{\|F(x_0)-F(x_0)\|+\|x_0-y_0\|} + \beta \frac{\|x_0-F(x_0)\|\|F(x_0)-x_0\|}{\|x_0-F(x_0)\|+\|F(x_0)-x_0\|}$$
$$+\gamma\|F(x_0)-F(x_0)\| + \emptyset\|x_0-x_0\|$$
$$+\delta\|x_0-F(x_0)\| + \varphi\|F(x_0)-x_0\|$$
$$+\eta\|F(x_0)-x_0\|$$
$$= \frac{\beta}{2}\|F(x_0)-x_0\| + (\delta+\varphi)\|F(x_0)-x_0\| + \eta\|F(x_0)-x_0\|$$
$$\leq (\tfrac{\beta}{2}+\delta+\varphi+\eta)\|F(x_0)-x_0\|$$

i.e.

$$\|F(x_0)-x_0\| \leq (\tfrac{\beta}{2}+\delta+\varphi+\eta)\|F(x_0)-x_0\|$$

Since $(\frac{\beta}{2}+\delta+\varphi+\eta) < 1$

We must have $\|F(x_0)-x_0\| = 0$

$\therefore F(x_0) = x_0$

$\therefore$ x_0 is fixed point of F.

But $F(x_0) = G(x_0)$

$\therefore G(x_0) = x_0$.

$\therefore$ x_0 is common fixed point of F and G.

Now we only to show that x_0 is Unique fixed point of F and G.

If possible suppose that y_0 is another fixed point of F and G.

$$\|x_0 - y_0\| = \|F^2(x_0) - F^2(y_0)\|$$
$$= \|FF(x_0) - FF(y_0)\|$$
$$\leq \alpha \frac{\|FF(x_0)-GF(x_0)\|\|FF(y_0)-GF(y_0)\|}{\|FF(x_0)-GF(x_0)\|+\|FF(x_0)-GF(y_0)\|}$$
$$+\beta \frac{\|FF(y_0)-GF(x_0)\|\|FF(x_0)-GF(y_0)\|}{\|FF(y_0)-GF(x_0)\|+\|FF(x_0)-GF(y_0)\|}$$
$$+\gamma\|FF(x_0)-GF(x_0)\| + \emptyset\|FF(y_0)-GF(y_0)\|$$
$$+\delta\|FF(y_0)-GF(x_0)\| + \varphi\|FF(x_0)-GF(y_0)\| + \eta\|GF(x_0)-GF(y_0)\|$$
$$= \alpha \frac{\|x_0-x_0\|\|y_0-y_0\|}{\|x_0-x_0\|+\|x_0-y_0\|} + \beta \frac{\|x_0-y_0\|\|y_0-x_0\|}{\|x_0-y_0\|+\|y_0-x_0\|}$$
$$+\gamma\|x_0-x_0\| + \emptyset\|y_0-y_0\| + \delta\|x_0-y_0\| + \varphi\|y_0-x_0\|$$
$$+\eta\|x_0-y_0\|$$
$$= \alpha(0) + \frac{\beta}{2}\|x_0-y_0\| + (\gamma+\emptyset)(0) + (\delta+\varphi)\|x_0-y_0\| + \eta\|x_0-y_0\|$$
$$= \left(\frac{\beta}{2}+\delta+\varphi+\eta\right)\|x_0-y_0\|$$

i.e. $\|x_0-y_0\| \leq \left(\frac{\beta}{2}+\delta+\varphi+\eta\right)\|x_0-y_0\|$

since $\left(\frac{\beta}{2}+\delta+\varphi+\eta\right) < 1$

we must have

$\|x_0-y_0\| = 0$

$\therefore x_0 = y_0$

$\therefore$ x_0 is Unique fixed point of F and G.

2.3 Let $\boldsymbol{X}$ be banach space and $\boldsymbol{K}$ be closed and convex subset of $\boldsymbol{X}$. Let $\boldsymbol{F}, \boldsymbol{G}$ **and** $\boldsymbol{H}$ be three mapping from $\boldsymbol{K}$ into itself such that

$FG = GF, GH = HG$ and $FH = HF$ ……………………………….2.10

$F^2 = I$, $G^2 = I$, and $H^2 = I$, where I is identity mapping ………..2.11

$$\|F(x)-F(y)\| \le \alpha \frac{\|F(x)-GH(x)\|\|F(y)-GH(y)\|}{\|F(x)-GH(x)\|+\|F(y)-GH(X)\|} + \beta \frac{\|F(y)-GH(x)\|\|F(x)-GH(y)\|}{\|F(y)-GH(x)\|+\|F(x)-GH(y)\|}$$
$$+\gamma\|F(x)-GH(x)\| + \emptyset\|F(y)-GH(y)\|$$
$$+\delta\|F(y)-GH(x)\| + \varphi\|F(x)-GH(y)\|$$
$$+\eta\|GH(x)-GH(y)\| \quad \ldots\ldots 2.12$$

For every $x, y \in K$, where $0 \le \alpha, \beta, \gamma, \emptyset, \delta, \varphi, \eta$ and $7\alpha + 8\beta + 4\gamma < 8$.

……………..2.13

Then there exist at least on fixed point $x_0 \in K$ such that
$F(x_0) = GH(x_0) = x_0$ and $FG(x_0) = H(x_0)$. Moreover

x_0 is Unique fixed point of mapping F, G and H, if $\frac{\beta}{2} + \delta + \varphi + \eta < 1$.

Proof:

From (2.10) and (2.11) it follows that Occasionally Weakly Compatible
$(FGH)^2 = I$.

Again from (2.11) and (2.12) we have

$$\|FGH(x)-FGH(y)\| \le \alpha \frac{\|FGH.G(x)-(GH)^2.G(x)\|\|FGH.G(y)-(GH)^2.G(y)\|}{\|FGH.G(x)-(GH)^2.G(x)\|+\|FGH.G(y)-(GH)^2.G(X)\|}$$
$$+\beta \frac{\|FGH.G(y)-(GH)^2.G(x)\|\|FGH.G(x)-(GH)^2.G(y)\|}{\|FGH.G(y)-(GH)^2.G(x)\|+\|FGH.G(x)-(GH)^2.G(y)\|}$$
$$+\gamma\|FGH.G(x)-(GH)^2.G(x)\| + \emptyset\|FGH.G(y)-(GH)^2.G(y)\|$$
$$+\delta\|FGH.G(y)-(GH)^2.G(x)\| + \varphi\|FGH.G(x)-(GH)^2.G(y)\|$$
$$+\eta\|(GH)^2.G(x)-(GH)^2.G(y)\|$$
$$= \alpha \frac{\|FGH.G(x)-G(x)\|\|FGH.G(y)-G(y)\|}{\|FGH.G(x)-G(x)\|+\|FGH.G(y)-G(X)\|}$$
$$+\beta \frac{\|FGH.G(y)-G(x)\|\|FGH.G(x)-G(y)\|}{\|FGH.G(y)-G(x)\|+\|FGH.G(x)-G(y)\|}$$
$$+\gamma\|FGH.G(x)-G(x)\| + \emptyset\|FGH.G(y)-G(y)\|$$
$$+\delta\|FGH.G(y)-G(x)\| + \varphi\|FGH.G(x)-G(y)\|$$
$$+\eta\|G(x)-G(y)\|$$

Taking $G(x) = u$ and $G(y) = v$, we get

$$\|FGH(u)-FGH(v)\| \le \alpha \frac{\|FGH(u)-u\|\|FGH(v)-v\|}{\|FGH(u)-u\|+\|FGH(v)-u\|}$$
$$+\beta \frac{\|FGH(v)-u\|\|FGH(u)-v\|}{\|FGH(v)-u\|+\|FGH(u)-v\|}$$
$$+\gamma\|FGH(u)-u\| + \emptyset\|FGH(v)-v)\|$$
$$+\delta\|FGH(v)-u\| + \varphi\|FGH(u)-v\|$$
$$+\eta\|u-v\|$$

Since $(FGH)^2 = I$ and $7\alpha + 8\beta + 4\gamma < 8$

So by theorem 2.1 FGH has at least one fixed point say $x_0 \in K$.

i.e. $FGH(x_0) = x_0$

$\therefore GH(FGH)(x_0) = GH(x_0)$ or $F(x_0) = GH(x_0)$.

Also $H(FGH)(x_0) = H(x_0)$ or $FG(x_0) = H(x_0)$

Now by using (2.10),(2.11),(2.12).

We have

$$\|H(x_0)-x_0\| = \|FG(x_0)-F^2(x_0)\|$$
$$= \|FG(x_0)-F(F(x_0))\|$$
$$= \propto \frac{\|H(x_0)-H(x_0)\|\|x_0-x_0\|}{\|H(x_0)-H(x_0)\|+\|x_0-H(x_0)\|}$$
$$+\beta \frac{\|x_0-H(x_0)\|\|H(x_0)-x_0\|}{\|x_0-H(x_0)\|+\|H(x_0)-x_0\|}$$
$$+\gamma\|H(x_0)-H(x_0)\| + \emptyset\|x_0-x_0\|$$
$$+\delta\|x_0-H(x_0)\| + \varphi\|H(x_0)-x_0\|$$

$$+\eta\|H(x_0)-x_0\|$$
$$=\left(\frac{\beta}{2}+\delta+\varphi+\eta\right)\|H(x_0)-x_0\|$$

$\therefore\ \|H(x_0)-x_0\|\le\left(\frac{\beta}{2}+\delta+\varphi+\eta\right)\|H(x_0)-x_0\|$

Since $\left(\frac{\beta}{2}+\delta+\varphi+\eta\right)<1$

It occasionally weakly compatible that $H(x_0)=x_0$.

i.e. x_0 is fixed point of H.

$\therefore G(x_0)=F(x_0)$.

Again

$$\|F(x_0)-x_0\|=\|F(x_0)-F^2(x_0)\|$$
$$=\|F(x_0)-F(F(x_0))\|$$
$$\le\alpha\frac{\|F(x_0)-GH(x_0)\|\|FF(x_0)-GHF(x_0)\|}{\|F(x_0)-GH(x_0)\|+\|FF(x_0)-GH(x_0)\|}$$
$$+\beta\frac{\|FF(x_0)-GH(x_0)\|\|F(x_0)-GHF(x_0)\|}{\|FF(x_0)-GH(x_0)\|+\|F(x_0)-GHF(x_0)\|}$$
$$+\gamma\|F(x_0)-GH(x_0)\|+\emptyset\|FF(x_0)-GHF(x_0)\|$$
$$+\delta\|FF(x_0)-GH(x_0)\|+\varphi\|F(x_0)-GHF(x_0)\|$$
$$+\eta\|GH(x_0)-GHF(x_0)\|$$
$$=\alpha\frac{\|F(x_0)-F(x_0)\|\|x_0-x_0\|}{\|F(x_0)-F(x_0)\|+\|x_0-F(x_0)\|}$$
$$+\beta\frac{\|x_0-F(x_0)\|\|F(x_0)-x_0\|}{\|x_0-F(x_0)\|+\|F(x_0)-x_0\|}$$
$$+\gamma\|F(x_0)-F(x_0)\|+\emptyset\|x_0-x_0\|$$
$$+\delta\|x_0-F(x_0)\|+\varphi\|F(x_0)-x_0\|$$
$$+\eta\|F(x_0)-x_0\|$$
$$=\left(\frac{\beta}{2}+\delta+\varphi+\eta\right)\|F(x_0)-x_0\|$$

$\therefore\|F(x_0)-x_0\|\le\left(\frac{\beta}{2}+\delta+\varphi+\eta\right)\|F(x_0)-x_0\|$

Since $\left(\frac{\beta}{2}+\delta+\varphi+\eta\right)<1$

$\therefore$ it occasionally weakly compatible that $F(x_0)=x_0$

But $F(x_0)=G(x_0)$.

$\therefore F(x_0)=G(x_0)=H(x_0)=x_0$

$\therefore\ x_0$ is the common fixed point of F, G and H.

Now we only to show that x_0 is unique fixed point.

If possible suppose that y_0 is another common fixed point of F, G and H.

From (3.2.7),(3.2.8) and (3.2.9) we get

$$\|x_0-y_0\|=\|F^2(x_0)-F^2(y_0)\|$$
$$=\|FF(x_0)-FF(y_0)\|$$
$$\le\alpha\frac{\|FF(x_0)-GHF(x_0)\|\|FF(y_0)-GHF(y_0)\|}{\|FF(x_0)-GHF(x_0)\|+\|FF(y_0)-GHF(x_0)\|}$$
$$+\beta\frac{\|FF(y_0)-GHF(x_0)\|\|FF(x_0)-GHF(y_0)\|}{\|FF(y_0)-GHF(x_0)\|+\|FF(x_0)-GHF(y_0)\|}$$
$$+\gamma\|FF(x_0)-GHF(x_0)\|+\emptyset\|FF(y_0)-GHF(y_0)\|$$
$$+\delta\|FF(y_0)-GHF(x_0)\|+\varphi\|FF(x_0)-GHF(y_0)\|$$
$$+\eta\|GHF(x_0)-GHF(y_0)\|$$
$$=\alpha\frac{\|x_0-x_0\|\|y_0-y_0\|}{\|x_0-x_0\|+\|y_0-x_0\|}$$
$$+\beta\frac{\|y_0-x_0\|\|x_0-y_0\|}{\|y_0-x_0\|+\|x_0-y_0\|}$$
$$+\gamma\|x_0-x_0\|+\emptyset\|y_0-y_0\|$$
$$+\delta\|y_0-x_0\|+\varphi\|x_0-y_0\|$$

$$+\eta\|x_0 - y_0\|$$
$$= \frac{\beta}{2}\|x_0 - y_0\| + (\delta + \varphi)\|x_0 - y_0\| + \eta\|x_0 - y_0\|$$
$$= \left(\frac{\beta}{2} + \delta + \varphi + \eta\right)\|x_0 - y_0\|$$

i.e $\|x_0 - y_0\| \le \left(\frac{\beta}{2} + \delta + \varphi + \eta\right)\|x_0 - y_0\|$

since $\left(\frac{\beta}{2} + \delta + \varphi + \eta\right) < 1$

∴ it occasionally weakly compatible that $x_0 = y_0$.

∴ F ,G and H has Unique common fixed point x_0.

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