On Trivial Zeros of Zeta Function and Problems with the Analytic Continuation

Manvendra Somvanshi

2G&H, 1st block, Doshi Llanstephen Apartments,

McNichols Road, Chetpet, Chennai 600 031,

Tamil Nadu, India.

Abstract- *This paper is based on the Riemann Hypothesis and trivial zeros of the zeta function. The paper shows the reasons of why the analytic continuation of zeta function is wrong in two peculiar ways. Identities are derived from the very definition of analytic functions that are contradictory to those given by Riemann hypothesis.*

I. Introduction

In the paper published by Bernhard Riemann in 1859 on 'the number of primes before a given quantity' he gave the famous Riemann Hypothesis on the zeta function. According to this hypothesis all the zeros of the zeta function are either negative even numbers or are imaginary numbers with real part ½.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \qquad (1.1)$$

This hypothesis was given by the analytic continuation of the zeta function. This paper will use these assumptions to derive contrary identities. The assumption of zeta function as an analytic function is based on the following expression:

$$\zeta(s) = \frac{\eta(s)}{1-2^{1-s}} \qquad (1.2)$$

Here $\eta(s)$ is the dirichlet eta function. Since the eta function is an analytic function, the relation would suggest that zeta function is also analytic. $\eta(s)$ is defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{s-1}}{n^s} \qquad (1.3)$$

The trivial zeros are the negative even integer roots of the zeta function. These were calculated by analytic continuation of the function. Trivial zeros are well understood by mathematicians yet they are the basis of this paper.

II. Trivial Zeros

The zeros of the zeta function and the eta function must be the same following from equation (1.3). This implies that trivial zeros are also the zeros of the eta function.

$$\eta(-2) = 0$$

Or $$1^2 - 2^2 + 3^2 \ldots. = 0 \qquad (2.1)$$

Using the above identity (2.1) we can determine the value of $\zeta(-1)$. This would include yet another identity which is would be derived in the next sub section.

A. Dirichlet Function at S= -1

At s=-1 the eta function takes a quite interesting form. At -1 the eta function can be written as

$$1 - 2 + 3 - 4 + 5\ldots. = \eta(-1)$$

Here η(-1) can be re-written as the square of η(0). By finding the value of eta at 0 and squaring it will give us the value of eta at -1.

$$\eta(0) = 1 - 1 + 1 - 1 + 1\ldots..$$

Since this is an infinite series the total sum can be calculated using the partial sum. There are two recurring partial sums 0 and 1. In this case the total sum would be the mean of the partial sums. Here the mean of the partial sum is ½. This implies that the sum is ½. The sum can also be calculated using the expression of the Geometric Progression. The reason it is not used here is that it is not completely correct. The range of the common ratio in that expression is 1>r>-1.

The value of eta at 0 is ½ therefore the sum of eta at -1 would be ¼.

$$\eta(-1) = \tfrac{1}{4} \qquad (2.2)$$

B. Finding Zeta at -1

The value of zeta at -1 can be found simply by using equation (1.3). By using this method, the value would be

$$\zeta(-1) = \frac{\eta(-1)}{1-2^{1+1}}$$

$$\text{Or} \quad \zeta(-1) = \frac{1/4}{-3}$$

$$\text{Or} \quad \zeta(-1) = \frac{-1}{12} \qquad (2.3)$$

This is the value of zeta function at -1 can also be seen in the famous letter written by S. Ramanujam to G.H. Hardy. But one more value can be obtained by adding the equations (2.1) and (2.2).

$$1^2 + 1 - 2^2 - 2 + 3^2 + 3\ldots.. = \tfrac{1}{4}$$

Or $$2 - 2(3) + 3(4) - 4(5)\ldots.. = \tfrac{1}{4}$$

Or $$2 + 3(2) + 5(2) + 7(2)\ldots\ldots = \tfrac{1}{4}$$

Or $$1 + 3 + 5 + 7\ldots\ldots = \tfrac{1}{8} \qquad (2.4)$$

Equation (2.4) gives us the sum of all positive odd integers. By substituting this in equation (2.1) sum of all even integers can be gained as $\frac{-1}{8}$. The sum of all even and odd numbers is simply $\zeta(-1)$.

$$\zeta(-1) = 0 \qquad (2.5)$$

There are two different values of zeta at -1, which are given by equations (2.3) and (2.5). Reductio ad Absurdum. We have reached at a point of contradiction. This suggests that there exists an assumption that might be wrong. After examining one can reach at the conclusion that the only assumption made is in the derivation was in equation (2.1). We had assumed that the roots of the zeta function are negative even integers. This contradiction might be suggesting that the Riemann hypothesis is wrong.

III. Problems with Analytic Continuation of Zeta Function

Analytic functions are functions which can be expressed as general polynomials. These polynomials might be finite or infinite depending on the function. The zeta function is an infinite series so, assuming it is analytic, it can be expressed as an infinite polynomial. Expressing the function as a polynomial would make it possible for us to get a general equation for the function.

$$\zeta(s) = a + bs^1 + cs^2 + ds^3 \ldots..$$

Or
$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = a + bs^1 + cs^2 + ds^3 \qquad (3.1)$$

A. General Equation of Zeta Function

For finding the coefficients of this polynomial we would differentiate equation (3.1) and substitute the value of s to be 0. The value the coefficients are:

$$a = 1 + 1 + 1 + 1 \ldots$$

Or
$$a = \zeta(0)$$

$$b = ln\, \frac{1}{2}\frac{1}{3}\frac{1}{4} \ldots \;\; ; \; c = \frac{b^2}{2!} \; ; \; d = \frac{b^3}{3!} \text{ ; and so on}$$

Substituting these coefficients in equation (3.1) we get the following expression

$$\zeta(s) = \zeta(0) + bs + \frac{(bs)^2}{2!} + \frac{(bs)^3}{3!} \ldots . \qquad (3.2)$$

This has a resemblance to Taylor expansion of e^x which is:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

By substituting the value of $x = bs$ the equation (3.2) can be written as:

$$\zeta(s) = \zeta(0) + e^{bs} - 1$$

$$\zeta(s) = \zeta(0) + e^{ln\, \frac{1}{2}\frac{1}{3}\frac{1}{4} \ldots ^{\,s}} - 1$$

$$\zeta(s) = \zeta(0) + \left(\frac{1}{2}\frac{1}{3}\frac{1}{4} \ldots\right)^s - 1 \qquad (3.3)$$

. The value of zeta at 0 can be determined using equation (1.3). Eta of 0 has a value ½ as shown before.

$$\zeta(0) = \frac{\frac{1}{2}}{1-2}$$

$$\zeta(0) = -\frac{1}{2}$$

The equation (3.3) after substituting the value of zeta at 0 is:

$$\zeta(s) = \frac{-3}{2} + \left(\frac{1}{2}\frac{1}{3}\frac{1}{4}\ldots\right)^{s} \qquad (3.4)$$

The above equation is a general equation of zeta function.

B. Contradictions from General Equation of Zeta Function

In the general form of zeta function, i.e. equation (3.4), $\frac{-3}{2}$ and $\frac{1}{2}\frac{1}{3}\frac{1}{4}\ldots$ are certainly constants. But this does not follow. By substituting different values of s, different values for $\frac{1}{2}\frac{1}{3}\frac{1}{4}\ldots$ can be gained.

For example:

For S=-2

$$0 = \frac{-3}{2} + \left(\frac{1}{2}\frac{1}{3}\frac{1}{4}\ldots\right)^{-2}$$

$$\frac{1}{2}\frac{1}{3}\frac{1}{4}\ldots = 0.8164$$

For S=-4

$$0 = \frac{-3}{2} + \left(\frac{1}{2}\frac{1}{3}\frac{1}{4}\ldots\right)^{-4}$$

$$\frac{1}{2}\frac{1}{3}\frac{1}{4}\ldots = 0.9036$$

There are infinite such values for the constant, these can be gained by substituting different values for s. This is again absurd. Mathematics does not allow such identities. This identity suggests that the general equation of the zeta function is wrong. But the general form was built over the assumption that zeta function is analytic. Therefore the zeta function is not an analytic function.

IV. Conclusion

The very foundations of mathematics are being challenged by considering the zeta function to be analytic. In this paper only two ways have been shown but their might exist many. From the Riemann hypothesis absurd and conspicuously wrong identities like 0= -1/12 and 0.8164=0.9036 emerge. These equations lead us to believe that the Riemann Hypothesis is wrong. This might have major implications in the field of mathematics. The equation that gives the distribution of prime numbers and many other studies are highly dependent on the Riemann hypothesis. This paper also suggests that *by equating a function to a different analytic function does not always imply that the function is analytic too.* This statement can reduced from the example of equation (1.3) where the zeta function was written in terms of an analytic function i.e. the eta function. This might seem intuitively wrong, but this example of the zeta function proves the statement.

References

[1] Bernhard Riemann, *The Number of Primes Less Than a Given Quantity,* 1859.