Heat Conduction and Sequence of Functions Containing Generalized Hypergeometric Function

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Abstract
The main aim of this paper is to obtain the solution of heat conduction partial differential equation pertaining to sequence of functions containing generalized hypergeometric function and the multivariable I- function defined by Prathima et. al. [7]. Some particular cases related to H- function of several variables and I- function of two variables given by Rathi et. al. [11] are mentioned.

Keywords: *Multivariable I-function, Heat conduction, Multivariable H-function, Sequence of functions.*

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1. Introduction

The Gauss Hypergeometric function is defined [8] as,

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} z^n \quad , \quad (|z| < 1\,, c \neq 0,-1,-2,...) \qquad ...(1.1)$$

A sequence $G_{(e,f)}^{(a,b,c,d)}(x;p,k,s)$ of functions containing generalized hypergeometric function is introduced by Rao et.al. [9] and represented in following manner:

$$G_{(e,f)}^{(a,b,c,d)}(x;p,k,s) = \frac{p^f}{f!}\sum_{g=0}^{\infty}\sum_{l=0}^{g}\left[\frac{(a)_{g-1}\,\Gamma c\,\Gamma\{(b+e(g-l)\}}{\Gamma\{(c+e(g-l)\}\,\Gamma b}\frac{x^{kg}}{(g-l)!\,l!}\frac{(-1)^l (a)_l\,\Gamma c\,\Gamma(b+el)}{\Gamma(c+el)\,\Gamma b}\left(\frac{d+s+kl}{p}\right)_f\right], \qquad ...(1.2)$$

where $x \in (0,\infty)$, p,k,s are constants and e$\in (0,\infty)$; $Re(a) > 0\,, Re(b) > 0, Re(c) > 0$.

Here we obtain the solution of partial differential equation concerned with heat conduction. Let the partial differential equation is

$$\frac{\partial f}{\partial t} = \zeta\frac{\partial^2 f}{\partial t^2} - \zeta f x^2 \qquad ...(1.3)$$

where $f(x,t) \to 0$ for large of t and when $x \to \infty$; this equation is concerned with the problem of heat conduction given by Churchil [5].

$$\frac{\partial f}{\partial t} = \zeta\frac{\partial^2 f}{\partial t^2} - \varphi(f - f_0) \ ,$$

provided that $f_0 = 0 \; and \; \varphi = fx^2$.

Here we consider that

$$f(x) = x^{2\theta} e^{-x^2} G_{(e,f)}^{(a,b,c,d)}(x;p,k,s)\, I\begin{pmatrix} z_1 x^{2\lambda_1} \\ \vdots \\ z_r x^{2\lambda_r} \end{pmatrix} \qquad \ldots(1.4)$$

The multivariable *I*-function introduced by Prathima et al. [7] is represented and defined in terms of Mellin –Barnes type integral as follows:

$$I(z_1,\ldots,z_r) = I^{0,I:(H_1,I_1);\ldots;(H_r,I_r)}_{J,K:(J_1,K_1);\ldots;(J_r,K_r)}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle| \begin{array}{l} (a_j,\theta_j',\ldots,\theta_j^{(r)};A_j)_{1,J}:(b_j',\phi_j';C_j')_{1,J_1};\ldots;(b_j^{(r)},\phi_j^{(r)};C_j^{(r)})_{1,J_r} \\ (c_j;\psi_j',\ldots,\psi_j^{(r)};B_j)_{1,K}:(d_j',\delta_j';D_j')_{1,K_1},\ldots,(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,K_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r} U_1(s_1)\ldots U_r(s_r)\, V(s_1,\ldots,s_r)\, z_1^{s_1}\ldots z_r^{s_r}\, ds_1\ldots ds_r, \qquad \ldots(1.5)$$

where $\omega = \sqrt{-1}$,

$$U_i(s_i) = \frac{\prod_{j=1}^{H_i}\Gamma^{D_j^i}(d_j^{(i)} - \delta_j^{(i)}s_i)\prod_{j=1}^{I_i}\Gamma^{C_j^i}(1 - b_j^{(i)} + \phi_j^{(i)}s_i)}{\prod_{j=H_i+1}^{K_i}\Gamma^{D_j^i}(1 - d_j^{(i)} + \delta_j^{(i)}s_i)\prod_{j=I_i+1}^{J_i}\Gamma^{C_j^i}(b_j^{(i)} - \phi_j^{(i)}s_i)}, \forall\, i \in \{1,\ldots, r\} \qquad \ldots(1.6)$$

and

$$V(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{I}\Gamma^{A_j}\left(1 - a_j + \sum_{i=1}^{r}\theta_j^{(i)}s_i\right)}{\prod_{j=I+1}^{J}\Gamma^{A}\left(a_j - \sum_{i=1}^{r}\theta_j^{(i)}s_i\right)\prod_{j=1}^{K}\Gamma^{C_j}\left(1 - c_j + \sum_{i=1}^{r}\psi_j^{(i)}s_i\right)} \qquad \ldots(1.7)$$

an empty product is interpreted as unity, the coefficients $\theta_j^{(i)}, j = 1,\ldots, J$; $\phi_j^{(i)}, j = 1,\ldots, J_r; \psi_j^{(i)}, j = 1,\ldots, K$; $\delta_j^{(i)}, j = 1,\ldots, K_r$ are positive numbers and I, H_i, I_i, J, J_i, K, K_i are integrals such that $0 \le I \le J,\; 1 \le H_i \le K_i, K \ge 0$ and $0 \le I_i \le J_i$, $i = 1,\ldots,r$.

The contour L_i in the complex s_i-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary in such a manner that all poles of $\Gamma^{D_j^i}(d_j^{(i)} - \delta_j^{(i)}s_i)$, $j = 1,\ldots, H_i$ are to the right and those of $\Gamma^{C_j^i}(1 - b_j^{(i)} + \phi_j^{(i)}s_i)$,j =1,...,I_r to the left of s_i, the various parameters being so restricted that these poles are all simple and none of them coincide and with the points $z_i = 0$, i = 1,…,r, being tacitly excluded.

For more details , one can go through Prathima et. al. [7].

For convenience we use $P = (H_1, I_1);\ldots;(H_r, I_r)$, $Q = (J_1, K_1);\ldots;(J_r, K_r)$ …(1.8)

$$R = (c_j;\psi_j',\ldots,\psi_j^{(r)};B_j)_{1,K}:(d_j',\delta_j';D_j')_{1,K_1},\ldots,(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,K_r} \qquad \ldots(1.9)$$

$$S = (a_j, \theta_j', \ldots, \theta_j^{(r)}; A_j)_{1,J} : (b_j', \phi_j'; C_j')_{1,J_1}; \ldots; (b_j^{(r)}, \phi_j^{(r)}; C_j^{(r)})_{1,J_r}$$

2. Main Result:

For proving the main result we required the following Lemma:

$$\int_{-\infty}^{\infty} x^{2\theta} e^{-x^2} H_{2\mu}(x) dx = \frac{\sqrt{\pi}\, 4^{\mu-\theta}\, \Gamma(2\theta+1)}{\Gamma(\theta-\mu+1)} \qquad \ldots(2.1)$$

Theorem 2.1

$$\int_{-\infty}^{\infty} x^{2\theta} e^{-x^2} H_{2\mu}(x) G_{(e,f)}^{(a,b,c,d)}(x; p, k, s)\, I\begin{pmatrix} z_1 x^{2\lambda_1} \\ \vdots \\ z_r x^{2\lambda_r} \end{pmatrix} dx$$

$$= \sqrt{\pi}\, 4^{\mu-\theta} \frac{p^f}{f!} \sum_{g=0}^{\infty} \sum_{l=0}^{g} \left[\frac{(a)_{g-1}\, \Gamma c\, \Gamma\{(b+e(g-l)\}}{\Gamma\{(c+e(g-l)\}\, \Gamma b} \frac{2^{-gk}}{(g-l)!\, l!} \frac{(-1)^l (a)_l\, \Gamma c\, \Gamma(b+el)}{\Gamma(c+el)\, \Gamma b} \left(\frac{d+s+kl}{p}\right)_f \right].$$

$$I_{J+1,K+1:Q}^{0,I+1:P}\left[\begin{matrix} z_1 4^{-\lambda_1} \\ \vdots \\ z_r 4^{-\lambda_r} \end{matrix} \left| \begin{matrix} (-2\theta - kg;\, 2\lambda_1, \ldots, 2\lambda_r; 1), R \\ (\mu - \theta - \frac{kg}{2}; \lambda_1, \ldots, \lambda_r; 1), S \end{matrix} \right.\right] \qquad \ldots(2.2)$$

provided that $\min(\theta, \lambda_i) > 0, i = 1, \ldots, r; Re\left[1 + \sum_{i=1}^{r} \lambda_i \min_{1 \le j \le H_i} \frac{d_j^{(r)}}{\delta_j^{(r)}}\right] > 0$, for the convergence condition and condition of analyticity and more details of I -function, see Prathima et. al. [7].

Proof:

To prove (2.2) ,first put the value of $G_{(e,f)}^{(a,b,c,d)}(x; p, k, s)$ in series form given by (1.2) and express I-function of several variables in Mellin –contour integral and interchanging the order of integration permissible . Then evaluating the integral by using lemma and interpreting the Mellin –Barnes contour integral in multivariable I-function, we get the required result (2.2).

3. Solution:

The solution of equation (1.3) is

$$\psi(x,t) = \sum_{\alpha=0}^{\infty} \sum_{g=0}^{\infty} \sum_{l=0}^{g} \frac{2^{\alpha-2\theta-kg}}{\alpha!} \frac{p^f}{f!} \left[\frac{(a)_{g-1}\, \Gamma c\, \Gamma\{(b+e(g-l)\}}{\Gamma\{(c+e(g-l)\}\, \Gamma b} \frac{1}{(g-l)!\, l!} \frac{(-1)^l (a)_l\, \Gamma c\, \Gamma(b+el)}{\Gamma(c+el)\, \Gamma b} \left(\frac{d+s+kl}{p}\right)_f \right]$$

$$e^{(1+2\alpha)\zeta t - \frac{x^2}{2}}\, I_{J+1,K+1:Q}^{0,I+1:P}\left[\begin{matrix} z_1 4^{-\lambda_1} \\ \vdots \\ z_r 4^{-\lambda_r} \end{matrix} \left| \begin{matrix} (-2\theta - kg;\, 2\lambda_1, \ldots, 2\lambda_r; 1), R \\ (\frac{\alpha}{2} - \theta - \frac{kg}{2}; \lambda_1, \ldots, \lambda_r; 1), S \end{matrix} \right.\right] \qquad \ldots(3.1)$$

provided that $\min(\theta, \lambda_i) > 0, i = 1, \ldots, r; Re\left[1 + \sum_{i=1}^{r} \lambda_i \min_{1 \le j \le H_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$

Proof:

The solution of (1.3) can be written as ([2] ,page 360, Eq.2.3)

$$\psi(x,t) = \sum_{\alpha=0}^{\infty} U_\alpha \, e^{(1+2\alpha)\zeta t - \frac{x^2}{2}} H_\alpha(x), \text{ where } H_\alpha(x) \text{ is the Hermite polynomial.} \quad \ldots(3.2)$$

If t = 0 ,then by (1.4), we have

$$x^{2\theta} e^{-x^2} G_{(e,f)}^{(a,b,c,d)}(x; p, k, s)\, I\begin{pmatrix} z_1 x^{2\lambda_1} \\ \vdots \\ z_r x^{2\lambda_r}\end{pmatrix} = \sum_{\alpha=0}^{\infty} U_\alpha \, e^{-\frac{x^2}{2}} H_\alpha(x) \quad \ldots(3.3)$$

Now multiplying both sides by $H_\beta(x)$ and integrating from $-\infty$ *to* ∞ with respect to x and using orthogonal property of Hermite Polynomials [6] and result (2.2) , We have

$$U_\beta = \sum_{g=0}^{\infty}\sum_{l=0}^{g} \frac{2^{\alpha-2\theta-kg}}{\alpha!}\frac{p^f}{f!}\left[\frac{(a)_{g-1}\,\Gamma c\,\Gamma\{(b+e(g-l)\}}{\Gamma\{(c+e(g-l)\}\,\Gamma b}\frac{1}{(g-l)!\,l!}\frac{(-1)^l (a)_l\,\Gamma c\,\Gamma(b+el)}{\Gamma(c+el)\,\Gamma b}\left(\frac{d+s+kl}{p}\right)_f\right].$$

$$e^{(1+2\alpha)\zeta t - \frac{x^2}{2}} I_{J+1,K+1:Q}^{0,I+1:P}\left[\begin{matrix} z_1 4^{-\lambda_1} \\ \vdots \\ z_r 4^{-\lambda_r}\end{matrix}\middle| \begin{matrix}(-2\theta - kg\,;2\lambda_1, \ldots, 2\lambda_r;1), R \\ (\frac{\alpha}{2} - \theta - \frac{kg}{2}; \lambda_1, \ldots, \lambda_r;1), S\end{matrix}\right] \quad \ldots(3.4)$$

Now using (3.2) and (3.4) , we obtain the solution (3.1)

4. Special Cases:

(i) If r = 2 then the multivariable I-Function reduces to I –function of two variables defined by Rathie et.al. [11]

and we obtain the solution as

$$\psi(x,t) = \sum_{\alpha=0}^{\infty}\sum_{g=0}^{\infty}\sum_{l=0}^{g} \frac{2^{\alpha-2\theta-kg}}{\alpha!}\frac{p^f}{f!}\left[\frac{(a)_{g-1}\,\Gamma c\,\Gamma\{(b+e(g-l)\}}{\Gamma\{(c+e(g-l)\}\,\Gamma b}\frac{1}{(g-l)!\,l!}\frac{(-1)^l (a)_l\,\Gamma c\,\Gamma(b+el)}{\Gamma(c+el)\,\Gamma b}\left(\frac{d+s+kl}{p}\right)_f\right]$$

$$e^{(1+2\alpha)\zeta t - \frac{x^2}{2}} I_{J+1,K+1:Q}^{0,I+1:P}\left[\begin{matrix} z_1 4^{-\lambda_1} \\ z_r 4^{-\lambda_2}\end{matrix}\middle| \begin{matrix}(-2\theta - kg\,;2\lambda_1, 2\lambda_r;1), R \\ (\frac{\alpha}{2} - \theta - \frac{kg}{2}; \lambda_1, \lambda_r;1), S\end{matrix}\right] H_\alpha(x) \quad ..(4.1)$$

Under the same conditions and notations that (3.1)

(ii) For $a = b = c = e = 1$ we get

$$\psi(x,t) = \sum_{\alpha=0}^{\infty}\sum_{g=0}^{\infty}\sum_{l=0}^{g} \frac{2^{\alpha-2\theta-kg}}{\alpha!}\frac{p^f}{f!}\left[(-1)^l\left(\frac{d+s+kl}{p}\right)_f\right].$$

$$e^{(1+2\alpha)\zeta t-\frac{x^2}{2}} I^{0,I+1:P}_{J+1,K+1:Q}\left[\begin{array}{c} z_1 4^{-\lambda_1} \\ \vdots \\ z_r 4^{-\lambda_r} \end{array}\middle| \begin{array}{c} (-2\theta-kg\ ; 2\lambda_1,\dots,2\lambda_r;1), R \\ (\frac{\alpha}{2}-\theta-\frac{kg}{2};\lambda_1,\dots,\lambda_r;1), S \end{array}\right] \quad ...(4.2)$$

(iii) If we replace

sequence $G^{(a,b,c,d)}_{(e,f)}(x;p,k,s)$ of functions by $\bar{I}$ -function introduced by Rathie [10], then we obtain the results given by Ayant [1].

5. Conclusion:

A large number of results can be evaluated by suitable selection of parameters of the multivariable I-function and the sequence of function used which are useful in Applied Mathematics, Physics, Mechanics and other field of Science and Technology. The result as obtained here are of general nature and may be useful in various existing situations in the literature of Science.

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