On (N+K) Power Class (Q) Operators in the Hilbert Space - I

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Abstract - *This article is focussed on characterizing new theorems and verifying examples on some properties of operators in (n+k) power class (Q)for any k ≥ 0 and for particular integer n in the Hilbert Space. Further we introduce a new class of n power quasi n normal operator acting on the complex Hilbert space H.*

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I. INTRODUCTION

Let H be a Hilbert space and L(H) be the algebra of all bounded linear operators acting on H. A.A.S. Jibril [1], in 2008 introduced the class of n – power normal operators as a generalization of normal operators. The operator T is called n – power normal if T^n commutes withT^*, i.e. $T^nT^* = T^*T^n$ and is denoted by [nN]. In 2010, Adnan A.S. Jibril [2], introduced class (Q) operators acting on the Hilbert space. An operator T ∈ L(H) is in class (Q) if it satisfies the condition $T^{*^2}T^2 = (T^*T)^2$. In 2012, S. Panayappan and N. Sivamani [3] defined n power class (Q) operators on the Hilbert space. An operator T ∈ L(H) is said to be n power class (Q) if it satisfies the condition $T^{*^2}T^{2n} = (T^*T^n)^2$. In 2013, Krutan Rasimi, Luigj Gjoka [4] gave some results related to n power class (Q) operators acting on infinite complex separable Hilbert space.In the year 2014 Dr. T. Veluchamy, K.M.Manikanadan and T.Ramesh [5] investigated some characterization of n power class (Q) operators on Hilbert space using MATLAB. A bounded linear operator T on a complex Hilbert space H is n-power quasi n-normal if $T^n(T^*T^n) = (T^*T^n)T^n$ that isT^ncommutes with T^*T^n for some integer n.

2. Main Results

In general (n+k) power class(Q) operator need not be a normal operator. Let us prove this result by giving the following example. By substituting n = 2 and k = 0 we claim that $T = \begin{pmatrix} -i & 0 \\ 2 & i \end{pmatrix}$ is 2 power class(Q) operator but not normal. Further we can prove that T is not 3 power class(Q)operator but 2 power quasi 2 normal operator.

Example2.1:

Consider the operator T =$\begin{pmatrix} -i & 0 \\ 2 & i \end{pmatrix}$ acting on C^2 which is 2 power class(Q) and T is not normal operator, but not 3 power class(Q) operator and 2 power quasi 2 normal operator

Solution:

On direct matrix multiplication operations, we show that $T^{*^2}T^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (T^*T^2)^2$. Hence, T is 2 power class(Q operator. Further we show that $T^*T = \begin{pmatrix} 5 & 2i \\ -2i & 1 \end{pmatrix}$ and $TT^* = \begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix}$. $T^*T \neq TT^*$ and therefore T is not normal operator. In addition we can verify that $T^{*^2}T^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(T^*T^3)^2 = \begin{pmatrix} 29 & 12i \\ -12i & 5 \end{pmatrix}$. Here $T^{*^2}T^6 \neq (T^*T^3)^2$, and therefore T is not 3 power class (Q) operator. Further, we can prove that T is 2 power quasi 2 normal operator by verifying $T^2(T^*T^2) = \begin{pmatrix} i & 2 \\ 0 & -1 \end{pmatrix} = (T^*T^2)T^2$

The following example shows that an operator of 3 power class (Q) need not be of 2 power class (Q).

Example 2.2:

Consider the operator T $=\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ acting on R^2 which is 3 power class(Q)but not class (Q)operator, not 2 power class(Q), but 3 – normal operator.

Solution:

On direct matrix multiplication operations, we can verify that $T^{*^2}T^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = (T^*T)^2$ and therefore T is not class (Q) operator and $T^{*^2}T^4 = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = (T^*T^2)^2$, and therefore T is not 2 power class (Q) operator. Further, we can show that, $T^{*^2}T^6 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = (T^*T^3)^2$ and therefore T is 3 power class (Q) operator. Also, $T^3T^* = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = T^*T^3$ this implies that T is 3 normal operator.

The following theorems and definitions are used to derive third part of proposition 2.3.

([6]: Theorem 3: Page No. 158)

Let M be a closed linear subspace of H.

(i) If M is invariant under the operator T, then $T/_M$ is an operator in the Hilbert space H and $\|T/_M\| \leq \|T\|$.

(ii) If M is invariant under the operator S and T, then it is invariant under S+T, ST and λTmoreover,
$(S+T)/_M = (S/_M) + (T/_M)$, $(ST)/_M = (S/_M)(T/_M)$, $(\lambda T)/_M = \lambda(T/_M)$ holds.

([6] Definition 2: Page no. 158)

Let M be a closed Linear subspace invariant under the operator T. The restriction of T to M is the mapping $T/_M : M \to M$ defined by $(T/_M)y = Ty$.Thus $T/_M$ is themapping y-> Ty restricted to M.

([6]: Theorem 5: Page No. 159) If M reduces T, then $(T/_M)^* = T^*/M$

([6] Definition 1: Page no. 156) A closed linear subspace M is said to be invariant under the operator T if T(M)⊂M.

([6] Definition 3: Page no. 158) A closed Linear subspace M is said to reduce the operator T incase both M and $M^\perp$
are invariant under T. ie., T(M) ⊂ M and T($M^\perp$) ⊂ $M^\perp$.

Proposition 2.3:

If T∈(n+k) power class(Q) then so are

(i) λT for any real numberλ.

(ii) for any S∈ $L(H)$ that is unitarily equivalent to T.

(iii) The restriction T/M of T to any closed subspace M of H that reduces T.

Proof:

(i) Since T∈(n+k) power class(Q) we get,$T^{*^2}T^{2(n+k)} = (T^*T^{n+k})^2$.

Replace T by λT we get

$$(\lambda T)^{*^2}(\lambda T)^{2(n+k)} = (\lambda T)^*(\lambda T)^*(\lambda T)^{n+k}(\lambda T)^{n+k}$$
$$= \bar{\lambda}T^*\bar{\lambda}T^*\lambda^{n+k}T^{n+k}\lambda^{n+k}T^{n+k}$$
$$= \lambda T^* \ \lambda T^* \ (\lambda^{n+k})^2 T^{n+k}T^{n+k} \quad [\because \lambda is real \lambda = \bar{\lambda}]$$
$$= \lambda^2((\lambda T)^*(\lambda T)^{n+k})^2. \ \therefore \lambda T \in (n+k) \text{ power class}(Q).$$

(ii) S∈ $L(H)$ is unitarily equivalent to T, then there is a unitary operator U∈ $L(H)$ such that
$S^{2n} = U^*T^{2n}U => S^* = U^*T^*U; S^{2(n+k)} = U^*T^{2(n+k)}U$

$$S^{*^2}S^{2(n+k)} = U^*T^*UU^*T^*UU^*T^{2(n+k)}U$$
$$= U^*T^*T^*T^{2(n+k)}U$$
$$= U^*T^{*^2}T^{2(n+k)}U \text{ ---------(1)}$$

and $(S^*S^{n+k})^2 = (U^*T^*UU^*T^{(n+k)}U)^2$

$$= U^*T^*UU^*T^{(n+k)}UU^*T^*UU^*T^{(n+k)}U$$
$$= U^*T^*T^{(n+k)}T^*T^{(n+k)}U$$
$$= U^*(T^*T^{(n+k)})^2U \quad \text{.............(2)}$$

Since T∈(n+k) power class(Q) we have

$T^{*^2}T^{2(n+k)} = (T^*T^{n+k})^2$ ...., (3)

From equations (1), (2) and (3) we get

$S^{*^2}S^{2(n+k)} = (S^*S^{n+k})^2$. Thus S∈(n+k) power class(Q).

(iii) By [1 Theorem 3, P.No. 158]

$$(T/_M)^{*^2}(T/_M)^{2(n+k)} = (T^{*^2}/_M)(T^{2(n+k)}/_M)$$
$$= (T^{*^2}T^{2(n+k)}/_M)$$

$= {(T^*T^{n+k})^2}/{M}$

$= (({T}/{M})^*({T}/{M})^{n+k})^2$

Thus ${T}/{M} \in$(n+k) power class(Q).

Theorem 2.4:

If the sum of two bounded linear operators is (2+k) normal then their sum is (2+k) power class (Q) operator.

The product of any two operators on the complex Hilbert space is (2+k) power class(Q) operator if their product is (2+k) normal.

Proof:

Let S and T be operators on Hilbert space H. Given S+T is (2+k) normal operator. Therefore from the definitionwe get$(S+T)^*(S+T)^{2+k} = (S+T)^{2+k}(S+T)^*$

(i) We have to prove that

$(S+T)^{*^2}(S+T)^{2(2+k)} = ((S+T)^*(S+T)^{2+k})^2$.

Consider

$$(S+T)^{*^2}(S+T)^{2(2+k)} = (S+T)^*(S+T)^*(S+T)^{2+k}(S+T)^{2+k}$$
$$= (S+T)^*(S+T)^{2+k}(S+T)^*(S+T)^{2+k}$$
$$= ((S+T)^*(S+T)^{2+k})^2 \qquad [\because \text{(S+T)is (2+k) normal }]$$

Therefore, (S+T) is (2+k) power class (Q) operator.

(ii) Given ST is (2+k) normal operator. Therefore from the definition we get $(ST)^*(ST)^{2+k} = (ST)^{2+k}(ST)^*$

Consider, $(ST)^{*^2}(ST)^{2(2+k)} = (ST)^*(ST)^*(ST)^{2+k}(ST)^{2+k}$

$$= (ST)^*(ST)^{2+k}\,(ST)^*(ST)^{2+k} \quad [\because \text{(ST) is (2+k)normal}$$
$$= (ST)^{*^2}(ST)^{2+k}(ST)^{2+k}$$

Hence ST is also (2+k) power class (Q) operator.

The following proposition shows that an operator T of 2 power class (Q) need not be of class (Q) such that $T^2 = 0$.

Proposition2.5:

If T$\in$ $(n+k)$ power class(Q) such that$T^2 = 0$, then it is not necessarily that T=0. Consider T=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ acting on R^2 which is not normal.

Proof:

We verify the theorem for n=2, k=0. On direct matrix multiplication operations we can verify that $T^{*^2}T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (T^*T)^2$. Therefore T is not class (Q). But$T^{*^2}T^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (T^*T^2)^2$.Hence T is 2 power class (Q). Further$T^*T \neq TT^*$, hence T is not normal. Hence, we conclude that, $T \notin (Q)$ But, T$\in$ 2 power class (Q) and T is not normal.

The following theorem shows that an operator of (n+k) normal is (n+k) power class (Q) operator.

Theorem2.6:

If T$\in$L(H) is (n+k) normal, then T$\in$(n+k) power class (Q).

Proof:

Since T is (n+k) normal then $T^*T^{n+k} = T^{n+k}T^*$

Pre multiply by T^* and post multiply by T^{n+k} on both sides we get

$T^*T^*T^{n+k}T^{n+k} = T^*T^{n+k}T^*T^{n+k}$

$\Rightarrow T^{*^2}T^{2(n+k)} = (T^*T^{n+k})^2$.

Therefore, we get T$\in$ (n+k) power class (Q).

Proposition 2.7:

Let T be an operator acting on three dimensional complex Hilbert space. Then T is (2+k). power class(Q) but it is not (2+k) normal. Example:= $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Solution:

For k = 0 and n = 2 we can prove that T is not normal but it is 2 power class (Q).

By direct decomposition we get $TT^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $T^*T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since, $TT^* \neq T^*T$, T is not normal. Further we can show that $T^{*^2}T^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (T^*T^2)^2$. Therefore T∈ 2 $powerclass(Q)$.

Proposition 2.8:

Let T be an operator acting on 2 dimensional complex Hilbert space. Then T is 2 power class(Q) but it is not 3 power class (Q). Example: T=$\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$.

Solution:

On direct matrix multiplication operations, we show that

$T^{*^2}T^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (T^*T^2)^2$. Hence, T is 2 power class (Q operator. Further we show that $TT^* = \begin{pmatrix} 5 & 2i \\ -2i & 1 \end{pmatrix}$ and $T^*T = \begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix}$. $T^*T \neq TT^*$ and therefore T is not normal operator. In addition we can verify that $T^{*^2}T^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(T^*T^3)^2 = \begin{pmatrix} 5 & -8i \\ 8i & 13 \end{pmatrix}$. Here $T^{*^2}T^6 \neq (T^*T^3)^2$, and therefore T is not 3 power class (Q) operator.

Definition 2.9:

Let E be a normal Linear space and L(E) denote the set of all bounded Linear operators on E.

An operator T∈ $L(E)$ is said to be regular if and only if there exists S ∈ $L(E)$ such that TS=ST=I such an operator S is called inverse of T so that S= T^{-1}.

In general, (n+k+1) power class(Q) operators need not be (n+k) power class (Q),

Lemma 2.10:

Let T∈ $L(E)$ the operator T is regular if and only if it is a linear transformation of E into itself. If T is regular then the inverse mapping T^{-1} is in L(E) and is the inverse of T in the sense of the definition.

We provethe following characterization using above lemma and definition.

Theorem2.11:

Let T be an operator acting on two dimensional complex Hilbert space H. Then T is (3+k) power class (Q) butit is not (2+k) power class (Q) operator. While attempting to prove this problem the generalized version for the above problem is formed and restated as follows.

Let T be the class of (n+1+k) power class (Q) operator acting on infinite complex separable Hilbert space H for any k≥0. If T is normal, then in T∈ $(n+k)$power class (Q).

Proof:

Given T is (n+1+k) power class (Q) operator.

∴ $T^{*^2}T^{2(n+1+k)}=(T^*T^{(n+1+k)})^2$ is true. Further, we have

$$\begin{aligned} T^{*^2}T^{2(n+k)}T^2 &= T^*T^{n+1+k}.T^*T^{n+1+k} \\ &= T^*T^{n+k}T.T^*T^{n+k}T \\ &= T^*T^{n+k}T^*TT^{n+k}T [\because \text{T is normal }] \\ &= T^*T^{n+k}T^*T^{1+n+k}T \\ &= T^*T^{n+k}T^*T^{n+k}T.T \\ &= T^*T^{n+k}T^*T^{n+k}T^2 \end{aligned}$$

If T is invertible, then post multiply by T^{-1} on both sides we get

$$T^{*^2}T^{2(n+k)}T^2T^{-1} = (T^*T^{n+k})^2T^2T^{-1}$$
$$T^{*^2}T^{2(n+k)}T = (T^*T^{n+k})^2T$$

Again post multiply byT^{-1} we get

$T^{*^2}T^{2(n+k)} = (T^*T^{n+k})^2$. Hence T is (n+k) power class (Q)

In the following example, we show that a 3 power class (Q) operator need not be 2 power class (Q).

The following examplesupports that (3+k) power class (Q) operator need not be (2+k) power class (Q) operator for k = 0.

Example 2.12:

An operator $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$acting on 2 dimensional complex separable Hilbert Space H is 3 power class(Q) but not 2 power class(Q).

Solution:

By using matrix multiplication we can show that obtain the values of $T^{*^2}T^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = (T^*T)^2$ and therefore$T \notin 2\, powerclass(Q)$.Also, $T^{*^2}T^6 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = (T^*T^3)^2$and hence$T \in 3\, powerclass(Q)$.

In the following example we show that an operator of 3 power class (Q) is 2 power class (Q).

Example 2.13:

Let $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be 3 power class (Q) operator acting on 2 dimensional complex separable Hilbert Space H. If T is regular then T is 2 power class (Q).

Solution:

$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, T^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $TT^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Further, $T^*T = I$ and $TT^* = T^*T = I$. $\therefore$ T is regular. Also,

$T^*T = TT^*$. $\therefore T$is normal. Further,$T^{*^2}T^4$=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (T^*T^2)^2$and therefore, $T \in 2\ powerclass(Q)$.Also,$T^{*^2}T^6$=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (T^*T^3)^2$ and therefore $T \notin 3$ power class (Q).

In the following theorem we show that an (n+k) power class (Q and quasi (n+k) normal operator T is (n+1+k) power class(Q) operator.

Conclusion:

The algebra of all bounded Linear operators on a Real (Complex) Hilbert space H is taken for the research work and the following findings on the class of (n+k) power class (Q) operators are given. 1. In general (n+k) power class(Q) operator need not be a normal operator. This result is verified through counter example. 2. A 2 power class (Q) operator is not 3 power class (Q) operator and vice versa. Hence we conclude that n power class (Q) operators are independent. 3. If T∈(n+k) power class(Q) then so are (i) λT for any real numberλ. (ii) for any S∈ $L(H)$ that is unitarily equivalent to T. (iii) The restriction T/M of T to any closed subspace M of H that reduces T. 4. The sum and product of any two commuting operators on the complex Hilbert space are (2+k) power class (Q) operator. 5. If T∈ $(n + k)$ power class (Q) such that$T^2 = 0$, then it is not necessarily that T = 0. This result is verified by considering the operator T= $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ acting on R^2 which is not normal, n = 2 and k = 0.6. If T∈L(H) is (n+k) normal, then T ∈(n+k) power class (Q). 7. Let T be an operator acting on three dimensional complex Hilbert space. Then T is (2+k). power class(Q) but it is not (2+k) normal. Example: = $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$8.Let T be an operator acting on 2 dimensional complex Hilbert space. Then T is 2 power class (Q) but it is not 3 power class (Q). Example: T =$\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$. 9. Let T be the class of (n+1+k) power class (Q) operator acting on infinite complex separable Hilbert space H for any k ≥ 0. If T is regular and normal, then in T∈ $(n + k)$power class (Q). 10. An operator $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$acting on 2 dimensional complex separable Hilbert Space H is 3 power class (Q) but not 2 power class(Q). 11. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be 3 power class (Q) operator acting on 2 dimensional complex separable Hilbert Space H. If T is regular then T is 2 power class (Q).

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