On (N+K) Power Class(Q) Operators in the Hilbert Space - II

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***Abstract** - This article is focussed on characterizing new theorems and verifying examples on some properties of operators in (n+k) power class (Q)for any k ≥ 0 and for particular integer n in the Hilbert Space. Also, we characterize a condition for an operator T in class (Q) on H , in addition it is complex conjugate operator on H. Finally we introduce quasi n – posi normal operators on the Hardy space and the new characterizations were done.*

Mathematics Subject Classification: 47B20, 47A05.

Keywords:_Normal operator, class (Q) operator, n – power class (Q) operator, (n+k) power class (Q) operatorand complex symmetric operator._

I. INTRODUCTION

Let H be a Hilbert space and L(H) be the algebra of all bounded linear operators acting on H. A.A.S. Jibril [1], in 2008 introduced the class of n – power normal operators as a generalization of normal operators. The operator T is called n – power normal if T^n commutes withT^*, i.e. $T^nT^* = T^*T^n$ and is denoted by [nN]. In 2010, Adnan A.S. Jibril [2], introduced class (Q) operators acting on the Hilbert space. An operator T ∈ L(H) is in class (Q) if it satisfies the condition ${T^*}^2T^2 = (T^*T)^2$. In 2012, S. Panayappan and N. Sivamani [3] defined n power class (Q) operators on the Hilbert space. An operator T ∈ L(H) is said to be n power class (Q) if it satisfies the condition ${T^*}^2T^{2n} = (T^*T^n)^2$. In 2013, Krutan Rasimi, Luigj Gjoka [4] gave some results related to n power class (Q) operators acting on infinite complex separable Hilbert space.In the year 2014 Dr. T. Veluchamy, K.M.Manikanadan and T.Ramesh [5] investigated some characterization of n power class (Q) operators on Hilbert space using MATLAB

In the year 2015, Sen Zhu and Jiayin Zhao [7] explored the structure of the skew symmetric operators with disconnected spectra. They used complex symmetric operators on the Complex separable Hilbert space H. A map C on H is said to be an anti unitary operator if C is conjugate, linear, invertible and $\langle Cx, Cy\rangle = \langle y, x\rangle forallx \in H$.If, in addition, C^{-1} = C, then C is called a conjugation. In our work we use the definitions skew symmetric and complex symmetric operators given by S.M. Zagorodnyuk in [8]. An operator $T \in B(H)$ is said to be skew symmetric if there exists conjugation C on H such that CTC = -T*. T is said to be complex symmetric if CTC = T* for some conjugation C on H.Finally we characterize a condition for an operator T in class (Q) on H , in addition it is complex conjugate operator on H.

In third section, we introduce Quasi n posinormal operator as an extension of Quasi posi normal operator. An operator T in a Hilbert space H is quasi n posinormal if $(TT^*)^{2n} \le \mu\left({T^*}^2T^2\right)^n$. We derive new theorems on quasi n posinormal operators acting on the weighted Hardy Hilbert space.

Let f be an analytic map on the open disk D given by the Taylors series f(z) = $a_0 + a_1z + a_2z_2 + ...$ Let β = $\{\beta_n\}_{n=0}^{\infty}$be a sequence of positive numbers with $\beta_0 = 1$ and $\beta_{n+1} / \beta_n \longrightarrow 1$ as n ⟶ ∞. The set $H^2(\beta)$ of formal complex power series f(z) = $\sum_{n=0}^{\infty} a_n z^n$ such that $||f||_\beta^2 = \infty \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$. is a Hilbert space of functions analytic in the unit disk with the inner product.$< f, g >_\beta = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2$ for f as above and g(z) = $\sum_{n=0}^{\infty} b_n z^n$. Let D be the open unit disc in the complex plane and let T : D ⟶ D be an analytic self map of the unit disc and consider the corresponding composition operator C_T acting on $H^2(\beta)$ ie., $C_T f = f \circ T$, $f \in H^2(\beta)$. The operator C_T is not necessarily defined on all of $H^2(\beta)$.

II. MAIN RESULTS

In general (n+k) power class(Q) operator need not be a normal operator. In the following theorem we show that an (n+k) power class (Q and quasi (n+k) normal operator T is (n+1+k) power class(Q) operator.

Theorem 2.1

If T is (n+k) power class(Q) and T is quasi (n+k) normal, then T is (n+1+k) power class(Q).

Proof:

Given T is (n+k) power class(Q). $\therefore T^{*2}T^{2(n+k)} = (T^*T^{n+k})^2$

Post multiply by T^2 on both sides,

$T^{*2}T^{2(n+k)}T^2 = (T^*T^{n+k})^2T^2$

$= T^*T^{n+k}T^*T^{n+k}TT$--------(1)

Since T is quasi (n+k) normal, we get

$T(T^*T^{n+k}) = (T^*T^{n+k})T$

Becomes $T^{*2}T^{2(n+k+1)} = T^*T^{n+k}T(T^*T^{n+k})T$

$= T^*T^{n+k+1}T^*T^{n+k+1}$

$= (T^*T^{n+k+1})^2$. Therefore T is (n+1+k) power class(Q) for any k≥0.

Theorem 2.2:

If an operator T on C^2 is both invertible and (2+k) normal operator then the following results hold. (i) T is (2+k) power class(Q), (ii) T is not quasi (n+k) normal and (iii) T is not (3+k) power class(Q).

Proof:

(i) To prove T is (2+k) power class(Q), we have to prove that $T^{*2}T^{2(2+k)} = (T^*T^{2+k})^2$

$T^{*2}T^{2(2+k)} = T^*T^*T^{2+k}T^{2+k}$

$= T^*T^{2+k}T^*T^{2+k}$ (∵T is (2+k) normal)

$= (T^*T^{2+k})^2$. $\therefore T \in (2+k)$ power class(Q).

(ii) First we prove the result for n = 2. We have to prove that $T(T^*T^{2+k}) \neq (T^*T^{2+k})T$

Now consider $T(T^*T^{2+k}) = TT^{2+k}T^*$ [∵ T is (2+k) normal]

$= T^{3+k}T^* \neq (T^*T^{2+k})T$. $\therefore T$ *is* not quasi (2+k) normal operator.

Next, we have to prove that $T(T^*T^{n+k}) \neq (T^*T^{n+k})T$

$T(T^*T^{n+k}) = T(T^*T^{2+k+n-2})$ [∵ T is invertible T^{-2} exists]

$= TT^*T^{2+k}T^{n-2}$

$= TT^{2+k}T^*T^{n-2}$[∵T is (2+k) normal]

$= T^{3+k}T^*T^{n-2}$ ----------(1)

$(T^*T^{n+k})T = T^*T^{n+k}T = T^*T^{2+k}T^{n-2}T$

$= T^{2+k}T^*T^{n-1}$ -----(2)

Equations (1) and (2) are not same. Therefore we get, T is not quasi (n+k) normal.

(iii) Next, we prove that T is not (3+k) power class(Q) operator.

$T^{*2}T^{2(3+k)} = T^*T^*T^{3+k}T^{3+k}$

$= T^*T^*T^{2+k}TT^{3+k}$

$= T^*T^{2+k}T^*TT^{3+k}$

$= T^*T^{2+k}T^*T^{2+k}T^2$

$= (T^*T^{2+k})^2T^2 \neq (T^*T^{3+k})^2$. Hence the proof.

The above theorem can be verified by putting k = 0 and T= $\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$.

Example2.3

Consider the operator T=$\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ acting on C^2 which is 2 normal, 2 power class(Q), not quasi 2 normal and not 3 power class(Q).

Solution:

Now$T^* = \begin{pmatrix} -i & 0 \\ 2 & i \end{pmatrix}$ and by direct decomposition we show that$T^2T^* = \begin{pmatrix} i & 0 \\ -2 & -i \end{pmatrix} = T^*T^2$ and therefore T is 2 normal operators. Now again by direct de composition we show that $T^{*2}T^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (T^*T^2)^2$ and therefore T is 2 power class (Q) operator. Further we show that $T(T^*T^2) = \begin{pmatrix} -5 & -2i \\ 2i & -1 \end{pmatrix} \neq \begin{pmatrix} -1 & 2i \\ -2i & -5 \end{pmatrix} = (T^*T^2)T$ and therefore T is not quasi 2 normal and finally we can verify that $T^{*^2}T^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq (T^*T^3)^2 = \begin{pmatrix} 5 & -12i \\ 12i & 29 \end{pmatrix}$ and therefore T is not 3 power class (Q) operator.

In the following theorem we prove that if an invertible operator T is (n+k) power class(Q) then it is (n+k) normal operator for any $k \geq 0$.

Theorem2.4:

Let T be bounded linear operator. If T is (n+k) power class(Q) operator and has inverse then T is (n+k) normal operator for any k≥0.

Proof:

Let T∈ $L(H)$ and let T be a (n+k) power class(Q) operator. ie., $T^{*^2}T^{2(n+k)} = (T^*T^{n+k})^2$

We can write that $T^{*^2}T^{2(n+k)} = T^*T^{n+k}T^*T^{n+k}$-------(1)

Since T has inverse, T^* has also inverse and it is $(T^*)^{-1} = (T^{-1})^*$

Multiply equation (1) from the left by $(T^*)^{-1}$, we get $T^*T^{2(n+k)} = T^{n+k}T^*T^{n+k}$

This relation is equivalent to $T^*T^{n+k}T^{n+k} = T^{n+k}T^*T^{n+k}$ -------(2)

Multiply equation (2) by (n+k) times with T^{-1} from right, we get $T^*T^{n+k} = T^{n+k}T^*$ which shows that T is (n+k) normal operator for any k≥0.

Theorem2.5

Let T be bounded linear operator on H. If T is (n+2k) power class(Q) operator and in the same time is 2 power quasi n normal operator, then T is (n+2k+2) power class(Q) operator for any k≥0.

Proof:

Let T be a bounded linear operator on H.

Suppose T is (n+2k) power class(Q) operator then from the definition we get

$T^{*^2}T^{2(n+k)} = (T^*T^{n+k})^2$ --------(1)

Also, given that, T is 2 power quasi n normal operator

$\therefore T^2(T^*T^n) = (T^*T^n)\,T^2$ ------(2)

We have to prove that $T^{*^2}T^{2(n+2k+2)} = (T^*T^{n+2k+2})^2$

$$
\begin{aligned}
T^{*^2}T^{2(n+2k+2)} &= T^{*^2}T^{2(n+2k)}T^4 \\
&= (T^*T^{n+2k})^2TTTT \\
&= T^*T^{n+2k}T^*T^{n+2k}TTTT \\
&= T^*T^nT^{2k}T^*T^nT^{2k}\text{TTTT} \\
&= T^*T^nT^{2k}T^*T^nT^{2k}T^2T^2 \\
&= T^*T^nT^{2k}T^*T^nT^2T^{2k}T^2 \qquad [\because T^{2k}T^2 = T^2T^{2k}] \\
&= T^*T^nT^{2k}T^*T^nT^2T^{2k}T^2 \\
&= T^*T^nT^{2k}T^2T^*T^nT^{2k}T^2 \qquad [\because T^*T^nT^2 = T^2T^*T^n] \\
&= T^*T^{n+2k+2}T^*T^{n+2k+2} \\
&= (T^*T^{n+2k+2})^2 .
\end{aligned}
$$

Hence T∈(n+2k+2) power class(Q) operator.

In the next theorem we prove that the product of doubly commuting (n+k) power class (Q) operators S and T is again (n+k) power class (Q) operator.

Theorem 2.6:

Let T and S be bounded Linear operators on L(H). If T and S are (n+k) power class(Q)operators that doubly commutes ie., TS=ST and TS* $= S^*T$ then operator TS is (n+k) power class(Q)operator.

Proof:

Let T and S be (n+k) power class(Q)operators. Thereefore, From the definition, we have $T^{*^2}T^{2(n+k)} = (T^*T^{n+k})^2$ and $S^{*^2}S^{2(n+k)} = (S^*S^{n+k})^2$ holds.Let T doubly commutes with S ie., TS=ST and TS* $= S^*T$. We have to prove that $(TS)^{*^2}(TS)^{2(n+k)} = [\,(TS)^*(TS)^{n+k})]^2$.

Consider, $(TS)^{*^2}(TS)^{2(n+k)} = S^*T^*S^*T^*TSTS \ldots\ldots TS$ upto 2(n+k) times

Since T doubly commutes with S, we can write

$(TS)^{*^2}(TS)^{2(n+k)} = S^*S^*T^*T^*TSTS \ldots\ldots TS$up to 2(n+k) times

= $S^*S^*T^*T^*STTS \ldots TS$up to $2(n+k)-1$ times [Since TS = ST]
= $S^*S^*T^*ST^*TTSTS \ldots\ldots TS$ $2(n+k)-1$ *times* [Since T*S = ST*]
= $S^*S^*ST^*T^*TTSTS \ldots\ldots TS$ $2(n+k)-1$ times [Since T*S = ST*]

By successive remove on the left of each of S we get

= [$S^*S^*SS \ldots\ldots S$ $2(n+k)$*times*]$T^*T^*TT \ldots\ldots T$ $2(n+k)$*times*]
= $S^{*^2}S^{2(n+k)}T^{*^2}T^{2(n+k)}$
= $(S^*S^{n+k})^2(T^*T^{n+k})^2$

[$\because$ *SandTare*$(n+k)$*powerclass*(Q)*operators*]

= $S^*S^{n+k}S^*S^{n+k}T^*T^{n+k}T^*T^{n+k}$
= $S^*S^{n+k}S^*T^*S^{n+k}T^{n+k}T^*T^{n+k}$ [$\because S^{n+k}T^* = T^*S^{n+k}$]
= $S^*S^{n+k}T^*S^*S^{n+k}T^{n+k}T^*T^{n+k}$ [$\because T^*S^* = S^*T^*$]
= $S^*T^*S^{n+k}S^*S^{n+k}T^{n+k}T^*T^{n+k}$ [$\because S^{n+k}T^* = T^*S^{n+k}$]
= $S^*T^*S^{n+k}S^*T^{n+k}S^{n+k}T^*T^{n+k}$ [$\because S^{n+k}T^{n+k} = T^{n+k}S^{n+k}$]
= $S^*T^*S^{n+k}T^{n+k}S^*S^{n+k}T^*T^{n+k}$ [$\because S^*T^{n+k} = T^{n+k}S^*$]
= $S^*T^*S^{n+k}T^{n+k}S^*T^*S^{n+k}T^{n+k}$ [$\because S^{n+k}T^* = T^*S^{n+k}$]
= $(TS)^*(ST)^{n+k}(TS)^*(ST)^{n+k}$
= $(TS)^*(TS)^{n+k}(TS)^*(TS)^{n+k}$ [$\because TS = ST$]
= $((TS)^*(TS)^{n+k})^2$. $\therefore$TS is (n+k) power class(Q).

Theorem 2.7

Let T and S be bounded Linear operators on L(H). If T and S are (n+k) power class(Q)operators that doubly commutes ie., TS=ST and T$S^* = S^*T$ under the same conditions operator (ST) is (n+k) power class(Q)operator.

Proof:

Given, T and S are (n+k) power class(Q)operators. ie.,$T^{*^2}T^{2(n+k)}$ =$(T^*T^{n+k})^2$ and $S^{*^2}S^{2(n+k)} = (S^*S^{n+k})^2$

Let T doubly commutes with S ie., TS=ST and T$S^* = S^*T$. We have to prove that $(ST)^{*^2}(ST)^{2(n+k)} = [(ST)^*(ST)^{n+k})]^2$

Consider L.H.S.

$(ST)^{*^2}(ST)^{2(n+k)}$=$T^*S^*T^*S^*STST \ldots\ldots ST$ $2(n+k)$*times.*

Since T doubly commutes with S, we can write

$(ST)^{*^2}(ST)^{2(n+k)}$ = $T^*T^*S^*S^*STST \ldots\ldots ST$2(n+k) times

= $T^*T^*S^*S^*TSST \ldots\ldots ST$ $2(n+k)-1$ times [$\because TS = ST$]
= $T^*T^*S^*TS^*SSTST \ldots\ldots ST$ $2(n+k)-1$ *times* [$\because$S*T = TS*]
= $T^*T^*TS^*S^*SSTST \ldots\ldots ST$ $2(n+k)-1$ times [$\because$S*T = TS*]

By successive remove on the left of each of T we get

= $T^*T^*TT \ldots\ldots T$ $2(n+k)$*times*$S^*S^*SS \ldots\ldots S$ $2(n+k)$*times*
= $T^{*^2}T^{2(n+k)}S^{*^2}S^{2(n+k)}$
= $(T^*T^{n+k})^2$ $(S^*S^{n+k})^2$

[$\because$ *S and T are* $(n+k)$*power class*(Q)*operators*]

= $T^*T^{n+k}T^*T^{n+k}S^*S^{n+k}S^*S^{n+k}$
= $T^*T^{n+k}T^*S^*T^{n+k}S^{n+k}S^*S^{n+k}$ [$\because T^{n+k}S^* = S^*T^{n+k}$]
= $T^*T^{n+k}S^*T^*T^{n+k}S^{n+k}S^*S^{n+k}$ [$\because T^*S^* = S^*T^*$]
= $T^*S^*T^{n+k}T^*T^{n+k}S^{n+k}S^*S^{n+k}$ [$\because T^{n+k}S^* = S^*T^{n+k}$]
= $T^*S^*T^{n+k}T^*S^{n+k}T^{n+k}S^*S^{n+k}$ [$\because T^{n+k}S^{n+k} = S^{n+k}T^{n+k}$]
= $T^*S^*T^{n+k}S^{n+k}T^*T^{n+k}S^*S^{n+k}$ [$\because S^{n+k}T^* = T^*S^{n+k}$]
= $T^*S^*T^{n+k}S^{n+k}T^*S^*T^{n+k}S^{n+k}$ [$\because T^{n+k}S^* = S^*T^{n+k}$]
= $(ST)^*(TS)^{n+k}(ST)^*(TS)^{n+k}$
= $(ST)^*(ST)^{n+k}(ST)^*(ST)^{n+k}$ [$\because TS = ST$]
= $[(ST)^*(ST)^{n+k})]^2$. Hence, ST is (n+k) power class(Q)operator.

Theorem 2.8:

If T is (n+k) power class(Q)operator such that T doubly commute with an isometric operator S, then TS is (n+k) power class(Q)operator.

Proof:

Given, T is (n+k) power class(Q)operator.

Therefore from the definition we have $T^{*^2}T^{2(n+k)} = (T^*T^{n+k})^2$

$(TS)^{*^2}(TS)^{2(n+k)}$=$S^*T^*S^*T^*TSTS\ldots..TS$2(n+k) times

$= S^*S^*T^*T^*STSTTSTS\ldots TS\ \ 2(n+k)-2$ times [$\because T^*S^* = S^*T^* and TS = ST$]

$= S^*S^*T^*ST^*TSTTS\ldots\ldots TS\ \ 2(n+k)-2\ times$ [∵ T*S = ST*]

$= S^*S^*ST^*T^*TSTTSTS\ldots..TS\ \ 2(n+k)-2$ times [∵ T*S = ST*]

$= S^*S^*ST^*T^*STTTSTS\ldots..TS\ \ 2(n+k)-2$ times [$\because TS = ST$]

$= S^*S^*ST^*ST^*T^2TSTS\ldots..TS\ \ 2(n+k)-2$ times [∵ T*S = ST*]

$= S^*S^*SST^*T^*T^2TSTS\ldots..TS\ \ 2(n+k)-2$ times [∵ T*S = ST*]

$= S^*IST^{*^2}T^2TSTS\ldots..TS\ \ 2(n+k)-2$ times [∵ S is isometry we get, S*S = I]

$= S^*ST^{*^2}T^2TSTS\ldots..TS\ \ 2(n+k)-2$ times

$= I\,T^{*^2}T^2TSTS\ldots..TS\ \ 2(n+k)-2$ times

$= T^{*^2}T^2TSTS\ldots..TS\ \ 2(n+k)-2$ times

Repeating the above procedure, we get

$= T^{*^2}T^2T^{2(n+k)-2}SS\ldots.S\ \ 2(n+k)-2\ times$

$= T^{*^2}T^{2(n+k)}SS\ldots.S\ \ 2(n+k)-2\ times$

$= (T^*T^{n+k})^2S^{2(n+k)-2}$ [∵T∈(n+ k) power class(Q)]

$= T^*T^{n+k}T^*T^{n+k}S^{2(n+k)-2}$

$= S^*ST^*S^{n+k-1}T^{n+k}S^*ST^*S^{n+k-1}T^{n+k}$

$= S^*T^*SS^{n+k-1}T^{n+k}S^*T^*SS^{n+k-1}T^{n+k}$ [∵ T*S = ST*]

$= (TS)^*(ST)^{n+k}(TS)^*(ST)^{n+k}$

$= (TS)^*(TS)^{n+k}(TS)^*(TS)^{n+k}$

$= ((TS)^*(TS)^{n+k})^2$. $\therefore TS \in$(n+k) power class(Q).

Further in this chapter we introduce a new class of operator called n power quasi n normal operator acting on a complex Hilbert space. This operator is the generalization ofn power quasi normal operator.

Definition 2.9

Let L(H) be the algebra of all bounded linear operators on a Hilbert space H. An operator T ∈ L(H) is said to be n power quasi n normal operator if it satisfies the following condition $T^n(T^*T^n)$= $(T^*T^n)\,T^n$ for some natural number n.

Example 2.10:

Consider the operator T= $\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ acting on which is 2 power class(Q)and 2 power quasi 2 normal operator.

Solution:

Now $T^* = \begin{pmatrix} -i & 0 \\ 2 & i \end{pmatrix}$ and by direct decomposition, we show that$T^{*^2}T^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (T^*T^2)^2$ and therefore T is 2 power class (Q) operator. Further we show that$T^2(T^*T^2) = \begin{pmatrix} -i & 0 \\ 2 & i \end{pmatrix} = T^*T^4$ and therefore T is 2 power quasi 2 normal operator.

Theorem 2.11:

Let T be an invertible operator acting on three dimensional complex Hilbert space for any k≥0. If T is (2+k) power class(Q) then T is (2+k) normal and hence T^{2+k} is normal.

Proof:

Suppose T is (2+k) power class(Q) then$T^{*^2}T^{2(2+k)} = (T^*T^{2+k})^2$ is true. We can write this equation as, $T^{*^2}T^{2(2+k)} = (T^*T^{2+k})^2 = T^*T^{2+k}T^*T^{2+k}$. If T has inverse then T^* also has inverse and $(T^*)^{-1} = (T^{-1})^*$. Pre multiply by $(T^*)^{-1}$ from the left, we get $T^*T^{2(2+k)} = T^{2+k}T^*T^{2+k}$ and this implies that$T^*T^{2+k}T^{2+k}$= $T^{2+k}T^*T^{2+k}$

Multiply (2+k) times with T^{-1} from the right we get, $T^*T^{2+k} = T^{2+k}T^*$ and hence T is (2+k) normal. Further, it is well known that T is n normal operator if and only if T^n is normal operator. Hence, here T is (2+k) normal implies that $T^{(2+k)}$ is normal.

Theorem 2.12 If a quasi normal operator T on C^2 is (2+k) normal operator then it is 2 power class (Q) operator.

Proof:

We prove that T is (3+k) power class (Q) operator.

$$\begin{aligned}(T^*T^{3+k})^2 &= (T^*T^{3+k})(T^*T^{3+k})\\ &= T^*T^{2+k}T(T^*T)T^{2+k}\\ &= T^*T^{2+k}(T^*T)TT^{2+k} && [\because T \text{ is quasi normal}]\\ &= T^*T^*T^{2+k}TTT^{2+k} && [\because T \text{ is } (2+k) \text{ normal}]\\ &= T^{*^2}T^{3+k}T^{3+k}\\ &= T^{*^2}T^{2(3+k)}.\end{aligned}$$

$(T^*T^{3+k})^2 = (T^*T^{3+k})^2$Hence T is (3+k) power class (Q) operator.

Theorem 2.13:

Let T be a class (Q) operator on H. If T is complex symmetric operator, then the following condition $T^2T^{*^2} = (TT^*)^2$ hold.

Proof:

If T is complex symmetric operator, then by definition we get, CTC = T* , T = CT*C and $C^2 = I$. It follows that $T^{*^2}T^2 =$ CTC CTC CT*C CT*C $= CT^2T^{*^2}C$ and $(T^*T)^2 =$ CTC CT*C CTC CT*C $= C(TT^*)^2$ C. Also it is given that T is class (Q) operator therefore $T^{*^2}T^2 = (T^*T)^2$ implies that $CT^2T^{*^2}C = C(TT^*)^2$ C. On pre multiply and post multiply by C on both sides and using the involuntary property of C ($C^2 = I$) we get the required condition.

Section 3.0:

In this section we derive theorems based on quasi n posinormal operator on Hardy space.

Definition 3.1:[9] An operator T in a Hilbert space H is called posinormal if TT* ≤ c^2T*T for some c >0.

Definition 3.2 :[9] An operator T in a Hilbert space H is quasi posinormal if $(TT^*)^2 \leq c^2 T^{*^2}T^2$

Definition 3.3 : An operator T in a Hilbert space H is quasi n posinormal if $(TT^*)^{2n} \leq \left(T^{*^2}T^2\right)^n$

Remark: quasi n – posinormal operator and n - quasi posinormal operator have the same definition.

Theorem 3.4: If C_T is a hyponormal composition operator then it is quasi n posinormal.

Proof:

$C_T$is quasi n posinormal if $(C_TC_T^*)^n \leq \mu C_T^{*^{2n}} C_T^{2n}$ for some $\mu > 1$

$$\begin{aligned}< ((C_TC_T^*)^n - \mu C_T^{*^{2n}} C_T^{2n})f, f > &= < (C_TC_T^*)^n f, f > - < \mu C_T^{*^{2n}} C_T^{2n} f, f >\\ &= \|C_T^* f\|^{2n} - \mu \|C_T f\|^{4n}, \forall f \in H^2(\beta)\end{aligned}$$

Let $f = k_o^\beta$ we have $\left\|C_T^* k_o^\beta\right\|_\beta^{2n} - \mu \left\|C_T k_o^\beta\right\|_\beta^{4n} = \left\|k_{T(o)}^\beta\right\|_\beta^{2n} - \mu \left\|k_o^\beta\right\|_\beta^{4n} = \left\|k_o^\beta\right\|_\beta^{2n} - \mu \left\|k_o^\beta\right\|_\beta^{4n} = 1 - \mu \leq 0$

$\left\|C_T^* k_o^\beta\right\|_\beta^{2n} \leq \mu \left\|C_T k_o^\beta\right\|_\beta^{4n}$ and therefore $C_T$is quasi n posinormal.

Theorem: 3.5 $C_T$is quasi n posinormal if and only if $\left\|(C_TC_T^*)^{*^n} k_o^\beta\right\|_\beta^2 \leq \mu \left\|C_T^2 k_o^\beta\right\|_\beta^2$

Proof:

$C_T$is n quasi posinormal if $(C_TC_T^*)^{2n} - \mu(C_T^{*^2} C_T^2)^n \leq 0 \; for \mu > 1$

$$\begin{aligned}&\Leftrightarrow < ((C_TC_T^*)^{2n} - \mu\left(C_T^{*^2} C_T^2\right)^n) f, f > \leq 0\\ &\Leftrightarrow < (C_TC_T^*)^{2n} f, f > - \mu < (C_T^{*^2} C_T^2)^n f, f > \leq 0\\ &\Leftrightarrow < (C_TC_T^*)^n \, (C_TC_T^*)^n f, f > - \mu < C_T^2 f, C_T^2 f > \leq 0\\ &\Leftrightarrow < (C_TC_T^*)^n f, \, (C_TC_T^*)^n f > - \mu \|C_T^2 f\|^2 \leq 0\\ &\Leftrightarrow \| (C_TC_T^*)^{*n} f\|^2 - \mu \|C_T^2 f\|^2 \leq 0 \; for all f \in H^2(\beta)\end{aligned}$$

Let $f = k_o^\beta$ we have, $\left\|(C_TC_T^*)^{*^n} k_o^\beta\right\|_\beta^2 - \mu \left\|C_T^2 k_o^\beta\right\|_\beta^2 \leq 0$

which implies that, $\left\|(C_TC_T^*)^{*^n} k_o^\beta\right\|_\beta^2 \leq \mu \left\|C_T^2 k_o^\beta\right\|_\beta^2 for all f \in H^2(\beta)$

Theorem: 3.6 C_r is n quasi posinormal if and only if $(P_v^2)^{2n} \leq \mu(h_2 E(\pi_2^2) o T^{-2})^n$ a.e

Proof:

C_r is n quasi posinormal if $(C_r C_r^*)^{2n} \leq \mu(C_r^{*^2} C_r^2)^n$

$$\Leftrightarrow <((C_r C_r^*)^{2n} - \mu(C_r^{*^2} C_r^2)^n) f, f > \leq 0$$

$$\Leftrightarrow \int [(P_v^2)^{2n} - \mu(h_2 E(\pi_2^2) o T^{-2})^n]\, |f|^2 d\lambda > \leq 0 \; for every E \in \Sigma$$

$$\Leftrightarrow (P_v^2)^{2n} \leq \mu(h_2 E(\pi_2^2) o T^{-2})^n \text{ a.e}$$

Theorem: 3.7 $C_{s,t}$ is n quasi posinormal if and only if $(P_{v(s,t)}^2)^{2n} \leq \mu h_{2n} E(\pi_{2(s,t)}^2) o T^{-2})^n$

Proof:

$C_{s,t}$ is n quasi posinormal if $(C_{(s,t)} C_{(s,t)}^*)^{2n} \leq \mu(C_{(s,t)}^{*^2} C_{(s,t)}^2)^n$

$$\Leftrightarrow <((C_{(s,t)} C_{(s,t)}^*)^{2n} - \mu(C_{(s,t)}^{*^2} C_{(s,t)}^2)^n) f, f > \leq 0$$

$$\Leftrightarrow \int [(P_{v(s,t)}^2)^{2n} - \mu(h_2 E(\pi_{2(s,t)}^2) o T^{-2})^n$$

$$\Leftrightarrow P_{v(s,t)}^{4n} \leq \mu(h_2 E(\pi_{2(s,t)}^2) o T^{-2})^n$$

Theorem: 3.8 $C_{s,t}{}^*$ is n quasi posinormal if and only if $h_{2n}(E(\pi_{(s,t)}^2) o T^{-1})^{2n} \leq \mu \pi_2^2 (h_2 o T^2)^n$

Proof:

$C_{s,t}{}^*$ is n quasi posinormal if $(C_{(s,t)}^* C_{(s,t)})^{2n} \leq \mu(C_{(s,t)}^2 C_{(s,t)}^{*^2})^n$

$$\Leftrightarrow <((C_{(s,t)}^* C_{(s,t)})^{2n} - \mu\left(C_{(s,t)}^2 C_{(s,t)}^{*^2})^n\right) f, f > \leq 0$$

$$\Leftrightarrow \int [h_{2n}(E((\pi_{(s,t)}^2) o T^{-1})^{2n} - \mu(\pi_2(h_2 o T^2) E(\pi_2 f)^n]\, |f|^2 d\lambda \leq 0 \text{ for every } E \in \Sigma$$

$$\Leftrightarrow h_{2n}(E(\pi_{(s,t)}^2) o T^{-1})^{2n} - \mu \pi_2^2 (h_2 o T^2)^n \leq 0$$

$$\Leftrightarrow h_{2n}(E(\pi_{(s,t)}^2) o T^{-1})^{2n} \leq \mu \pi_2^2 (h_2 o T^2)^n$$

Characterzation on the generalized Aluthge Transformation

Definition: Generalized Aluthge Transform

For an operator T = U|T|, define $\tilde{T}$ as follows $\tilde{T}_{s,t} = |T|^s U |T|^t$ for s and t > 0 which is called the generalized Aluthge transform of T. In particular, $\tilde{T} = |T|^{1/2} U |T|^{1/3}$ is called Aluthge transform of T.

Theorem: 3.9 $C_r{}^*$ is n quasi posinormal if and only if $h_{2n}(E(\prod_r^2) o T^{-1})^{2n} \leq \mu(\prod_2^2 (h_2 o T^2))^n$

Proof:

$C_r{}^*$ is n quasi posinormal if $(C_r^* C_r)^{2n} \leq \mu(C_r^2 C_r^{*^2})^n$

$$\Leftrightarrow <((C_r^* C_r)^{2n} - \mu\left(C_r^2 C_r^{*^2})^n\right) f, f > \leq 0$$

$$\Leftrightarrow \int [h_{2n}(E(\pi^2 o T^{-1})^{2n} - \mu(\pi_2(h_2 o T^2) E(\pi_2 f))^n]\, |f|^2 d\lambda \leq 0 \text{ for every } E \in \Sigma$$

$$\Leftrightarrow \int [h_{2n}(E(\pi^2 o T^{-1})^{2n} - \mu(\pi_2^2 (h_2 o T^2))^n]\, |f|^2 d\lambda \leq 0$$

$$\Leftrightarrow h_{2n}(E(\pi^2 o T^{-1})^{2n} \leq \mu(\pi_2^2 (h_2 o T^2))^n$$

CONCLUSION

The algebra of all bounded Linear operators on a Hilbert space H is taken for the research work and the following findings on the class of (n+k) power class (Q) operators are given.

1. If T is (n+k) power class (Q) and T is quasi (n+k) normal, then T is (n+1+k) power class Q). 2. If an operator T on C^2 is both invertible and (2+k) normal operator then the following results hold. (i) T is (2+k) power class (Q), (ii) T is not quasi (n+k) normal and (iii) T is not (3+k) power class (Q).3. Consider the operator T= $\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ acting on C^2 which is 2 normal, 2 power class (Q), not quasi 2 normal and not 3 power class(Q). 4. Let T be bounded linear operator. If T is (n+k) power class (Q) operator and has inverse then T is (n+k) normal operator for any k ≥ 0. 5. Let T be bounded linear operator on H. If T is (n+2k) power class(Q)operator and in the same time is 2 power quasi n normal operator, then T is (n+2k+2) power class(Q)operator for any k ≥ 0. 6. Let T and S be bounded Linear operators on L(H). If T and S are (n+k) power class (Q) operators that doubly commutes ie., TS = ST and $TS^* = S^*T$ then operator TS is (n+k) power

class(Q)operator. 7. Let T and S be bounded Linear operators on L(H). If T and S are (n+k) power class (Q) operators that doubly commutes ie., TS = ST and T$S^* = S^*T$ under the same conditions operator (ST) is (n+k) power class(Q)operator. 8. If T is (n+k) power class (Q)operator such that T doubly commute with an isometric operator S, then TS is (n+k) power class(Q)operator. 9. n power quasi n normal operator on a Hilbert space is introduced. An operator T ∈ L(H) is said to be n power quasi n normal operator if it satisfies the following condition $T^n(T^*T^n)$= $(T^*T^n)\,T^n$ for some natural number n. 10. An n power class (Q) operator need not be n power quasi n normal operator. This result is verified by considering the operator T=$\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ acting on H which is 2 power class (Q)and it is not 2 power quasi 2 normal operator. 11. Let T be an invertible operator acting on three dimensional complex Hilbert space for any k ≥ 0. If T is (2+k) power class (Q) then T is (2+k) normal and hence T^{2+k} is normal. 12. If a quasi normal operator T on C^2 is (2+k) normal operator then it is 2 power class (Q) operator. 13. Let T be a class (Q) operator on H. If T is complex symmetric operator, then the following condition $T^2T^{*^2} = (TT^*)^2$ hold. Let $H^2(\beta)$ be a Hilbert space of functions analytic in the unit disk. Then the following charatcerizations hold. 13. If C_T is a hyponormal composition operator then it is quasi n posinormal. 14. $C_T$is quasi n posinormal if and only if $\left\|(C_TC_T^*)^{*^n}k_o^\beta\right\|_\beta^2 \leq \mu\left\|C_T^2k_o^\beta\right\|_\beta^2$. 15. $C_r$is n quasi posinormal if $(C_rC_r^*)^{2n} \leq \mu(C_r^{*^2}C_r^2)^n$ 16. $C_{s,t}$is n quasi posinormal if and only if $(P_{v(s,t)}^2)^{2n} \leq \mu\hbar_{2n}E\left(\pi_{2(s,t)}^2\right)oT^{-2})^n$. 17. $C_{s,t}{}^*$is n quasi posinormal if and only if $\hbar_{2n}(E\left(\pi_{(s,t)}^2\right)oT^{-1})^{2n} \leq \mu\pi_2^2(\hbar_2oT^2)^n$.18. $C_r{}^*$is n quasi posinormal if and only if $\hbar_{2n}(E(\prod_r^2)oT^{-1})^{2n} \leq \mu(\prod_2^2(\hbar_2oT^2))^n$.

ACKNOWLEDGEMENTS

This article is supported by UGC - Minor Research Project during 2013 - 2015.

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