Hydromagnetic Mixed Convection Flow Through Horizontal Channel : Analysis with Viscous Dissipation, Joule Heating, Variable Viscosity and Thermal Conductivity

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Abstract

Laminar mixed convection flow of an incompressible, electrically conducting, viscous fluid with variable viscosity and variable thermal conductivity through two parallel horizontal walls under the influence of variable magnetic field is studied. Arrhenius model is used to express variable viscosity and thermal conductivity. In this model, the variable viscosity, and also the thermal conductivity decrease exponentially with temperature. The fluid is subjected to a constant pressure gradient and an external magnetic field perpendicular to the plates. The plates are maintained at different but constant temperatures. Approximation technique is used to obtain the solution of the coupled non-linear equations of the velocity field and the temperature distribution. The expressions for skin-friction and heat transfer rate are also derived. The effects of parameters of engineering importance on velocity field and temperature distribution are discussed graphically, while effects on skin-friction and rate of heat transfer are presented in tabular form and discussed.

Nomenclature

B	variable magnetic field,
B_0	constant magnetic field, when $T' = T_1'$,
C_p	specific heat at constant pressure,
Ec	Eckert number,
h	width of the channel,
k_m'	dimensional permeability,
k_m	permeability parameter,
K_T	variable thermal conductivity,
K_{T_0}	thermal conductivity, when $T' = T_1'$,
M	magnetic parameter,
Nu	rate of heat transfer,
$P'\left(= -\frac{\partial p'}{\partial x'}\right)$	constant pressure gradient,
P	non-dimensional pressure,

Pr	Prandtl number,
T'	dimensional fluid temperature,
T	non-dimensional fluid temperature,
T_1', T_2'	temperatures of the lower and upper walls,
u'	velocity of the fluid along the channel,
u	non-dimensional velocity of the fluid,
u_m	mean velocity of the fluid,
x', y'	dimensional coordinate system,
x, y	non-dimensional coordinates,

Greek symbols

α	viscosity parameter,
α', β'	small positive constants,
β	thermal conductivity parameter,
μ_0	constant viscosity, when $T' = T_1'$,
μ'	variable viscosity of the fluid,
ρ	density of the fluid,
σ	electrical conductivity of the fluid,
τ	non-dimensional skin-friction.

Introduction

The convection flow of an incompressible, electrically conducting fluid between two infinite parallel stationary plates in the presence of magnetic field has been studied in numerous ways due to its important applications in MHD pumps, MHD generators, flow meters etc. Most of these studies are based on constant physical properties of the fluid. However, some physical properties of the fluid are function of temperature. Therefore, consideration of constant properties is a good approximation so long as small differences in temperature are involved. More accurate prediction of the flow and heat transfer properties can be achieved by considering the variation of physical properties with temperature. In fact, viscosity of many fluids vary with temperature. Therefore, the results drawn from flow of such fluids with constant viscosity are not applicable to the fluid flows with temperature dependent viscosity. Hence, it is necessary to take into account the variation of viscosity, to predict a better estimation of the flow and heat transfer behavior. The flow of fluids considering temperature dependent viscosity are of immense importance in chemical engineering, bio-chemical engineering and petroleum industries [1,2] to predict the results more accurately.

Ling and Lybbs [3] presented a very interesting theoretical investigation of the temperature dependent fluid viscosity influence on the forced convection through a porous medium bounded by an isothermal flat plate. The fluid viscosity was modeled as an inverse linear function of the fluid temperature, which is a suitable model for many liquids including water and crude oil. Rao and Pop [4] investigated the same model envisaging transient free convection flow over a plate submersed in fluid saturated porous medium. Kafoussius and Williams [5] studied the effect of temperature dependent viscosity on the free convection boundary layer flow past a vertical isothermal plate. Kafoussius and Rees [6] extended the work [5] and examined numerically, the effect of temperature dependent viscosity on the mixed convection laminar boundary layer flow along a vertical isothermal plate. Singh et al. [7] investigated effects of temperature dependent viscosity on heat transfer rate envisaging unsteady free convection flow along a vertical isothermal and non-isothermal plate embedded in a fluid saturated porous medium. Hazarika and Phukan [8] extended the study [7] to investigate effects of variable temperature dependent viscosity considering continuous moving isothermal and non-isothermal plate using

Karmann-Pohlhausen integral method. Hussain et al. [9] discussed the effects of radiation on free convection flow with temperature dependent viscosity in presence of magnetic field past a vertical porous plate. Bagai [10] obtained a similarity solution for the analysis of the steady free convection boundary layers over a non-isothermal axi-symmetric body embedded in a fluid saturated porous medium and discussed the effect of temperature dependent viscosity on heat transfer rate with internal heat generation.

Barakat [11] investigated the effect of variable viscosity on the flow and heat transfer about a fluid underling axi-symmetric spreading surface in the presence of an axial magnetic field envisaging that the viscosity of the fluid vary as an inverse linear function of temperature and the magnetic field strength is inversely proportional to the radial coordinate . Sequentially, Cheng [12] discussed effect of temperature dependent viscosity on natural convection heat transfer from a horizontal isothermal cylinder of elliptic cross-section, whereas Molla et al. [13] investigated natural convection flow from an isothermal circular cylinder with temperature dependent viscosity. Attia [14] discussed the effect of temperature dependent viscosity on steady Hartmann flow with ion-slip. Attia [15] also studied effect of temperature dependent viscosity on transient MHD flow and heat transfer between two parallel plates. Kankane and Gokhale [16] used Arrhenius model (commonly known as exponential model) to study fully developed flow through a horizontal channel. Pantokratoras [17] discussed effects of variable viscosity with variable Prandtl number on forced and mixed convection boundary layer flow along a flat plate. Pantokratoras [18] further discussed the effects of variable viscosity and variable Prandtl number on non-Darcian forced convection heat transfer over a flat plate. Attia [19] investigated unsteady Couette flow and heat transfer of an electrically conducting fluid envisaging temperature dependent viscosity and thermal conductivity. Recently, Singh et al. [20, 21] have extended the work [19] to discuss the flow in a horizontal channel embedded in a homogeneous porous medium envisaging Arrhenius model. Recently, Singh and coworkers [21-24] have examined MHD convective flow in horizontal channel with different flow and thermal restrictions. More recently, Singh et al [25, 26] also investigated MHD convective flow past a vertical porous plate and discussed the effects of variable suction/injection and variable permeability and also effects of variable suction/injection as well as radiation respectively. However, in these studies the variation in viscosity with temperature is not taken into account.

In fact, for most realistic fluids, the viscosity shows a rather pronounced variation with temperature. A decrease in temperature causes the viscosity of the liquid to increase and there is a substantial correlation between the viscosity and the corresponding thermal expansion of the fluid. Therefore, the object of the present work is to study free convection flow of a viscous, electrically conducting, incompressible fluid with temperature dependent viscosity and variable thermal conductivity through a long horizontal channel under the influence of magnetic field envisaging viscous dissipation and Joule heating. Arrhenius model is considered in order to account for the temperature dependent viscosity [19] as well as for variation in thermal conductivity. The coupled non-linear equations of momentum and energy are solved using approximation technique following Ganji et al. [21]. The variations in velocity field and temperature distribution are discussed graphically, while skin-friction and rate of heat transfer are discussed with the help of tables for different numerical values of the parameters of engineering importance. The results of the study are in well agreement with those of Kankane and Gokhale [16], Singh et al. [20] and Gupta [22] have been deduced as particular case of the present study. The configuration suggested in this model enhances the utility of the model of Attia [19] and is a good approximation in some practical situations such as heat exchangers, flow meters and pipes that connect system components.

1. Formulation of the problem

We consider fully developed laminar flow of an electrically conducting, viscous, incompressible fluid taking into account the temperature dependent viscosity and temperature dependent thermal conductivity. It is assumed that the fluid flows between two long parallel horizontal channel walls. Let $2h$ be the width of the horizontal channel walls, assumed to be electrically non-conducting and kept at two constant temperatures, T_1' for the lower cold wall and T_2' for the upper hot wall respectively ($T_2' > T_1'$). The heat transfer takes place from upper hot wall to the lower cold wall by conduction through the fluid [14]. Also, there is a heat generation due to both, the Joule and viscous dissipations [19]. The viscosity and also the thermal conductivity of the fluid is assumed to vary with temperature. These are defined as $\mu' = \mu_0 f_1(T')$ and $K_T = K_{T_0} f_2(T')$, respectively. For practical reasons, which are found to be suitable for many realistic fluids of engineering interest [27, 28], the viscosity and thermal conductivity is assumed to vary exponentially with temperature. Hence, functions $f_1(T')$ and $f_2(T')$ take the form $f_1(T') = e^{-\alpha'}(T' - T_1')$ and

$f_2(T') = e^{-\beta'}(T' - T_1')$, respectively [16, 29]. A constant pressure gradient $\left(-\dfrac{\partial p'}{\partial x'} = P'\right)$ is applied in the x' - direction and the magnetic field $B = B_0\, exp\left\{-\dfrac{\alpha'}{2}(T' - T_1')\right\}$ is applied in the positive y' - direction, i.e., normal to the flow field. The constant magnetic field B_0 is chosen such that the induced magnetic field is neglected [30]. The no-slip condition at the walls implies that the fluid velocity has neither a z' - component nor an x' - component at the wall $y' = 0$ and the wall $y' = h$. Since the walls are long enough in the x' - and z' - directions, the physical variables are invariant in these directions, the problem is essentially one dimensional with velocity component $u'(y')$ along the x' - axis. The physical model and the coordinate system of the problem are shown in Fig. 1.

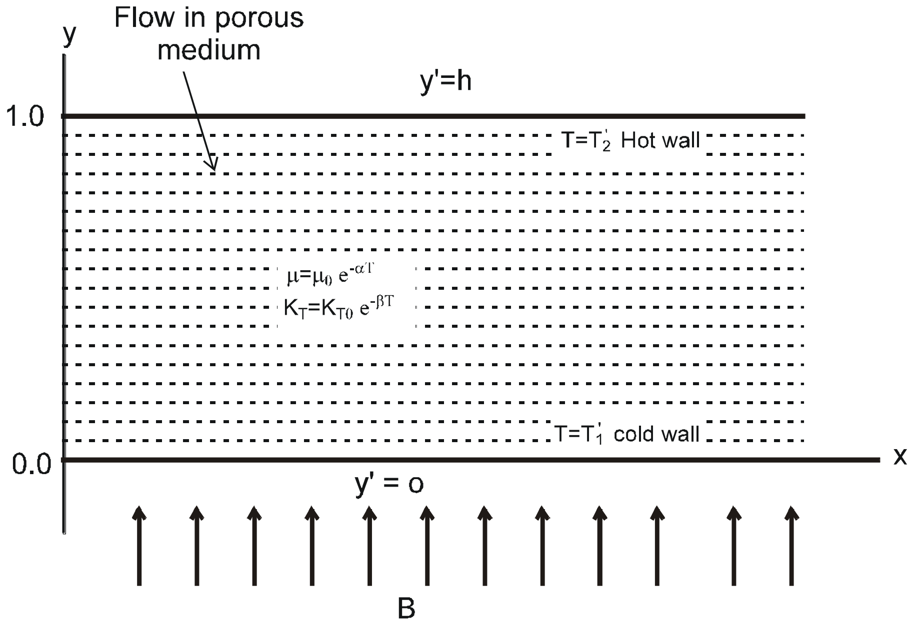

Fig.1: Physical model and coordinate system of problem

Under the present configuration, the flow can be shown to be governed by the following system of coupled non-linear equations [30].

$$P' + \frac{d}{dy'}\left(\mu'\frac{du'}{dy'}\right) - \frac{\mu'}{k_m'}u' - \sigma B^2 u' = 0 . \quad (1)$$

$$\frac{d}{dy'}\left(K_T'\frac{dT'}{dy'}\right) + \mu'\left(\frac{du'}{dy'}\right)^2 + \sigma B^2 u'^2 = 0 . \quad (2)$$

The terms in the left hand side of Eq. (1) represent, respectively, the pressure gradient, viscous forces, Darcy velocity and Lorentz force, while in Eq. (2), the terms in the left-hand side represent, respectively, the thermal diffusion, viscous dissipation and Joule dissipation.

The boundary conditions of the velocity field and the temperature distribution relevant to the problem [31] are:

$$u' = 0 , \qquad T' = T_1' \qquad \text{at} \qquad y' = 0 ,$$

$$u' = 0 , \qquad T' = T_2' \qquad \text{at} \qquad y' = h . \quad (3)$$

The problem is simplified by writing Eqs. (1) - (2) in the non-dimensional form. We define the following non-dimensional variables and parameters:

$$u=\frac{u'}{u_m},\qquad y=\frac{y'}{h},\qquad \mu=\frac{\mu'}{\mu_0},\ K_T=\frac{K'_T}{K_{T_0}},\qquad k_m=\frac{k'_m}{h^2},$$

$$\frac{\partial p}{\partial x}=P',\qquad T=\frac{T'-T'_1}{T'_2-T'_1}\qquad P=\frac{P'h^2}{\mu_0 u_m},\qquad \alpha=\alpha'\left(T'_2-T'_1\right),$$

$$\beta=\beta'\left(T'_2-T'_1\right),\qquad Ec=\frac{u_m^2}{C_p\left(T'_2-T'_1\right)},\qquad M=\frac{\sigma}{\mu_0}B_0^2h^2,\quad Pr=\frac{\mu_0 C_p}{K_{T_0}}.$$

The symbols and parameters are defined in the nomenclature.

In terms of the above non-dimensional variables and parameters, the Eqs. (1) and (2), take the form:

$$\frac{d}{dy}\left(e^{-\alpha T}\frac{du}{dy}\right)-\frac{1}{k_m}e^{-\alpha T}u-Me^{-\alpha T}u+P=0. \qquad (4)$$

$$\frac{d}{dy}\left(e^{-\beta T}\frac{dT}{dy}\right)+PrEc\,\mu\left(\frac{du}{dy}\right)^2+PrEcMe^{-\alpha T}u^2=0. \qquad (5)$$

Eqs. (4) - (5), can be written as follows:

$$\frac{d^2u}{dy^2}-\alpha\frac{dT}{dy}\frac{du}{dy}-\left(M+k_m^{-1}\right)u+Pe^{\alpha T}=0. \qquad (6)$$

$$\frac{d^2T}{dy^2}-\beta\left(\frac{dT}{dy}\right)^2+PrEc\,\mu\left(\frac{du}{dy}\right)^2e^{\beta T}+PrEcMe^{(\beta-\alpha)T}u^2=0. \qquad (7)$$

The boundary conditions (3) in non-dimensional form are:

$$u=0\ ,\qquad T=0\qquad \text{at}\qquad y=0,$$

$$u=0,\qquad T=1\qquad \text{at}\qquad y=1. \qquad (8)$$

2. Solution of the problem

Eqs. (4) - (5) represent a system of coupled non-linear differential equations. In order to solve the non-linear system, we expand u and T in powers of Ec, under the assumption $Ec<<1$, which is valid for incompressible fluids [27]. Hence, the velocity and temperature can be expressed as follows:

$$u=u_0+Ecu_1+o\left(Ec\right)^2+\ldots\ldots \qquad \text{and}\qquad T=T_0+EcT_1+o\left(Ec\right)^2+\ldots\ldots. \qquad (9)$$

Introducing (9) in (4) - (5) and equating the constant term, as well as the coefficients of Ec, neglecting the coefficients of $o\left(Ec\right)^2$, we obtain:

$$\frac{d}{dy}\left(e^{-\alpha T_0}\frac{du_0}{dy}\right)-\left(k_m^{-1}+M\right)e^{-\alpha T_0}u_0+P=0, \qquad (10)$$

$$\frac{d}{dy}\left\{e^{-\alpha T_0}\left(\frac{du_1}{dy}-\alpha T_1\frac{du_0}{dy}\right)\right\}-\left(k_m^{-1}+M\right)e^{-\alpha T_0}\left(u_1-\alpha u_0T_1\right)=0 \qquad (11)$$

$$\frac{d}{dy}\left(e^{-\beta T_0}\frac{dT_0}{dy}\right)=0, \tag{12}$$

$$\frac{d}{dy}\left\{e^{-\beta T_0}\left(\frac{dT_1}{dy}-\beta T_1\frac{dT_0}{dy}\right)\right\}+Pr\,e^{-\alpha T_0}\left(\frac{du_0}{dy}\right)^2+Pr\,M\,e^{-\alpha T_0}u_0^2=0 \tag{13}$$

Introducing (9), the boundary conditions (6) are transformed to:

$$u_0=0,\quad u_1=0,\quad T_0=0,\quad T_1=0 \quad \text{at} \quad y=0,$$

$$u_0=0,\quad u_1=0,\quad T_0=1,\quad T_1=0 \quad \text{at} \quad y=1. \tag{14}$$

The solution of Eqs. (10) - (13) satisfying the corresponding boundary conditions (14) are obtained as follows:

$$T_0=1-y \tag{15}$$

$$u_0=C_1e^{m_1y}+C_2e^{m_2y}+K_1e^{-\alpha y} \tag{16}$$

$$T_1=\left(C_3+C_4y\right)e^{-\beta y}+K_2e^{(2m_1+\alpha-\beta)y}+K_3e^{(2m_2+\alpha-\beta)y}+K_4e^{-(\alpha+\beta)y}$$
$$+K_5e^{(m_1+m_2+\alpha-\beta)y}+K_6e^{(m_1-\beta)y}+K_7e^{(m_2-\beta)y} \tag{17}$$

$$u_1(y)=C_5e^{m_3y}+C_6e^{m_4y}+K_{27}e^{(m_1-\beta)y}+K_{28}e^{(m_2-\beta)y}$$
$$+K_{29}e^{-(\alpha+\beta)y}+\left(K_{30}y+K_{31}\right)e^{(m_1-\beta)y}+\left(K_{32}y+K_{33}\right)e^{(m_2-\beta)y}$$
$$+\left(K_{34}y+K_{35}\right)e^{-(\alpha+\beta)y}+K_{36}e^{(3m_1+\alpha-\beta)y}+K_{37}e^{(3m_2+\alpha-\beta)y}$$
$$+K_{38}e^{-(2\alpha+\beta)y}+K_{39}e^{(2m_1+m_2+\alpha-\beta)y}+K_{40}e^{(m_1+2m_2+\alpha-\beta)y}$$
$$+K_{41}e^{(2m_1-\beta)y}+K_{42}e^{(2m_2-\beta)y}+K_{43}e^{(m_1-\alpha-\beta)y}$$
$$+K_{44}e^{(m_2-\alpha-\beta)y}+K_{45}e^{(m_1+m_2-\beta)y}. \tag{18}$$

The constants are defined in the appendix.

3. Skin-friction and rate of heat transfer

The skin-friction (τ) at the lower wall ($y=0$) and upper wall ($y=1$) is given by:

$$(\tau)_{y=0,1}=\left(\frac{du}{dy}\right)_{y=0,1}=\left(\frac{du_0}{dy}\right)_{y=0,1}+Ec\left(\frac{du_1}{dy}\right)_{y=0,1} \tag{19}$$

$$(\tau)_{y=0}=\left(m_1C_1+m_2C_2-\alpha K_1\right)+EcK_{48}. \tag{20}$$

$$(\tau)_{y=1}=\left(m_1C_1e^{m_1}+m_2C_2e^{m_2}-\alpha K_1e^{-\alpha}\right)+EcK_{49}. \tag{21}$$

The rate of heat transfer (Nu) at the lower wall ($y=0$) and upper wall ($y=1$) is given by:

$$(Nu)_{y=0,1} = \left(\frac{dT}{dy}\right)_{y=0,1} = \left(\frac{dT_0}{dy}\right)_{y=0,1} + Ec\left(\frac{dT_1}{dy}\right)_{y=0,1} \qquad (22)$$

$$(Nu)_{y=0} = -1 + EcK_{50}. \qquad (23)$$

$$(Nu)_{y=1} = -1 + EcK_{51}. \qquad (24)$$

4. Verification of the results for simple cases

1. When hydromagnetic force is zero, i.e., $M = 0$, the fluid flow is simulated by in the presence of homogeneous porous medium, the results obtained are similar to those of Singh et al. [20].
2. In the limit, when $k_m \to \infty$, the fluid flows in purely fluid regime, i.e., in absence of porous medium. In addition, if the term of Joule heating is ignored in Eq. (2), the results obtained are exactly the same to those obtained by Singh et al. [29], except notations.
3. In absence of magnetic field, i.e., when $M = 0$, the term due to Joule dissipation is ignored and $k_m \to \infty$, the results obtained are exactly the same to those obtained by Kankane and Gokhale [16], except notations.

Table-1

Comparison of present numerical values of the velocity with numerical values of Attia [19] at middle of the channel walls for different values of α and β ($M = 0.0$, $Pr = 1.0$, $Ec = 0.001$ and $k_m = 100$)

	Attia [19]			Present case		
β \ α	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.5$
$\beta = 0.0$	2.0777	2.1551	2.5245	2.18719	2.38176	3.02167
$\beta = 0.1$	2.0777	2.1542	2.5163	2.22943	2.44187	3.26463
$\beta = 0.5$	2.0777	2.1514	2.4920	2.31027	2.60609	3.53169

Table-2

Variations in velocity u at middle ($y = 0.5$) of the channel walls ($M = 0$) for different values of α and β ($Pr = 1.0$, $Ec = 0.001$ and $k_m = 100$)

β \ α	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
$\beta = 0.0$	2.18719	2.38176	2.63431	3.94594	3.02167
$\beta = 0.1$	2.22943	2.44187	2.78352	3.15179	3.26463
$\beta = 0.3$	2.26614	2.51852	2.80474	3.17318	3.31942
$\beta = 0.4$	2.29196	2.57347	2.89942	3.30964	3.48630
$\beta = 0.5$	2.31027	2.60609	2.93796	3.34102	3.53169

Table3

Variations in velocity u at middle ($y = 0.5$) of the channel walls ($M = 1.0$) for different values of α and β ($Pr = 1.0$, $Ec = 0.001$ and $k_m = 100$)

β \ α	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
$\beta = 0.0$	1.97863	2.12184	2.30164	2.50708	2.56641
$\beta = 0.1$	2.00978	2.18016	2.38861	2.61025	2.68071
$\beta = 0.3$	2.04934	2.24634	2.47192	2.73976	2.81148
$\beta = 0.4$	2.07976	2.31743	2.57207	2.87163	2.91921

$\beta = 0.5$	2.08014	2.35465	1.60928	2.61997	3.20864

Table-4

Variations in temperature T at middle ($y = 0.5$) of the channel walls ($M = 0$) for different values of α and β ($M = 0.0$, $Ec = 0.001$ and $Pr = 1.0$)

β \ α	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
$\beta = 0.0$	2.18719	2.36654	2.57916	2.81972	2.88789
$\beta = 0.1$	2.22738	2.44721	2.69854	2.97793	3.08938
$\beta = 0.3$	2.26935	2.52693	2.81763	3.13927	3.28386
$\beta = 0.4$	2.29198	2.56942	2.88937	3.23275	3.41579
$\beta = 0.5$	2.31053	2.60819	2.94991	3.31989	3.51397

Table-5

Variations in temperature T at middle of the channel walls for different values of α and β ($y = 0.5$, $M = 1.0$, $Ec = 0.01$ and $Pr = 1.0$)

β \ α	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
$\beta = 0.0$	1.97863	2.19348	2.47819	2.83107	2.91687
$\beta = 0.1$	2.00811	2.25174	2.56782	2.95996	3.06198
$\beta = 0.3$	2.06973	2.34682	2.68591	3.10989	3.24682
$\beta = 0.4$	2.09728	2.40963	2.77864	3.22619	3.39189
$\beta = 0.5$	2.11934	2.42109	2.80715	3.26827	3.44524

Table-6

Variations in skin-friction (τ) and heat transfer rate (*Nu*) at the lower channel wall ($y = 0$) for different values of α and β

($Pr = 1.0$, $k_m = 100$ and $Ec = 0.001$ **)**

α	β	τ		*Nu*	
		$M = 0$	$M = 1$	$M = 0$	$M = 1$
0.0	0.0	2.94817	2.18986	0.78594	0.89178
0.3	0.0	2.88985	2.13687	0.83719	0.95675
0.6	0.0	2.80769	2.06132	0.90189	1.04153
0.9	0.0	2.69965	1.96293	0.00992	1.17384
1.0	0.0	2.65819	1.93967	1.04193	1.22937
0.0	0.0	2.94817	2.18986	0.78594	0.89178
0.0	0.3	2.92795	2.17718	1.89346	1.01714
0.0	0.6	2.89817	2.15963	2.04893	1.19935
0.0	0.9	2.85173	2.12347	2.29932	1.45198
0.0	1.0	2.83287	2.08109	2.23189	1.53342

Table-7

Variations in skin-friction (τ) and heat transfer rate (*Nu*) at the upper channel wall ($y = 1$) for different values of α and β

($Pr = 1.0$, $k_m = 100$ and $Ec = 0.001$ **)**

α	β	τ		*Nu*	
		$M = 0$	$M = 1$	$M = 0$	$M = 1$
0.0	0.0	4.98376	3.96892	1.89365	1.71649
0.3	0.0	5.10957	4.04387	1.76047	1.62735

0.6	0.0	5.21782	4.16718	1.61853	1.50982
0.9	0.0	5.33192	4.28953	1.43698	1.41306
1.0	0.0	5.39758	4.32984	1.24917	1.39918
0.0	0.0	4.98376	3.96892	1.89365	1.71649
0.0	0.3	5.87254	5.02914	1.78934	1.59912
0.0	0.6	6.08971	5.19048	1.67819	1.48109
0.0	0.9	6.33185	5.31576	1.56975	1.37004
0.0	1.0	6.65672	5.45901	1.49837	1.29956

6. Results and discussion

Analytical solutions of the non-dimensional equations of momentum and energy (4)-(5) are obtained and expressed in (15)-(18). The modified equations governing the flow (6)-(7) show that both the fluid velocity as well as the temperature distribution are governed by viscosity parameter (α), permeability parameter (k_m), constant pressure gradient (*P*), thermal conductivity parameter (β), Prandtl number (*Pr*), Eckert number (*Ec*) and magnetic parameter (M). The thermal conductivity parameter (β) may take positive values for liquids such as water, benzene and crude oil, while for gases like air, helium or methane it has negative values. In order to get physical insight into the problem, numerical calculations are performed and the effects of different parameters on velocity field and temperature distribution are observed. Variations in the velocity distribution and temperature field are presented graphically, while variations in the skin-friction and heat transfer rate at the cold wall ($y = 0$) and the hot wall ($y = 1$) are presented in tabular form. The values of Prandtl number (*Pr*) are chosen to be 0.7 and 1.0 respectively, which correspond to air and electrolyte solutions; important fluids, which are used as energy systems and aero-space technologies [6,11]. The numerical values of the remaining parameters are chosen arbitrarily, but do retain physical significance in real energy system applications [14]. Besides, Eckert number (*Ec*) is included to add the dissipative effect in all flow computations with nominal values *Ec* = 0.001, 0.002, 0.003. The value of pressure gradient is constant, as such, in all the cases, a flow regime under constant pressure gradient is studied. The software mathematica is used for computation of the numerical values used in graphs and tables.

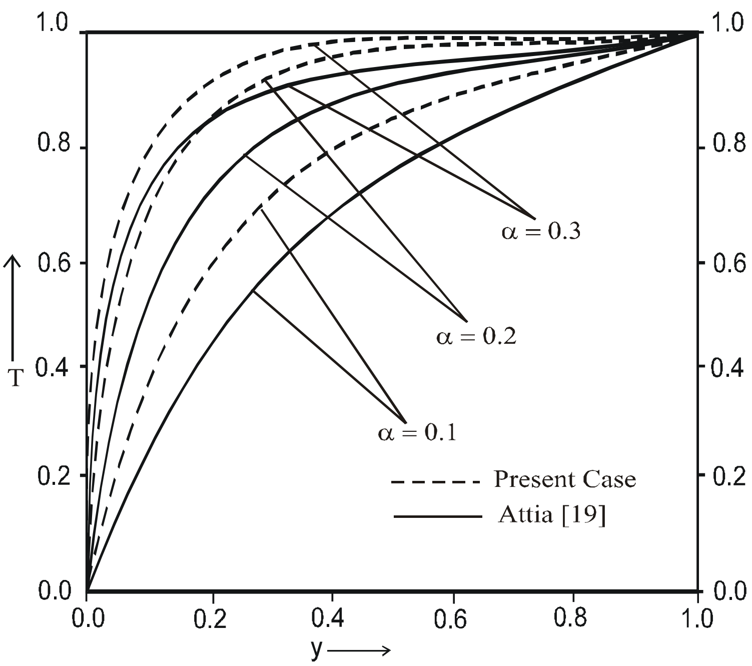

Fig. 2 : Effect of thermal conductivity parameter β on the profile of the fluid temperature T at the centre of the channel.
(y = 0.5, M = 0.0, Pr = 1.0, E_c=0.001, β = 0.0)

Fig. 2 shows effects of thermal conductivity parameter (β) on the profiles of temperature at the centre of the channel for different values of viscosity parameter (α) for $M = 0$ and $\beta = 0$. We observe that at the centre of the channel walls our results are in excellent agreement with Attia [19] in the absence of dust particles. In the figure, the dotted curves are for present case and solid curves are for Attia [19]. In fact, the solid particles gain heat energy from the fluid by conduction through their spherical surface, so that temperature is increased. In absence of solid dust particle, there exists pure flow region and the profiles overlap.

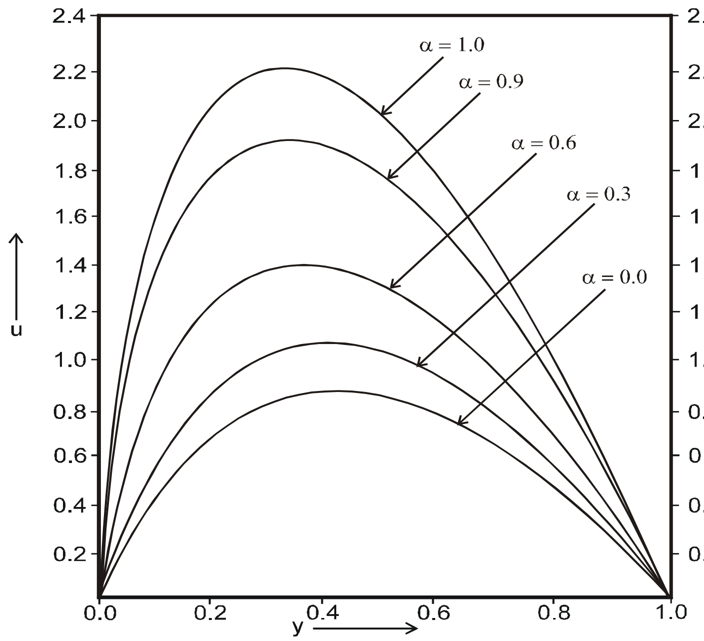

Fig. 3(a) :Effect of viscosity parameter on velocity field
($M = 0.0$, $k_m = 100$, Pr = 1.0, E_c=0.001, β = 0.0)

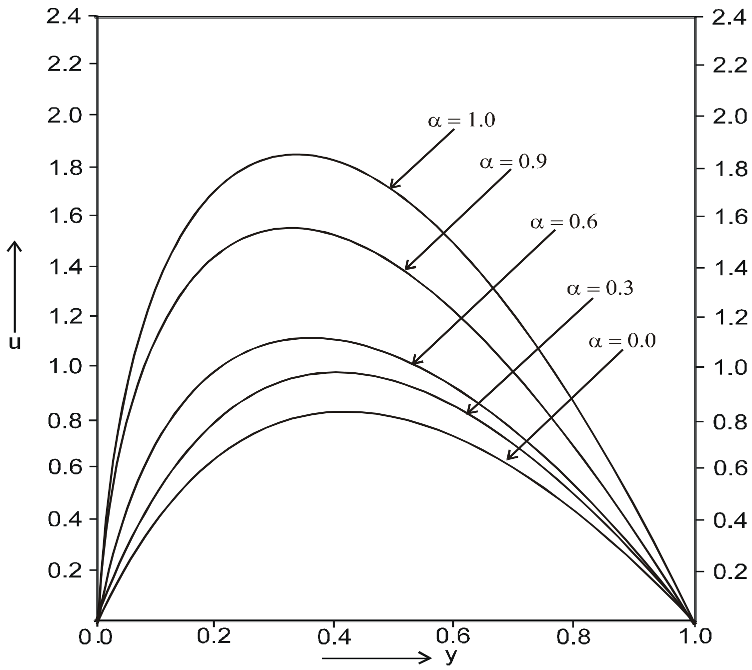

Fig. 3(b) :Effect of viscosity parameter on velocity field
($M = 1.0$, $k_m = 100$, Pr = 1.0, E_c=0.001, β = 0.0)

Fig. 3(a) and 3(b) show the effects of viscosity parameter (α) on velocity field in absence of magnetic field ($M = 0$) and in presence of magnetic field ($M = 1$), respectively, when $k_m = 100$, $Ec = 0.001$, $\beta = 0.0$ and $Pr = 1.0$. It is observed that increasing viscosity parameter increases the velocity and shift the profiles toward the lower wall. The shifting of peak of the velocity profiles toward the lower cold wall is due to the contribution of second term $-\alpha \frac{dT}{dy}\frac{du}{dy}$ in the left hand side of the Eq. (6), which results from the variation of the viscosity with temperature. Actually speaking, this term is equivalent to a variable suction / injection normal to the channel walls. This implies that the suction is acted onto the lower cold wall, while the injection in acted onto upper hot wall: Thus, the velocity increases in the vicinity of the upper wall, which ultimately shifts the peak of the velocity profiles toward the lower wall. We also note that increase in magnetic parameter (M) reduces the velocity due to its damping effect. The application of uniform magnetic field adds an resistance term to the momentum equation and the Joule dissipative term to the energy equation. In fact, the hydromagnetic body force reduces the velocity due to presence of the term $-Mu$ of Eq. (6). This implies that the Lorentz force creates resistance in the fluid, which reduces the fluid velocity. That is why, hydromagnetic force is used as an important controlling mechanism for heat transfer processes in nuclear energy systems, where momentum can be reduced in temperature dependent viscosity regimes, by enhancing the magnetic field [32].

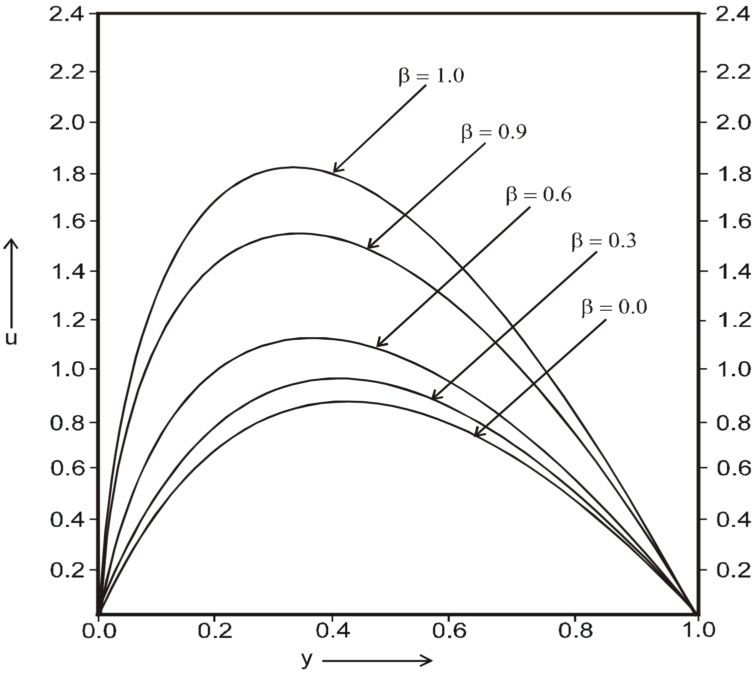

Fig. 4(a) :Effect of thermal conductivity parameter on velocity field (M = 0.0, k_m = 100, Pr = 1.0, E_c=0.001, α = 0.0)

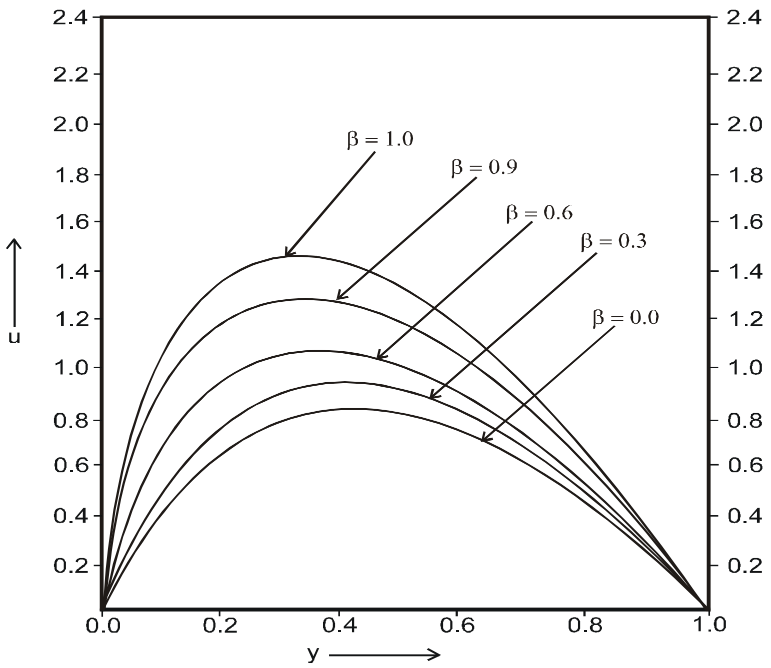

Fig. 4(b) :Effect of thermal conductivity parameter on velocity field
(M = 1.0, k_m = 100, Pr = 1.0, E_c=0.001, α = 0.0)

Fig. 4(a) and 4(b) represent the effect of thermal conductivity parameter (β) on velocity field in absence of magnetic field ($M = 0$) and in presence of magnetic field ($M = 1$), respectively, when $k_m = 100$, $Ec = 0.001$, $\alpha = 0$ and $Pr = 1.0$. It is observed that increasing thermal conductivity parameter (β) increases the velocity, so that the velocity profiles shift toward the lower wall. The shifting of the peak of velocity profiles with increasing β is due to the contribution of third term in the left hand side of the Eq. (7), where β exists in exponential power as a multiple of temperature. This term is in existence due to the variation of the thermal conductivity parameter and viscous dissipation. Hence, the velocity increases with increase in thermal conductivity parameter. We also note that increase in magnetic parameter (M) reduces the velocity. In fact, the additional resistance created by the magnetic force decreases the velocity and increases the temperature.

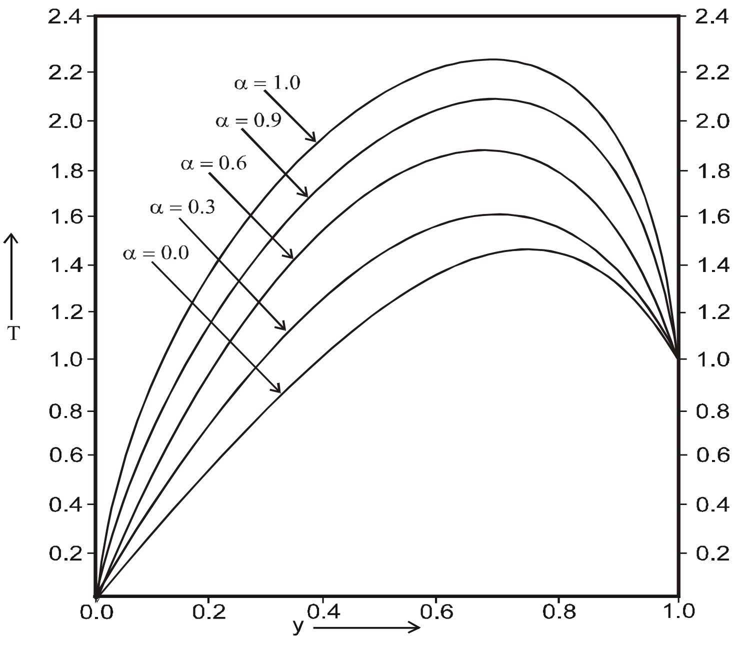

Fig. 5(a) :Effect of viscosity parameter on temperature field
(M = 0.0, k_m = 100, Pr = 1.0, E_c=0.001, β = 0.0)

Fig. 5(a) and 5(b) show the effect of viscosity parameter (α) on temperature distribution in absence of magnetic field ($M = 0$) and in presence of magnetic field ($M = 1$), respectively, when $k_m = 100$, $Ec = 0.001$, $\beta = 0.0$ and $Pr = 1.0$. It is observed that increasing viscosity parameter increases the temperature and profiles shift toward the upper hot wall. It is notable that increasing thermal conductivity parameter (β) increases the velocity (u) and its gradient, which in turn, increases the viscous dissipation and then increases the temperatures. The shifting of the peak of temperature profiles are due to the contribution of second term in the left hand side of the Eqs. (6) and (7), namely $-\alpha \frac{dT}{dy}.\frac{du}{dy}$ and $-\beta\left(\frac{dT}{dy}\right)^2$ respectively. These terms are in existence due to the variation of the viscosity and thermal conductivity with temperature. The term $-\alpha \frac{dT}{dy}.\frac{du}{dy}$, as explained, is equivalent to a variable suction / injection normal to the channel walls. This implies that injection is acted upon the upper hot wall, while the suction is acted upon the lower cold wall. Thus, the temperature in the vicinity of the upper wall increases more rapidly and the temperature profiles shift toward upper hot wall. We also note that increase in magnetic parameter (*M*) increases the temperature. The physics behind this phenomenon is that the viscous dissipation and Joule dissipation terms contribute a heat addition, which increases the temperature throughout the region. This confirms the useful properties of magnetism in controlling transient temperatures, by adjusting hydromagnetic force suitably, in naval, nuclear and energy systems [33].

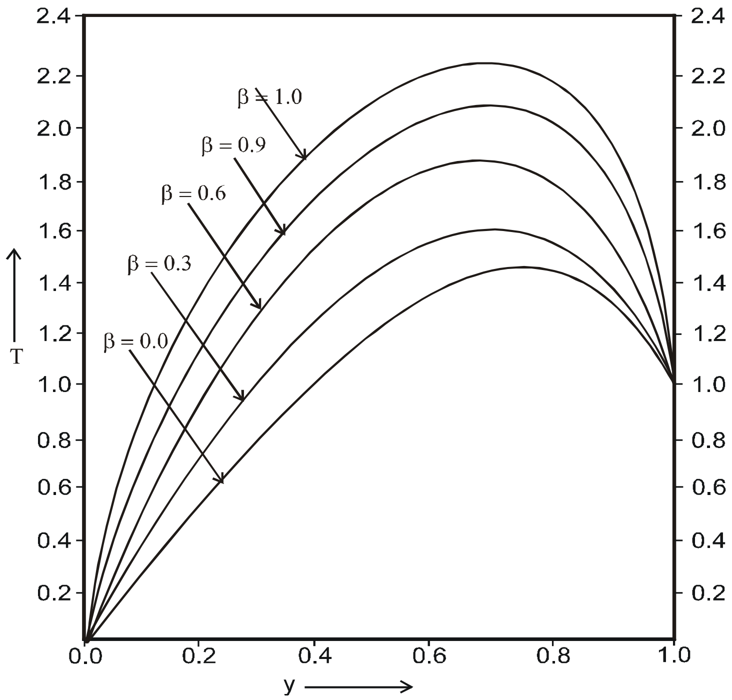

Fig. 6(a) :Effect of thermal conductivity parameter on temperature field ($M = 0.0$, $k_m = 100$, Pr = 1.0, E_c=0.001, $\alpha = 0.0$)

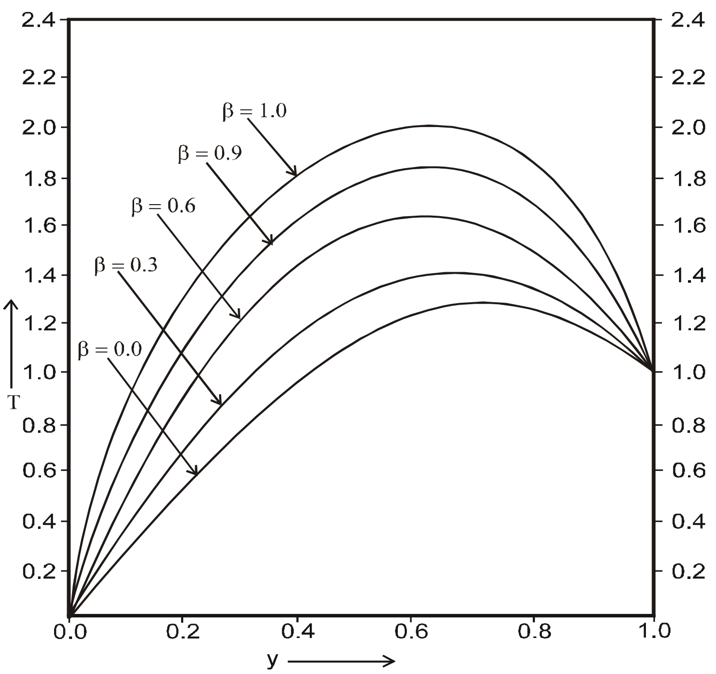

Fig. 6(b) :Effect of thermal conductivity parameter on temperature field (M = 1.0, k_m = 100, Pr = 1.0, E_c=0.001, α = 0.0)

. Fig. 6(a) and 6(b) show the effect of thermal conductivity parameter (β) on temperature distribution in absence of magnetic field ($M = 0$) and in presence of magnetic field ($M = 1$), respectively, when $k_m = 100$, $Ec = 0.001$, $\alpha = 0.0$ and $Pr = 1.0$. It is observed that increasing thermal conductivity parameter increases temperature. Again, as explained, the temperature profiles shift toward the upper hot wall and the shifting of the peak of temperature profiles is due to the contribution of second term in the left hand side of the Eqs. (6) and (7). We also note that increase in magnetic parameter (M) increases the temperature. The physics behind this phenomenon is that the viscous dissipation and Joule dissipation terms contribute a heat addition, which increases the temperature.

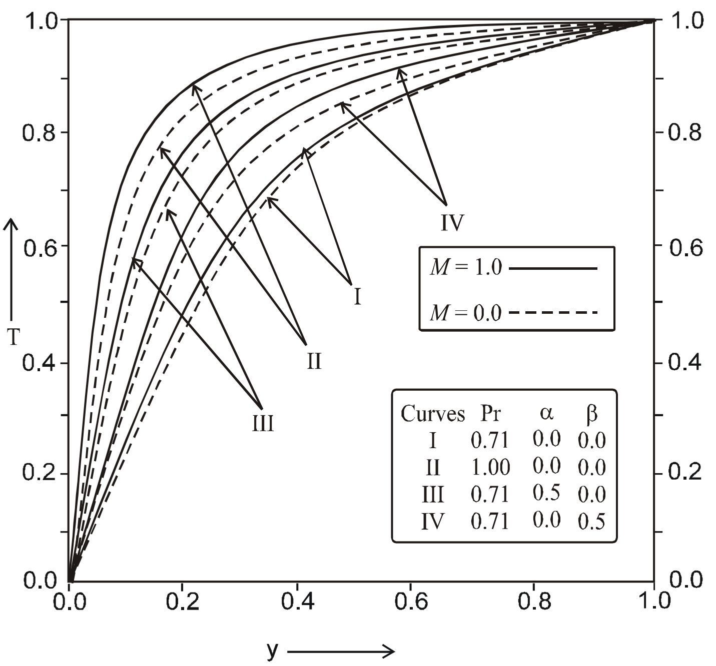

Fig. 7: Effects of Prandtl number, viscosity parameter and thermal conductivity parameter on temperature field (k_m = 100 and E_c=0.001)

. Fig. 7 shows the effect of Prandtl number (Pr) on temperature distribution versus non-dimensional y-coordinate, when $k_m = 100$ and $Ec = 0.001$. It is observed that increasing Prandtl number decreases the temperature. Mathematically, the Prandtl number (*Pr*) defines the ratio of the momentum diffusivity to the thermal diffusivity. Hence, higher *Pr*- fluids transfer heat less effectively as compared to lower *Pr*- fluids. Consequently, lower temperatures are observed in profiles II in comparison with profile I, Also it is observed that increasing of the viscosity parameter (α) or thermal conductivity parameter (β) increases the temperature (*T*). This can be attributed to the fact that the centre of the channel is colder than the upper half flow region and acquires heat by conduction from the upper hot plate, which increases the temperature at the centre. Hence, the temperature increases with increasing α or β . It is interesting to note that the Prandtl number (*Pr*) plays dominant role in declining transient temperature.

Table-1 represents comparison of present numerical values of the velocity (*u*) with numerical values of Attia [19] at the middle of the channel walls ($y = 0.5$) for different values of α and β choosing $M = 0.0$, $Pr = 1.0$, $Ec = 0.001$ and $k_m = 100$. We observe that at the centre of the channel walls our results are in good agreement with Attia [19].

Table-2, 3 show variations in the velocity (*u*) at the middle of channel walls for different values of α and β for $M = 0$ and $M = 1$, respectively, when $y = 0.5$, $Pr = 1.0$, $Ec = 0.001$ and $k_m = 100$. It is observed that an increase in α or β increases the velocity at the middle of the channel wall, but increase in α is more pronounced than β . At the surface of lower cold plate $\alpha = 0$ and $\beta = 0$, so that the viscosity of the fluid as well as thermal conductivity of the fluid becomes constant under the influence of uniform magnetic field. Hence, the velocity at the centre of the channel is more when $M = 0$ as compared to $M = 1$.

Table-4 and Table-5 show the variations in temperature *T* at the middle of channel walls for different values of α and β at $M = 0$ and $M = 1$, respectively, when $y = 0.5$, $Pr = 1.0$, $Ec = 0.001$ and $k_m = 100$. Again, it is observed that increase in α or β increases the temperature at the middle of the channel wall but increase in α is more pronounced than β . The presence of magnetic parameter (*M*) increases the temperature at the centre of the channel wall due to Joule dissipation.

Table-6, 7 present numerical values of skin-friction (τ) and heat transfer rate (*Nu*) for the lower channel wall $y = 0$ and the upper channel wall $y = 1$ due to change in α and β at the magnetic parameter $M = 0$ and $M = 1$, when $y = 0$, $Pr = 1.0$, $Ec = 0.001$ and $k_m = 100$. These tables are self explanatory, therefore any discussion about them seems to be redundant.

7. Conclusions

In this paper, the flow and heat transfer of an electrically conducting, incompressible viscous fluid through a horizontal channel with parallel walls embedded in a homogeneous porous medium is studied in the presence of an external uniform magnetic field. The variations of the viscosity and thermal conductivity of the fluid with temperature are taken into account using Arrhenius model. The effects of different parameters governing the convection flow are observed. The conclusions of the study are as follows:

- An increase in viscosity parameter (α) increases the velocity.
- An increase in thermal conductivity parameter (β) increases the velocity and the effect of viscosity parameter (α) is more pronounced in comparison with thermal conductivity parameter.
- An increase in uniform magnetic field decreases the velocity.
- An increase in viscosity parameter (α) increases the temperature.
- An increase in thermal conductivity parameter (β) increases the temperature.
- An increase in uniform magnetic field increases the temperature.
- An increase in Prandtl number (*Pr*) decreases temperature.

References

[1]. Herwig, H. Wicken, G., The effect of variable properties on laminar boundary layer flow, *Warme-und Stoffubertragung*, 1986, **20**, pp. 47-57.

[2]. Klemp, K. Herwig, H. Selmann, M., Entrance flow in channel with temperature dependent viscosity including viscous dissipation effects, *Proc. Third Int. Cong. Fluid Mech. Cairo, Egypt*., 1990, **3**, pp. 1257-1266.

[3]. Ling, J. X. Lybbs, A., The effect of variable viscosity on forced convection over a flat plate submersed in a porous medium, *Trans. ASME, J. Heat Transfer*, 1992, **114**, pp. 1063-1065.

[4]. Rao, K. N. Pop, I., Transient free convection in fluid saturated porous medium with temperature dependent viscosity, *Int. Comm. Heat Mass Transfer*, 1994, **21**, pp. 573-581.

[5]. Kafoussius, N. G. Williams, E. M., The effect of temperature dependent viscosity on the free convective laminar boundary layer flow past a vertical isothermal plate, *Acta Mechanica*, 1995, **110**, pp. 123-137.

[6]. Kafoussius, N. G. Rees, D. A. S., Numerical study of the combined free and forced convective laminar boundary layer flow past a vertical isothermal flat plate with temperature dependent viscosity, *Acta Mechanica*, 1998, **127**, pp. 39-50.

[7]. Singh, N. P. Singh, A. K. Yadav, M. K. Singh, A. K., Unsteady free convection flow in a fluid saturated porous medium with temperature dependent viscosity, *Bull. Cal. Math. Soc*., 2000, **92**, pp. 351-356.

[8]. Hazarika, G. C. Phukan, P., The effect of variable viscosity on magneto-hydrodynamic flow and heat transfer to a continuous moving flat plate, *The Mathematics Education*, 2001, **35**, pp. 234-241.

[9]. Hossain, M. A. Khanafer, K. Vafai, K., The effect of radiation on free convection of fluid with variable viscosity from a porous vertical plate, *Int. J. Therm. Sci*., 2001, **40**, pp. 115-124.

[10]. Bagai, S., Effect of variable viscosity on free convection over a non-isothermal axisymmetric body in a porous medium with internal heat generation, *Acta Mechanica*., 2004, **169**, pp. 187-194.

[11]. Barakat, E. I. I., Variable viscosity effect on hydromagnetic flow and heat transfer about a fluid underlying the axi-symmetric spreading surface, *Acta Mechanica*., 2004, **169**, pp. 195-202.

[12]. Cheng, Y. C., The effect of temperature dependent viscosity on the natural convection heat transfer from a horizontal isothermal cylinder of elliptic cross- section, *Int. Comm. Heat and Mass Transfer*, 2006, **33**, pp. 1021-1028.

[13]. Molla, M. M. Hossain, M. A. Gorla, R. S. R., Natural convection flow from an isothermal circular cylinder with temperature dependent viscosity, *Heat and Mass Transfer*, 2005, **41**, pp. 594-598.

[14]. Attia, H. A., Steady Hartmann flow with temperature dependent viscosity and the ion-slip, *Int. Comm. Heat Mass Transfer*, 2003, **30**, pp. 881-890.

[15]. Attia, H. A., Transient MHD flow and heat transfer between two parallel plates with temperature dependent viscosity, *Mech. Res. Comm*., 1999, **26**, pp. 115-121.

[16]. Kankane, N. Gokhale, M. Y., Fully developed channel flow with variable viscosity, *Proc. Math. Soc. BHU*., 2005, **21**, pp. 1-9.

[17]. Pantokratoras, A., Forced and mixed convection boundary layer flow along a flat plate with variable viscosity and variable Prandtl number: New results, *Heat Mass Transfer*, 2005, **41**, pp. 1085-1094.

[18]. Pantokratoras, A., Non-Darcian forced convection heat transfer over a flat plate in a porous medium with variable viscosity and variable Prandtl number, *J. Porous Media*, 2007, **10**, pp. 201-208.

[19]. Attia, H. A., Unsteady hydromagnetic couette flow of dusty fluid with temperature dependent viscosity and thermal conductivity, *Int. J. Non-Linear Mech*., 2008, **43**, pp. 707-715.

[20]. Singh, A. K. Sharma, P. K. Singh, N. P., Free convection flow with variable viscosity through horizontal channel embedded in porous medium, *The Open Applied Physics Journal*, 2009, **2**, pp. 11-19.

[21]. Ganji, D. D. Varedi, S. M. Rahimi, M., Application of analytical methods to some systems of non-linear equations arising in fluid flows with variable viscosity, *Journal of Energy, Heat and Mass Transfer*, 2008, **30**, pp. 287-310.

[22]. Singh, A. K. Raghav, S. Singh, N. P., Temperature distribution in generalized Couette flow of two immiscible fluids in presence of uniform magnetic field and naturally permeable boundary, *Journal of Energy, Heat and Mass Transfer*, 2010, **32**, pp. 179-198.

[23]. Singh, N. P. Singh, A. K. Agnihotri, P., Effect of load parameter on unsteady MHD convective flow of viscous immiscible liquids in a horizontal channel: Two fluid model, *Journal of Porous Media*, 2010, **13**, pp. 439-455.

[24]. Singh, N. P. Singh, A. K. Singh, D. Singh, A. K., Hydromagnetic convective flow of viscous fluid through porous medium in horizontal channel with constant rate of heat addition and radiation, *Proc. Nat. Acad. Sci. India*, 2011, **81**, pp. 165-171.

[25]. Singh, N. P. Singh, Atul K. Singh, Ajay K. Singh, U. Kumar, R., Hydro-dynamically and thermally fully developed MHD flow between two horizontal parallel plates: Analysis with viscous dissipation and constant heat flux boundary conditions, *Proc. Nat. Acad. Sci. India*, 2011, **81**, pp. 295-302.

[26]. Singh, Ajay K. Singh, U. Singh, H. Singh, N. P., Transient micropolar fluid flow and heat transfer past a semi-infinite vertical porous plate with variable suction/injection and non-homogeneous porous medium, *J*. Energy, Heat Mass Transfer, 2011, **33**, pp.251-270.

[27]. Singh, N. P. Singh, Ajay K. Singh, Atul K. Agnihotri, P., Effects of thermophoresis on hydromagnetic mixed convection and mass transfer flow past a vertical permeable plate with variable suction and thermal radiation, Commun. Nonlinear Sci. Numer. Simul., 2011, **16**, pp. 2519-2534.

[28]. Schlichting, H., *Boundary Layer Theory*, New York, McGraw-Hill, Book Co., 1968.

[29]. White, M. F., *Viscous Fluid Flow*, New York, McGraw-Hill, Book Co., 1991.

[30]. Singh, N. P. Singh, Atul K. Singh, D., Hydromagnetic free convection flow with variable viscosity through horizontal channel: analysis with viscous dissipation, Impact J. Sci. Techno., 2008, **2**, pp. 231-241.

[31]. Crammer, K. R. Pai, S. I., *Magneto-Fluid Dynamics for Engineers and Physicists*, New York, McGraw-Hill, Book Co., 1973.

[32]. Kaviany, M., *Principles of Heat Transfer in Porous Media*, Springer, 1995.

[33]. Rosa, R. J., Magnetohydrodynamic Energy Conversion, New York, McGraw-Hill, Book Co., 1968.

[34]. Blums, E., *Heat and Mass Transfer in MHD Flow*, Singapore, World Science, 1987.

Appendix

$$m_1 = \frac{-\alpha - \sqrt{\alpha^2 + 4m_1}}{2}, \quad m_2 = \frac{-\alpha + \sqrt{\alpha^2 + 4m_1}}{2}, \quad M_1 = k_m^{-1} + M, \quad K_1 = \frac{P e^{\alpha}}{M_1},$$

$$C_1 = \frac{P\left(e^{(\alpha + m_2)} - 1\right)}{M_1\left(e^{m_1} - e^{m_2}\right)}, \quad C_2 = \frac{P\left(1 - e^{(\alpha + m_1)}\right)}{M_1\left(e^{m_1} - e^{m_2}\right)}, \quad K_2 = \frac{-Pr\,C_1^2 e^{\beta-\alpha}\left(m_1^2 + M\right)}{\left(2m_1 + \alpha\right)^2},$$

$$K_3 = \frac{-Pr\,C_2^2 e^{\beta-\alpha}\left(m_2^2 + M\right)}{\left(2m_2 + \alpha\right)^2}, \quad K_4 = \frac{-Pr\,K_1^2 e^{\beta-\alpha}\left(\alpha^2 + M\right)}{\alpha^2},$$

$$K_5 = \frac{-2\,Pr\,C_1 C_2 e^{\beta-\alpha}\left(m_1 m_2 + M\right)}{\left(m_1 + m_2\right)^2}, \quad K_6 = \frac{2\,Pr\,C_1 K_1 e^{\beta-\alpha}\left(m_1\alpha - M\right)}{m_1^2},$$

$$K_7 = \frac{2\,Pr\,C_2 K_1 e^{\beta-\alpha}\left(m_2\alpha - M\right)}{m_2^2}, \quad C_3 = -\left(K_2 + K_3 + K_4 + K_5 + K_6 + K_7\right),$$

$$C_4 = K_2\left(1 - e^{2m_1+\alpha}\right) + K_3\left(1 - e^{2m_2+\alpha}\right) + K_4\left(1 - e^{-\alpha}\right)$$

$$+ K_5\left(1 - e^{(m_1+m_2+\alpha)}\right) + K_6\left(1 - e^{m_1}\right) + K_7\left(1 - e^{m_2}\right),$$

$$K_8 = \alpha C_1\left(m_1^2 + \alpha m_1 - M_1\right), \quad K_9 = \alpha C_2\left(m_2^2 + \alpha m_2 - M_1\right), \quad K_{10} = -\alpha K_1 M_1,$$

$$K_{11} = K_8 C_3 + \alpha m_1 C_1\left(C_4 - \beta C_3\right), \quad K_{12} = K_9 C_3 + \alpha m_2 C_2\left(C_4 - \beta C_3\right),$$

$$K_{13} = K_{10} C_3 - \alpha^2 K_1\left(C_4 - \beta C_3\right), \quad K_{14} = C_4\left(K_8 - \alpha\beta m_1 C_1\right),$$

$$K_{15} = C_4\left(K_9 - \alpha\beta m_2 C_2\right), \quad K_{16} = C_4\left(K_{10} + \alpha^2\beta K_1\right),$$

$$K_{17} = K_2\left[K_8 + \alpha C_1 m_1\left(2m_1 + \alpha - \beta\right)\right], \quad K_{18} = K_3\left[K_9 + \alpha C_2 m_2\left(2m_2 + \alpha - \beta\right)\right],$$

$$K_{19} = K_4\left[K_{10} + \alpha^2 K_1\left(\alpha + \beta\right)\right],$$

$$K_{20} = K_8 K_5 + K_9 K_2 + \alpha m_1 C_1 K_5\left(m_1 + m_2 + \alpha - \beta\right) + \alpha m_2 C_2 K_2\left(2m_1 + \alpha - \beta\right),$$

$$K_{21} = K_8 K_3 + K_9 K_5 + \alpha m_2 C_2 K_5\left(m_1 + m_2 + \alpha - \beta\right) + \alpha m_1 C_1 K_3\left(2m_2 + \alpha - \beta\right),$$

$$K_{22} = K_8 K_6 + K_{10} K_2 + \alpha m_1 C_1 K_6\left(m_1 - \beta\right) - \alpha^2 K_1 K_2\left(2m_1 + \alpha - \beta\right),$$

$$K_{23} = K_9 K_7 + K_{10} K_3 + \alpha m_2 C_2 K_7\left(m_2 - \beta\right) - \alpha^2 K_1 K_3\left(2m_2 + \alpha - \beta\right),$$

$$K_{24} = K_8 K_4 + K_{10} K_6 - \alpha^2 K_1 K_6\left(m_1 - \beta\right) - \alpha m_1 C_1 K_4\left(\alpha + \beta\right),$$

$$K_{25} = K_9 K_4 + K_{10} K_7 - \alpha^2 K_1 K_7\left(m_2 - \beta\right) - \alpha m_2 C_2 K_4\left(\alpha + \beta\right),$$

$$K_{26} = K_8K_7 + K_9K_6 + K_{10}K_5 - \alpha^2 K_1K_5\left(m_1 + m_2 + \alpha - \beta\right) + \alpha m_2C_2K_6\left(m_1 - \beta\right) + \alpha m_1C_1K_7\left(m_2 - \beta\right),$$

$$K_{27} = K_{11}\left[\left(m_1 - \beta\right)^2 + \alpha\left(m_1 - \beta\right) - M_1\right]^{-1}, \quad K_{28} = K_{12}\left[\left(m_2 - \beta\right)^2 + \alpha\left(m_2 - \beta\right) - M_1\right]^{-1},$$

$$K_{29} = K_{13}\left[\beta\left(\alpha + \beta\right) - M_1\right]^{-1}, \qquad K_{30} = \frac{K_{14}}{A_1}, \qquad K_{31} = K_{14}A_1^{-2}\left(-2m_1 + 2\beta - \alpha\right),$$

$$K_{32} = \frac{K_{15}}{A_2}, \quad K_{33} = K_{15}A_2^{-2}\left(-2m_2 + 2\beta - \alpha\right), \quad K_{34} = \frac{K_{16}}{A_3}, \quad K_{35} = 2K_{16}A_3^{-2}\left(\alpha + \beta\right),$$

$$K_{36} = K_{17}\left[\left(3m_1 + \alpha - \beta\right)^2 + \alpha\left(3m_1 + \alpha - \beta\right) - M_1\right]^{-1},$$

$$K_{37} = K_{18}\left[\left(3m_2 + \alpha - \beta\right)^2 + \alpha\left(3m_2 + \alpha - \beta\right) - M_1\right]^{-1},$$

$$K_{38} = K_{19}\left[\left(2\alpha + \beta\right)^2 - \alpha\left(2\alpha + \beta\right) - M_1\right]^{-1},$$

$$K_{39} = K_{20}\left[\left(2m_1 + m_2 + \alpha - \beta\right)^2 + \alpha\left(2m_1 + m_2 + \alpha - \beta\right) - M_1\right]^{-1},$$

$$K_{40} = K_{21}\left[\left(m_1 + 2m_2 + \alpha - \beta\right)^2 + \alpha\left(m_1 + 2m_2 + \alpha - \beta\right) - M_1\right]^{-1},$$

$$K_{41} = K_{22}\left[\left(2m_1 - \beta\right)^2 + \alpha\left(2m_1 - \beta\right) - M_1\right]^{-1},$$

$$K_{42} = K_{23}\left[\left(2m_2 - \beta\right)^2 + \alpha\left(2m_2 - \beta\right) - M_1\right]^{-1},$$

$$K_{43} = K_{24}\left[\left(m_1 - \alpha - \beta\right)^2 + \alpha\left(m_1 - \alpha - \beta\right) - M_1\right]^{-1},$$

$$K_{44} = K_{25}\left[\left(m_2 - \alpha - \beta\right)^2 + \alpha\left(m_2 - \alpha - \beta\right) - M_1\right]^{-1},$$

$$K_{45} = K_{26}\left[\left(m_1 + m_2 - \beta\right)^2 + \alpha\left(m_1 + m_2 - \beta\right) - M_1\right]^{-1},$$

$$K_{46} = K_{27} + K_{28} + K_{29} + K_{31} + K_{33} + K_{35} + K_{36} + K_{37} + K_{38} + K_{39} + K_{40} + K_{41} + K_{42} + K_{43} + K_{44} + K_{45},$$

$$K_{47} = \left(K_{27} + K_{30} + K_{31}\right)e^{\left(m_1 - \beta\right)} + \left(K_{28} + K_{32} + K_{33}\right)e^{\left(m_2 - \beta\right)} + \left(K_{29} + K_{34} + K_{35}\right)e^{\left(\alpha + \beta\right)} + K_{36}e^{\left(3m_1 + \alpha - \beta\right)} + K_{37}e^{\left(3m_2 + \alpha - \beta\right)} + K_{38}e^{-\left(2\alpha + \beta\right)} + K_{39}e^{\left(2m_1 + m_2 + \alpha - \beta\right)} + K_{40}e^{\left(m_1 + 2m_2 + \alpha - \beta\right)} + K_{41}e^{\left(2m_1 - \beta\right)} + K_{42}e^{\left(2m_2 - \beta\right)} + K_{43}e^{\left(m_1 - \alpha - \beta\right)}$$

$$+K_{44}e^{(m_2-\alpha-\beta)}+K_{45}e^{(m_1+m_2-\beta)},$$

$$C_5=\frac{K_{47}-K_{46}e^{m_4}}{e^{m_4}-e^{m_3}},\quad C_6=\frac{K_{46}e^{m_3}-K_{47}}{e^{m_4}-e^{m_3}},\quad A_1=(m_1-\beta)^2+\alpha(m_1-\beta)-M_1,$$

$$A_2=(m_2-\beta)^2+\alpha(m_2-\beta)-M_1,\qquad A_3=\beta(\alpha+\beta)-M_1,$$

$$\begin{aligned}K_{48}={}&m_3C_5+m_4C_6+(m_1-\beta)K_{27}+(m_2-\beta)K_{28}-(\alpha+\beta)K_{29}+K_{30}\\&+(m_1-\beta)K_{31}+K_{32}+(m_2-\beta)K_{33}+K_{34}-(\alpha+\beta)K_{35}\\&+(3m_1+\alpha-\beta)K_{36}+(3m_2+\alpha-\beta)K_{37}-(2\alpha+\beta)K_{38}\\&+(2m_1+m_2+\alpha-\beta)K_{39}+(m_1+2m_2+\alpha-\beta)K_{40}\\&+(2m_1-\beta)K_{41}+(2m_2-\beta)K_{42}+(m_1-\alpha-\beta)K_{43}\\&+(m_2-\alpha-\beta)K_{44}+(m_1+m_2-\beta)K_{45},\end{aligned}$$

$$\begin{aligned}K_{49}={}&m_3C_5e^{m_3}+m_4C_6e^{m_4}+(m_1-\beta)K_{27}e^{(m_1-\beta)}+(m_2-\beta)K_{28}e^{(m_2-\beta)}\\&+K_{30}e^{(m_1-\beta)}+(m_1-\beta)(K_{30}+K_{31})e^{(m_1-\beta)}\\&+K_{32}e^{(m_2-\beta)}+(m_2-\beta)(K_{32}+K_{33})e^{(m_2-\beta)}\\&+K_{34}e^{-(\alpha+\beta)}-(\alpha+\beta)(K_{34}+K_{35})e^{-(\alpha+\beta)}\\&+(3m_1+\alpha-\beta)K_{36}e^{(3m_1+\alpha-\beta)}+(3m_2+\alpha-\beta)K_{37}e^{(3m_2+\alpha-\beta)}\\&-(2\alpha+\beta)K_{38}e^{-(2\alpha+\beta)}+(2m_1+m_2+\alpha-\beta)K_{38}e^{(2m_1+m_2+\alpha-\beta)}\\&+(m_1+2m_2+\alpha-\beta)K_{40}e^{(m_1+2m_2+\alpha-\beta)}+(2m_1-\beta)K_{41}e^{(2m_1-\beta)}\\&+(2m_2-\beta)K_{42}e^{(2m_2-\beta)}+(m_1-\alpha-\beta)K_{43}e^{(m_1-\alpha-\beta)}\\&+(m_2-\alpha-\beta)K_{44}e^{(m_2-\alpha-\beta)}+(m_1+m_2-\beta)K_{45}e^{(m_1+m_2-\beta)},\end{aligned}$$

$$\begin{aligned}K_{50}={}&C_4-\beta C_3+(2m_1+\alpha-\beta)K_2+(2m_2+\alpha-\beta)K_3-(\alpha+\beta)K_4\\&+(m_1+m_2+\alpha-\beta)K_5+(m_1-\beta)K_6+(m_2-\beta)K_7,\end{aligned}$$

$$\begin{aligned}K_{51}={}&-\beta(C_3+C_4)e^{-\beta}+C_4e^{-\beta}+(2m_1+\alpha-\beta)K_3e^{(2m_1+\alpha-\beta)}\\&+(2m_2+\alpha-\beta)e^{(2m_2+\alpha-\beta)}-(\alpha+\beta)e^{-\beta}+C_4e^{-\beta}\\&+(m_1+m_2+\alpha-\beta)e^{(m_1+m_2+\alpha-\beta)}+(m_1-\beta)K_6e^{(m_1-\beta)}\\&+(m_2-\beta)K_7e^{(m_2-\beta)}.\end{aligned}$$