Comparision of Some Operator Graphs

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Abstract

*Let (G, *) be a group. We define Operator power graph $\Gamma op(G)$, Intersection operator graph $\Gamma_{IO}(G)$, Operator Intersection graph $\Gamma_{OI}(G)$ of G. In this paper we want to explore how the group theoretical properties of G can effect on the graph theoretical properties of $\Gamma op(G)$, $\Gamma_{IO}(G)$, $\Gamma_{OI}(G)$. Some characterizations for fundamental properties of these graphs have been obtained and we characterize certain classes of Operator graphs corresponding to some groups of special order then compare these graphs finally.*

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1.INTRODUCTION

To any group G ,we assign a graph and investigating algebraic properties of the group using graph theoretical concepts. There are many papers on assigning a graph to a group or ring and thereby investigating algebraic properties of the group or ring using the ssociated graph. The Cayley Graph of finite groups was first introduced by Arthur Cayley in 1878. Max Dehn in his unpublished lectures on group theory from 1909-10 reintroduced Cayley graphs under the name Gruppenbild (group diagram), which led to the development of geometric group theory of today. The non-commuting graph Γ_G was first introduced by Paul Erdos, In 2002, the directed power graph of a semigroup was defined by Kelarev and Quinn . Andrea Lucchini and Attila Maroti have worked on the graph called generating graph .In 2011, the Subgroup intersection graph of a group was defined by T.Tamilchelvam and M. Sattanathan.

Let G be a group with identity e . The order of the group G is the number of elements in G and is denoted by $O(G)$. The order of an element a in a group G is the smallest positive integer k such that $a^k = e$. If no such integer exists, we say a has infinite order. The order of an element a is denoted $O(a)$. Let p be a prime number. A group G with $o(G) = p^k$ for some $k \in \mathbb{Z}+$, is called a p-group. We consider simple graphs which are undirected, with no loops or multiple edges. For and graph $\Gamma=(V,E)$, V denotes the set of all vertices and E denotes the set of all edges in Γ. The degree of a vertex v in Γ is the number of edges incident to v and we denote by $deg(v)$. A vertex of degree 0 is known as an *isolated vertex of* Γ. A simple graph Γ is said to be complete if every pair of distinct vertices of Γ are adjacent in Γ.

2. 1. OPERATOR POWER GRAPH

Definition 2.1.1 Let (G,*) be a group with binary operation *. The Operator power graph Γop (G) is graph with $V(\Gamma op(G)) = G$ and the distinct vertices x and y are adjacent in Γop (G) iff either $x=(x*y)^n$ or $y = (x*y)^m$

Proposition 2.1.2 Let (G,*) be a group with n elements.The identity element e of G has degree n-1 and for any non- self inverse element $x \in G$, x and x^{-1} are non- adjacent in Γop (G)

Theorem 2.1.3 $\Gamma op(G)$ is complete iff $G \cong Z_1$ or Z_2

If G has a self-inverse element, $\Gamma op(G)$ cannot be complete. Suppose every element of G is self-inverse. Let $x,y \in G$ such that $x \neq e$; $y\neq e$ .then $o(x)= o(y) =o(x* y)$ and hence x and y are non-adjacent which is a contradiction.

Proposition 2.1.4 Let G be a finite group of order n with no self-inverse element. Then number of edges in $\Gamma op (G) \leq (n-1)^2 /2$

Proof: Deg Γop (G)(e) = n-1 and deg Γop (G) (x) $\leq$ n-2 . The degree sum $\leq(n-1)^2$ and hence the result.

Theorem 2.1.5 Let G be a finite group order in and q be the number of edges in Γop (G)

q= $(n-1)^2/2$ iff G is a group of prime order.
Proof: deg Γop(a) = n-2 for all vertices a∊ G-e . Every element of G has a unique prime order and hence G is a group of prime order.
Conversely, let G=<a>. G=<a>=<a^2>.=<a^3>=… = <a^{n-1}> . Then deg Γop (G)(a^i) = n-2 and the result follows.

Theorem 2.1.6 Let G be a cyclic graph of order 2p. Where p is a prime and p≥3. Then the number of edges of Γop (G) is equal to $(3p^2-4p+3)/2$
Proof: Let A={0}, B={1,3,5…..p-2,….2p-1} , B={2,4,6…,2p-2},D={p}. The elements of B are generators of G. The elements of C have an order p and the element in D has order 2. Every element $x \in B$ is adjacent to all elements in C other than p-x and the elements in A. Every element in C is adjacent to all in elements in C other than its inverse and adjacent to the element in D.
Sum of the degrees of all vertices = 2p-1+(p-1) (p-1) +(p-1) (2p-3)+ p = $3p^2-4p+3$.

Theorem 2.1.7 Let G be a cyclic group of order P^2 where p is an odd prime. Then the number of edges of Γop(G) is $(p-1)(p^3-1))/2$
Proof: The vertex set of G can be partitioned into three sets A,B and C such that A= {0} B= {p,2p,3p, … (p-1)p} and C= G-A-B. degΓ_{OP} (G) (0) =p^2-1 .The elements of C are generators of G and the elements of B have order p. Let x, y ∊ B. Since A∪B is a subgroup of G, either x+y ∊ B or x+y= 0. Hence for all x in B, x is adjacent to all elements other than its inverse in B. Let x ∊ B and y ∊ C, then x + y ∊ C. Therefore for all x in B, x is adjacent to all elements in C. Therefore for all x ∊ B,deg Γ_{OP} (x) = 1 + p −3 + p(p −1) = p^2 −2. Clearly an element in C is adjacent to all elements in A and B. For each x in C, x is not adjacent to p -x, 2p -x, . . ., p^2 -x, which are in C and hence the result is got by taking the degree sum .

2.2. INTERSECTION OPERATOR GRAPH

Definition 2.2.1 Let (G,*) be a group. The Intersection operator graph Γ_{IO} (G) of G is a graph with V(Γ_{IO} (G))= G and the vertices x &y are adjacent iff <x> ∩<y> ⊆ <x*y>

Result 2.2.2 The identity element e of G has degree n-1 and for any non- self inverse element x ∊ *G*, x and x^{-1} are non- adjacent in Γ_{IO} (G)

Proposition 2.2.3 Let (G, *) be a group. Any two elements of distinct prime order are adjacent in Γ_{IO} (G)
Proof: Let x and y be any two elements O(x)=p and O(y)=q, then <x>∩<y>={e}. Hence x & y are adjacent.

Remark 2.2.4 The converse of the Proposition need not be true. In Γ_{IO} (Z_5), 2 and 4 are adjacent but o(2) = o(4) =5

Theorem 2.2.5 Let G be any group. Γ_{IO} (G) is complete if and only if every element of G is a self-inverse element.
Proof: Assume that every element of G is a self-inverse element .Let x, y∊ G .< x >∩< y >= {e}. Therefore < x >∩<y >⊆ <x*y>. Hence x and y are adjacent in Γ_{IO} (G) .Since x and y are arbitrary , any two elements in G are adjacent in Γ_{IO} (G). Suppose G has a non- self-inverse element x, x and x^{-1} are non-adjacent in Γ_{IO} (G) , which is a contradiction and hence the result follows.

Theorem 2.2.6 Let G be a group. Γ_{IO} (G) is a star graph if and only if G ≅ Z_2 or Z_3
Proof: Clearly if G ≅ Z_2 or Z_3, then the corresponding Γ_{IO} (G) is a star graph.Conversely assume that Γ_{IO} (G) = $K_{1,n}$. Since the identity element 'e' has a full degree, any two non-identity elements are non-adjacentin Γ_{IO} (G). Suppose that G has an element x of order k such that k >3. < x >={e, x, x^2, x^3, . . . , x^{k-2}, x^{k-1}}. Since x * x^{k-2} = x^{k-1} = x^{-1}, < x >=<x *x^{k-2} > ,x and x^{k-2} are adjacent, which is a contradiction. Suppose that G has atleast two distinct subgroups of order either 2 or 3. Let x, y ∊ G be any two elements of order 2 such that < x >≠< y >. Clearly,< x >∩< y >= {e}. Therefore < x >∩< y >⊆< x * y >. Hence x and y are adjacent in Γ_{IO} (G) which is a contradiction. Therefore G has a unique subgroup of order either 2 or 3. Hence G ≅ Z_2 or Z_3

Theorem 2.2.7 Let G a group. The girth of Γ_{IO} (G) is 3 if and only if G ≇ Z_2 or Z_3

Proof: Assume that G ≇ Z_2 or Z_3 . By the previous Theorem G is not a star graph. Also the identity element 'e' has a full degree, there exits two elements a, b in G such that they are adjacent in Γ_{IO} (G)). Therefore e, a, b form a cycle in Γ_{IO} (G). Hence the girth of Γ_{IO} (G) is 3

Conversely, assume that the girth of Γ_{IO} (G) is 3. Suppose not, G ≅ Z_2 or Z_3. Then Γ_{IO} (G) is a star graph, which is a contradiction to the assumption that the girth of Γ_{IO} (G) is 3.

Proposition 2.2.8 Let G be a finite group of order n with no self-inverse element and q be the number of edges in Γ_{IO} (G) . Then $q \leq (n-1)^2/2$.Moreover, this bound is sharp.
We characterize the groups G for which the associated graph Γ_{IO} (G) attains this bound.

Theorem 2.2.9 Let G be a finite abelian group of order n and q be the number of edges in Γ_{IO} (G). Then $q = (n-1)^2/2$ if and only if every element of G is of order p, where p is an odd prime number.
Proof: Deg Γ_{IO} (G)(a) = n −2 for all vertices a G−{e} and deg Γ_{IO} (G) (e) = n−1.
Let a ϵ G−{e} be any element of order k. Then k is a prime number and every element of G is of unique prime order.Conversely, assume that every element other than identity is of order p. Since G is abelian, G ≅ Zp×Zp×. . . ×Zp. deg Γ_{IO} (G)(e) = n −1. Let a and b two elements of G such that $b \neq a^{-1}$. clearly < a >∩< b >=< a > or{e}. Therefore, < a >∩< b > ⊆ < a *b >. a and b are adjacent.Hence a is adjacent to all other elements in G other than its inverse.

Theorem 2.2.10 Let G be an abelian group of order p^n. Γ_{IO} (G) ≈$K_{1,2,2,...,}$k times, where $k = p^n$-1/2 if and only if G ≅ Zp×Zp×. . .×Zp.
Proof: Let G be an abelian group of order p^n. Assume that Γ_{IO} (G) ≈$K_{1,2,2,...,ktimes}$, where k = p^n-1/2 .
Clearly the number of edges of the graph Γ_{IO} (G) is $(p^n-1)^2/2$.Every element of G is of order p and hence G ≅ Zp×Zp×. . . ×Zp.
Conversely, assume that G ≅ Zp×Zp×. . .×Zp. Therefore, for every a∊ G − {e}, a is not adjacent to a^{-1} only. Therefore, we can partition the vertex set of (G) into k + 1 sets, suchthat the identity element e belongs to a single partition and for the remaining k sets, each set contains the pair of elements a and a^{-1}.

2.3 OPERATOR INTERSECTION GRAPH

Definition 2.3.1 Let (G, *) be a group with binary operation '∗'. The Operator Intersection graph Γ_{OI}(G) of G is a graph with V (Γ_{OI}(G)) = G−e , where e is an identity element of G and two distinct vertices x and y are adjacent in Γ_{OI}(G) if and only if < x ∗ y >⊆< x > ∩ < y > .

Proposition 2.3.2 For any non self- inverse element x ∈ G, x and x^{-1} are adjacent in Γ_{OI}(G)

Theorem 2.3.3 Let (G, ∗) be a group. Let x ∈ G. x is an isolated vertex in Γ_{OI}(G) if and only if x is a self-inverse element of G .

Proof: Let (G, *) be a group. Let x be an isolated vertex in Γ_{OI}(G) . We have to prove that x is a self- inverse element of G. Suppose not, x is non self- inverse element of G. x is adjacent to x^{-1}, which is a contradiction. Conversely assume that, x is a self- inverse element of G. Clearly < x > ∩ < y >= {e} or < x > for all y ∈ G. Also for all y ∈ G ,< x * y>⊈< x > or {e}. Hence x is an isolated vertex in Γ_{OI}(G)

Proposition 2.3.4 Let (G, *) be a group. Any two elements of distinct prime order are non-adjacent in Γ_{OI}(G).

Proof: Let (G, ∗) be a group with identity element e. Let x, y ∈ G be any two elements such that O(x) = p and O(y) = q, where p, q are distinct prime. Clearly < x > ∩ < y >= {e}. Therefore < x ∗ y >⊈< x > ∩ < y >. Hence the result follows.

Theorem 2.3.5 Let G be any group. Γ_{OI}(G) is complete if and only if G is a cyclic group of prime order.

Proof: Let G be a cyclic group of prime order p. Clearly, every element of G other than identity is a generator of G. Let x ∈ G − e. x and x^{-1} are adjacent. Let y ∈ G − e be an element other than x^{-1}. Clearly < x ∗ y >=< x > ∩ < y >. Therefore x and y are adjacent in Γ_{OI}(G). Hence Γ_{OI}(G) is complete.Conversely assume that Γ_{OI}(G) is complete. Let p and q be two distinct prime such that p|O(G) and q|O(G). By Cauchy's Theorem, G has two elements x, y such that O(x) = p and O(y) = q. Clearly < x > ∩ < y >= {e}. Therefore x and y are non adjacent in Γ_{OI}(G) and O(G) = p for some prime p.

Proposition 2.3.6 Let (G, ∗) be a cyclic group. Any two generators of G are adjacent in Γ_{OI}(G)

Proof: Let (G, ∗) be a cyclic group. Let x, y ∈ G be any two generators of G. Clearly < x > ∩ < y > = G. Therefore < x ∗ y >⊆< x > ∩ < y >. Hence, the result follows.

Remark 2.3.7 The converse of the Proposition is not true. Consider the group $\mathbb{Z}_6$. In $\Gamma_{OI}(\mathbb{Z}_6)$, 2 and 4 are adjacent but they are not generators of G .

Theorem 2.3.8 Let G be a finite group of order n= $p_1\ p_2 \ldots p_k$,where $p_1, p_2, \ldots p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. Then the independence number $\beta_0(\Gamma_{OI}(G)) \geq k$.

Proof: Since each pi divides o(G), by Cauchy's Theorem, G contains the elements a_i such that $o(a_i) = p_i$, for $1 \leq i \leq k$. Note that$\langle a_i \rangle \cap \langle a_j \rangle = \{e\}$ for all $i \neq j$. From this $\{a_1, a_2, \ldots a_k\}$ is an independent set of $\Gamma_{OI}(G)$ and hence the result follows.

Proposition2.3.9 Let G be a finite group of order n with no self inverse element and q be number of edges in $\Gamma_{OI}(G)$. Then $q \geq \frac{n-1}{2}$. Moreover, this bound is sharp.

Proof: x and x^{-1} are adjacent for all $x \in G$ -e. Hence $q \geq \frac{n-1}{2}$. Moreover, for the group $\mathbb{Z}_3$, $\Gamma_{OI}(\mathbb{Z}_3) \cong K_{1,1}$ and for this graph the bound is sharp.

Theorem 2.3.10 Let G be a group of order n and no self inverse element. Let q be number of edges in $\Gamma_{OI}(G)$. $q = \frac{n-1}{2}$ if and only if $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$.

Proof: Assume that $\Gamma_{OI}(G)$ is a graph with $\frac{n-1}{2}$ edges. we get $\Gamma_{OI}(G)$ is an union of K_2 . Suppose $p \geq 5$ be a prime number such that p|O(G), then G has an element of order p and so K_{p-1} is a subgraph of $\Gamma_{OI}(G)$, which is a contradiction. Since G has no self inverse element, O(G) must be 3^n . Suppose G has an element of order 3^k for some $k \geq 2$, then $\Gamma_{OI}(G)$ contains K_9 as a sub graph, which is a contradiction. Therefore every element of G has order 3. Hence $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$. Conversely assume that $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$. Let $x, y \in G$ such that $< x > \cap < y > = \{e\}$. Clearly x and y are non adjacent. Therefore the adjacent vertices of x is $x{-}1$ only. Hence $\Gamma_{OI}(G)$ is union of K_2 .

Theorem 2.3.11 Let G be a cyclic group of order p^2. Then $\Gamma_{OI}(G) \cong K_{\phi(p^2)} \cup K_{p-1}$, where p is a prime and φ(n) is an Euler function.

Proof: Let G be a cyclic group of order p^2. Let A be the set of elements of order p and B be the set of elements of order p^2. Since G is a cyclic group, the elements in B are generator of G. Clearly $|A|=p-1$ and $|B|= \Phi(p^2)$. The graph induced by the set B is $K_{\phi(p^2)}$ and by the graph induced by the set A is K_{p-1}. Let $x \in A$ and $y \in B$. Suppose $x * y \in A$. Since $A \cup \{e\}$ is a subgroup of order p, $y \in A$,gives a contradiction. suppose $x * y \in B$. $< x > \cap < y > = < x >$. Therefore $< x * y > \not\subseteq < x > \cap < y >$. So x and y are non- adjacent.

3. RELATION BETWEEN ISOMORPHISMS OF GROUPS AND THEIR ASSOCIATED GRAPHS

In this section, we study about relation between graph isomorphism and group isomorphism and compare the graphs.

Theorem 3.1.1 Let G_1 and G_2 be two groups. If $G_1 \cong G_2$, then $\Gamma_{OP}(G_1) \cong \Gamma_{OP}(G_2)$.
Proof: Assume that $f : G_1 \rightarrow G_2$ is a group isomorphism. Let x and y be any two elements in G_1 such that x and y are adjacent in $\Gamma_{OP}(G_1)$. Therefore either $x = (x*y)^n$ or $y = (x*y)^m$. Withoutloss of generality we assume that $x = (x*y)^n$. Therefore $f(x) = f((x*y)^n)$. Since f is an isomorphism, $f(x) = (f(x)*f(y))^{n,}$ which implies that f(x), f(y) are adjacent in $\Gamma_{OP}(G_2)$. Hence $\Gamma_{OP}(G_1) \cong \Gamma_{OP}(G_2)$.

Theorem 3.1.2 Let G_1 and G_2 be two groups. If $G_1 \cong G_2$ then $\Gamma_{OI}(G_1) \cong \Gamma_{OI}(G_2)$.
Proof: Let x and y be any two elements in G_1 such that x and y are adjacent in $\Gamma_{OI}(G_1)$. Therefore $< x*y > \subseteq < x > \cap < y >$. Let $a \in < f(x)*f(y) >$. There exists an element $t \in G_1$ such that $a = f(t)$. As f is an isomorphism, $t \in < x*y >$. Therefore $f(t) \in < f(x) > \cap < f(y) >$ which implies that f(x), f(y) are adjacent in $\Gamma_{OI}(G_2)$.
Theorem 3.1.3 Let G be a group of prime order then $\Gamma op(G) \cong \Gamma_{IO}(G)$.

Proof. Let f: G→G be an identity function.Let a & b are adjacent in Γop (G).
Then <a> ⊆ <a*b> or <b> ⊆ <a*b> Since G is a group of prime order, <a>= <b> = <a*b> and hence a & b are adjacent in Γ_{IO} (G).

Theorem 3.1.4 Let G be a cyclic Group of order 2p;p≥3 is prime then Γop (G) ≅ Γ_{IO} (G).
Proof: Let f: G→G be an identify function & a and b are adjacent in Γop(G) .The vertex of G can be partitioned into four sets namely A,B,C and D such that A={0}, B = {1,3,5………p-2,….2p-1} C= {2,4,6…..2p-2} and D ={p}. a and b are adjacent in Γ_{op} (G) in the following cases.

Case (i). a , b ∈ C.

<a> = <b> implies <a>∩<b> ⊆ <a*b>

Case(ii) . a ∈ B ,b ∈ C

<a*b> = G implies <a> ∩<b>⊆ <a*b>

Case (iii). a ∈ C ,b ∈ D

<a*b> = G implies <a> ∩<b>⊆ <a*b>

Hence a & b are adjacent in Γop (G) iff they are adjacent in Γ_{IO} (G).

Theorem 3.1.5 Let G be a cyclic group of order p^2 where p≥3 is a prime then

Γop (G) ≅ Γ_{IO} (G).

Proof: Let f be an identify function and a & b be two non-identity element such that a & b are adjacent in Γop (G).

Let A={0}, B={p, 2p, 3p…..(p-1)p} and C=G-A-B .a and b are adjacent in Γop(G) in the following cases.

Case (i) a,b ∈ B

In this case <a> = <b> and hence <a> ∩<b>⊆ <a*b>

Case (ii) a,b ∈ C

<a> = <b> = <a*b> = G

<a> ∩<b> ⊆ <a*b>

Case (iii) a ∈ B , b ∈ C

<b> = <a*b> = G

<a> ∩<b>⊆ <a*b>

a & b are adjacent in Γ_{IO} (G) and so Γop (G) ≅ Γ_{IO} (G).

Theorem 3.1.6 Let G be a group such that every non-identity element of G is of order 2 or 3, then
$\Gamma_{IO}(G) - \{e\} \cong \overline{\Gamma_{OI}(G)}$.
Proof: Let G be a group such that every non-identity element ofG is of order either 2 or 3.Let a and b be two non-identity elements in G. We discuss the adjacency in the following cases.
Case (i): o(a) = o(b) = 2. In this case < a >∩< b >= {e}. Clearly a and b are adjacent in Γ_{IO}(G) but a and b are not adjacent in Γ_{OI}(G).
Case (ii) : o(a) = o(b) = 3 and b ≠ a^{-1} .In this case < a >∩< b >= {e}. Clearly a and b are adjacent in Γ_{IO}(G) but a and b are not adjacent in Γ_{OI}(G).
Case (iii) : o(a) = o(b) = 3 and b = a^{-1} .In this case < a >∩< b >=< a >and < a *b >= {e}. Clearly a and b are not adjacent in Γ_{IO}(G) but a and b are adjacent in Γ_{OI} (G).
Case (iv) :o(a) = 2 and o(b) = 3

In this case $< a > \cap < b > = \{e\}$. Clearly a and b are adjacent in $\Gamma_{IO}(G)$ but a and b are not adjacent in $\Gamma_{OI}(G)$.
From the above cases it is clear that $\Gamma_{IO}(G) - \{e\} \cong \overline{\Gamma_{OI}(G)}$.

CONCLUSION

Characterized the properties of Operator Power Graph, Intersection operator graph and Operator Intersection graphs of groups of some special order, compare them and the isomorphic relation between these graphs has been established.

REFERENCES

1. A.Abdollahi, S. Akbari , H.R Maimani, Non-Commuting graph of a group, Journal of Algebra, 2006, 298: 468-492.
2. S.Akbari, A.Mohammadian ,On the zero-divisor graph of commutative ring, Journal of Algebra, 2004, 274(2): 847-855.
3. J.A.Bondy and U.S.R. Murty, Graph theorey with applications, Elsevier,(1977)
4. P.J. Cameron and S.Ghosh, the power graph of a finite group,Discrete Math.,311(2011),1220- 1222.
5. I.Chakrabarty, S .Ghos and M.K. Sen, Undirected power graphs of semigroups, Semigroup Forum, 2009, 78: 410-426.
6. I.N.Herstein, Topics in Algebra ,Wiley.(2013).
7. Joseph A.Gallian., Contemporary Abstract algebra, Narosa Publishing House,(1999)
8. A.V.Kelarev and S.J. Quinn ,Directed graph and combinatorial properties of semi groups,J.Algebra, 251(2002),16-26.
9. T.Tamizh Chelvam, M.Sattanathan, Subgroup intersection graph of group, Journal of Advanced Research in Pure Mathematics, 2011, 3(4): 44-49.
10. A.Vethamanickam, D. Premalatha ,Operator Power Graph of a Group, Int. J. Contemporary Mathematical Sciences, 9(12) (2014), 563- 568.