Hyperbolic Valued Signed Measure

Chinmay Ghosh[1], Sanjoy Biswas[2], Taha Yasin[3]

[1]*Kazi Nazrul University, Nazrul Road,*
Asansol-713340, Paschim Bardhaman,
West Bengal, India.
[2,3]*Guru Nanak Institute of Technology,*
157/F, Nilgunj Road, Panihati, Sodepur,
Kolkata-700114, West Bengal, India.

Abstract: *In this article we introduced two symbols* $+\infty_D$ *and* $-\infty_D$ *as extended hyperbolic numbers and modified the definition of hyperbolic valued measure. Also we defined hyperbolic valued signed measure and proved some theorems on it.*

AMS Subject Classification (2010) : 28A12.

Keywords: *Hyperbolic numbers, hyperbolic conjugation, idempotent representation, hyperbolic modulus, D-bounded, D-Cauchy sequence, D-valued measure, D-measure space, D-valued signed measure.*

I. Introduction

Albert Einstein gave the theory of special relativity at the beginning of the 20th Century which is based on Lorentzian geometry. The hyperbolic numbers (or duplex numbers or perplex numbers), put Lorentzian geometry to an analogous mathematical concept with Euclidean geometry. The hyperbolic numbers play a similar role in the Lorentzian plane that the complex numbers do in the Euclidean plane. Different structures of these numbers can understood well in [2],[3].[5],[6],[7],and [9].

D. Alpay et. al. [1] gave an idea of hyperbolic valued probabilities. This idea motivate to construct hyperbolic valued measure. Before the present work R. Kumar et. al. [4] had given the similar concept which is modified here by introducing two symbols $+\infty_D$ and $-\infty_D$ as extended hyperbolic numbers. Also the concept of hyperbolic valued signed measure is introduced and the hyperbolic version of Hahn and Jordan decomposition theorems are proved in this article. The proofs in this article are based on the book of I. K. Rana [8].

II. BASIC DEFINATIONS

The set of hyperbolic numbers D is defined as

$$\mathrm{D} = \{x + yk : x, y \in R\}.$$

where k is an imaginary element such that $k^2=1$ but $k \notin R$. D forms a commutative ring with respect to $+_D$ and $\cdot_D$ defined by

$$(x_1 + y_1 k) +_D (x_2 + y_2 k) = (x_1 + x_2) + (y_1 + y_2)k$$
$$(x_1 + y_1 k) \cdot_D (x_2 + y_2 k) = (x_1 x_2 + y_1 y_2) + (x_1 y_2 + x_2 y_1)k$$

A. Hyperbolic conjugation

The conjugate of a hyperbolic numbers $\zeta = x + yk$ is $\overline{\zeta} = x - yk$. Different algebraic operations (additive, involutive and multiplicative) on D with respect to the hyperbolic conjugation are as follows:

1. $\overline{\zeta + \eta} = \overline{\zeta} + \overline{\eta}$
2. $\overline{(\overline{\zeta})} = \zeta$
3. $\overline{(\zeta\eta)} = \overline{\zeta}\,\overline{\eta}$

Note that,

$$\zeta\overline{\zeta} = x^2 - y^2 \in R$$

The modulus of a hyperbolic number $\zeta = x + yk \in D$, is defined by

$$|\zeta| = \sqrt{|\zeta\overline{\zeta}|} = \sqrt{|x^2 - y^2|} \in R$$

Any hyperbolic number ζ with $\zeta\overline{\zeta} \neq 0$ is said to be invertible (or non-singular), and its inverse is given by

$$\zeta^{-1} = \frac{\overline{\zeta}}{|\zeta|}$$

If $\zeta \neq 0$ but $\zeta\overline{\zeta} = x^2 - y^2 = 0$ then $\zeta(\neq 0)$ is a zero divisor. The only zero divisors in D are those $\zeta(\neq 0)$ for which $\zeta\overline{\zeta} = x^2 - y^2 = 0$. We denote the set of zero divisors by O_D. Thus, $O_D = \{\zeta = x + yk : \zeta \neq 0, \zeta\overline{\zeta} = x^2 - y^2 = 0\}$

B. Idempotent representation

There are two important zero divisors in D, $\frac{1}{2} + \frac{1}{2}k$ and its conjugate $\frac{1}{2} - \frac{1}{2}k$.

Set $e_1 = \frac{1+k}{2}$ and $e_2 = \frac{1-k}{2}$

One can check that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$

$e_1^2 = e_1$ and $e_2^2 = e_2$

So these two elements are called mutually orthogonal idempotent elements in D. Thus $\{e_1, e_2\}$ forms a basis of D

The two sets

$$D_{e_1} = e_1.D \quad \text{and} \quad D_{e_2} = e_2.D$$

are (principal) ideals in the ring D and they have the properties:

$$D_{e_1} \cap D_{e_2} = \{0\} \quad \text{and}$$

$$D = D_{e_1} + D_{e_2} \qquad (1)$$

Formula (1) is called the idempotent decomposition of D. Every hyperbolic number $\zeta = x + yk$ can be written as

$$\zeta = (x+y)e_1 + (x-y)e_2 = v_1 e_1 + v_2 e_2, v_1, v_2 \in R \qquad (2)$$

Formula (2) is called the idempotent representation of a hyperbolic number. It has a remarkable feature: the algebraic operations of addition, multiplication, taking of inverse, etc. can be realized component-wise. Observe that the sets D_{e_1} and D_{e_2} can be written as

$D_{e_1} = \{se_1 : s \in R\} = Re_1$; $D_{e_2} = \{te_2 : t \in R\} = Re_2$ Remark1. One should keep in mind the following properties:

a) $\zeta \in D_{e_1}$ if and only if $\zeta_{e_1} = \zeta$

b) $\zeta \in D_{e_2}$ if and only if $\zeta_{e_2} = \zeta$

C. Partial order on D

The two idempotent orthogonal axes divide the whole hyperbolic plane into four quadrants, namely

$$D^{+} = \{v_1 e_1 + v_2 e_2 : v_1, v_2 \geq 0\}$$

$$D^{\pm} = \{v_1 e_1 + v_2 e_2 : v_1 \geq 0, v_2 \leq 0\}$$

$$D^{-} = \{v_1 e_1 + v_2 e_2 : v_1, v_2 \leq 0,\}$$

$$D^{\mp} = \{v_1 e_1 + v_2 e_2 : v_1 \leq 0, v_2 \geq 0\}$$

Observe that

$$D^{+} \cap D^{-} = \{0\}$$

A hyperbolic number ζ is said to be (strictly) positive if $z\,\hat{\imath}\; D^{+} \setminus \{0\}$ and (strictly) negative if $z\,\hat{\imath}\; D^{-} \setminus \{0\}$ The set of non-negative hyperbolic numbers is also defined as

$$D^{+} = \{x + yk : x^2 - y^2 \geq 0, x \geq 0\}$$

On the realization of D^+, M.E. Luna-Elizarraras et.al. [6] defined a partial order relation on D. For two hyperbolic numbers ζ_1,ζ_2 the relation $\preceq_D$ is defined as $\zeta_1 \preceq_D \zeta_2$ if and only if $\zeta_2 - \zeta_1 \in D^+$. One can check that this relation is reflexive, transitive and anti-symmetric. Therefore $\preceq_D$ is a partial order relation on D. This partial order relation $\preceq_D$ on D is an extension of the total order relation $\leq$ on R. We say $\zeta_1 \prec_D \zeta_2$ if $\zeta_1 \preceq_D \zeta_2$ but $\zeta_1 \neq \zeta_2$. Also we say $\zeta_2 \succeq_D \zeta_1$ if $\zeta_1 \preceq_D \zeta_2$ and $\zeta_2 \succ_D \zeta_1$ if $\zeta_1 \prec \zeta_2$. If neither $\zeta_1 \preceq_D \zeta_2$ nor $\zeta_2 \preceq_D \zeta_1$ then ζ_1 and ζ_2 are not comparable.

Definition 1: For any hyperbolic number $\zeta = v_1 e_1 + v_2 e_2$ the D-valued (or hyperbolic valued) modulus of ζ is defined by

$$|\zeta|_D = |v_1 e_1 + v_2 e_2|_D = |v_1| e_1 + |v_2| e_2 \in D^+$$

where $|v_1|$ and $|v_2|$ are the usual modulus of real numbers.

Definition 2: A subset A of D is said to be D-bounded if $\exists$ M$\in D^+$ such that $|\zeta|_D \preceq_D$ M for any $\zeta \in A$

Set $A_1 = \{x \in R : \exists\, y \in R,\, xe_1 + ye_2 \in A\}$ $A_2 = \{y \in R : \exists\, x \in R,\, xe_1 + ye_2 \in A\}$.

It is immediate task to verify that if A is D-bounded then A_1 and A_2 are bounded subset of R.

Definition 3: For a D-bounded subset A of D, the supremum of A with respect to the D-valued (or hyperbolic valued) modulus is defined by

$$\sup_D A = \sup A_1 e_1 + \sup A_2 e_2$$

Definition 4: A sequence of hyperbolic numbers $\{\zeta_n\}_{n\geq 1}$ is said to be convergent to $\zeta \in D$ if for $\varepsilon \in D^+\setminus\{0\}$ $\exists$ k $\in$ N such that $|\zeta_n - \zeta|_D \prec_D \varepsilon$ for all $n \geq k$

Then we write,

$$\lim_{n\to\infty} \zeta_n = \zeta$$

Definition 5: A sequence of hyperbolic numbers $\{\zeta_n\}_{n\geq 1}$ is said to be D-Cauchy sequence $\zeta \in D$ if for $\varepsilon \in D^+\setminus\{0\}$ $\exists$ N $\in$N such that

$$|\zeta_{N+m} - \zeta_N|_D \prec_D \varepsilon$$

for all m=1,2,3,.....

Note that a sequence of hyperbolic numbers $\{\zeta_n\}_{n\geq 1}$ is said to be convergent if and only if it is a D-Cauchy sequence.

Definition 6: A hyperbolic series $\sum_{n=1}^{\infty} \zeta_n$ is convergent if and only if its partial sums is a D-Cauchy sequence, i.e., for any $\varepsilon \in D^+\setminus\{0\}$ $\exists$ N $\in$N such that

$$\left|\sum_{k=1}^{m} \zeta_{N+k}\right|_D \prec_D \varepsilon \qquad \text{for any } m \in N.$$

Definition 7: A hyperbolic series $\sum_{n=1}^{\infty} \zeta_n$ is D-absolutely convergent if the series $\sum_{n=1}^{\infty} |\zeta_n|_D$ is convergent.

Every D-absolutely convergent series is convergent.

III. Main result

A. D-valued (or hyperbolic valued) measures:

We introduce two symbols $+\infty_D$ and $-\infty_D$ as

$$+\infty_D = v_1 e_1 + v_2 e_2 : v_1 = v_2 = +\infty$$

$$-\infty_D = v_1 e_1 + v_2 e_2 : v_1 = v_2 = -\infty$$

One can check that $\zeta = x + yk = \pm\infty_D$ only if x = $\pm\infty$ and y $\in$ R. If we take y = 0 then ζ = x= $\pm\infty_D$ if and only if x = $\pm\infty$. Thus $\pm\infty_D$ are the extension of $\pm\infty$ on the extended real number system.

We call the two symbols $+\infty_D$ and $-\infty_D$ as plus hyperbolic infinity and minus hyperbolic infinity. The following two sets are of special interests:

$$\overline{D}^+ = D^+ \cup \{+\infty_D\}$$
$$\overline{D}^- = D^- \cup \{-\infty_D\}$$

The extension of the algebraic operations and the partial order on $D^+ \cup D^-$ to $\overline{D}^+ \cup \overline{D}^-$

1. For every $\zeta \in D^+ \cup D^-$, $-\infty_D \prec_D \zeta \prec_D +\infty_D$
2. For every $\zeta \in D^+ \cup D^-$,

   $(-\infty_D) + \zeta = -\infty_D$ and $(+\infty_D) + \zeta = +\infty_D$

   $(-\infty_D) + (-\infty_D) = -\infty_D$ and $(+\infty_D) + (+\infty_D) = +\infty_D$

3. For every $\zeta \in D^+ \setminus \{0\}$,

   $x(+\infty_D) = (+\infty_D)x = +\infty_D$

   $x(-\infty_D) = (-\infty_D)x = -\infty_D$

4. For every $\zeta \in D^- \setminus \{0\}$

   $x(+\infty_D) = (+\infty_D)x = -\infty_D$

   $x(-\infty_D) = (-\infty_D)x = +\infty_D$

   Further,

   $(+\infty_D)0 = (-\infty_D)0 = 0$

   $(\pm\infty_D)(-\infty_D) = (\mp\infty_D)$

   Note that the relations $-\infty_D + (+\infty_D)$ and $+\infty_D + (-\infty_D)$ are not defined.

If $\zeta \in D^+$ but $\zeta \notin \overline{D}^+$ we say $0 \preceq_D \zeta \prec_D +\infty_D$ and if $\zeta \in D^+ \setminus \{0\}$ but $\zeta \notin \overline{D}^+$ we say $0 \prec_D \zeta \prec_D +\infty_D$. Also if $\zeta \in D^-$ but $\zeta \notin \overline{D}^-$ we say $-\infty_D \prec_D \zeta \preceq_D 0$ and if $\zeta \in D^- \setminus \{0\}$ but $\zeta \notin \overline{D}^-$ we say $-\infty_D \prec_D \zeta \prec_D 0$.

Definition 8: Let X be a non empty set and S be the sigma algebra of subsets of X. A set function $\mu_D : S \to \overline{D}^+$ is said to be a D-valued (or hyperbolic valued) measures on S if

(i) $\mu_D(\phi) = 0$

(ii) μ_D is countably additive on S i.e.,

$$\mu_D\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_D(A_n)$$

Whenever $\{A_n\}_{n \ge 1}$ is a sequence of pairwise disjoint sets in S. The triplet (X,S, μ_D) is called a D-measures space.

Remarks 2: Every D-valued (or hyperbolic valued) measure can be written as

$$\mu_D(A) = \nu_1(A) + \nu_2(A)k = \mu_1(A)e_1 + \mu_2(A)e_2$$

where ν_1 ,ν_2, μ_1, μ_2 are measures on S with

$$\mu_1(A) = \nu_1(A) + \nu_2(A)$$

And

$$\mu_2(A) = \nu_1(A) - \nu_2(A)$$ for every A∈S

Lemma 1 : A D-valued (or hyperbolic valued) measure satisfies the following properties:

(i) μ_D is finitely additive on S i.e.,

$$\mu_D\left(\bigcup_{i=1}^{n} A_n\right) = \sum_{i=1}^{n} \mu_D(A_i)$$

whenever A_i for i = 1,2,...,n are pairwise disjoint sets in S.

(ii) μ_D is monotonic increasing on S i.e., $\mu_D(A) \preceq \mu_D(B)$ for A,B∈S with A⊆B.

(iii) $m_D(B \setminus A) = m_D(B) - m_D(A)$ for A,B∈S with A⊆B.

(iv) Let $\{A_n\}_{n\geq 1}$ be an increasing sequence of sets in S i.e., $A_1,A_2,....\in S$ with $A_1\subseteq A_2\subseteq A_3\subseteq.....$ Let

$A = \bigcup_{i=1}^{n} A_n \in$ S, then

$$\lim_{n\to\infty} \mu_D(A_n) = \mu_D(A)$$

(v) Let $\{A_n\}_{n\geq 1}$ be a decreasing sequence of sets in S i.e., $A_1,A_2,....\in S$ with $A_1\supseteq A_2\supseteq A_3\supseteq.....$Let

$A = \bigcap_{n=1}^{\infty} A_n$ and $\mu_D(A_k) \in D^+$ but $\mu_D(A_k) \notin D^+$ for some k ∈ N, then

$$\lim_{n\to\infty} \mu_D(A_n) = \mu_D(A)$$

Proof:

(i) Let $A_1, A_2, ...A_n$ be a pairwise disjoint sequence of sets in S: Then

$$\begin{aligned}
\mu_D\left(\bigcup_{i=1}^{n} A_i\right) &= \mu_D\left(\bigcup_{i=1}^{\infty} A_i\right) \\
&= \sum_{i=1}^{\infty} \mu_D(A_i) \\
&= \sum_{i=1}^{n} \mu_D(A_i) + \sum_{i=1}^{n} \mu_D(\phi) \\
&= \sum_{i=1}^{n} \mu_D(A_i)
\end{aligned}$$

(ii) $A \subseteq B$ implies that $B = A \cup (B \setminus A)$ where $A \cap (B \setminus A) = \phi$

Therefore $\mu_D(B) = \mu_D(A) + \mu_D(B \setminus A)$

$$\Rightarrow \mu_D(B) - \mu_D(A) = \mu_D(B \setminus A)$$

$$\Rightarrow 0 \preceq_D \mu_D(B) - \mu_D(A)$$

$$\Rightarrow \mu_D(A) \preceq_D \mu_D(B)$$

(iii) A⊆B implies that B = A∪(B\A), where A∩(B\A)=φ.Therefore $m_D(B) = m_D(A) + m_D(B \setminus A)$ This implies that $m_D(B \setminus A) = m_D(B) - m_D(A)$

(iv) Set $B_1=A_1$ and $B_n = A_n \setminus A_{n-1}$ for n=2,3,4,......Then $B_n \in$S, $B_i \cap B_j$ =φ for i≠j; i,j=1,2,3,... $A_n = \bigcup_{i=1}^{n} B_i$ and $A = \bigcup_{i=1}^{\infty} B_i$ Thus using countable and finite additivity of μ_D , we have

$$\begin{aligned}
\mu_D(A) = \sum_{n=1}^{\infty} \mu_D(B_i) &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu_D(B_i) \\
&= \lim_{n\to\infty} \mu_D\left(\bigcup_{i=1}^{n} B_i\right) \\
&= \lim_{n\to\infty} \mu_D(A_n)
\end{aligned}$$

This proves (iv).

(v) Set $C_n = A_k \setminus A_n$ for every n ≥ k. Then C_i ∈S for all i ∈ N and $C_1 \subseteq C_2 \subseteq C_3 \subseteq$..... Thus using (iii) we get

$$\begin{aligned}\mu_D(A_k) - \mu_D(A) &= \mu_D(A_k \setminus A) \\ &= \lim_{n\to\infty} \mu_D(C_n) \\ &= \lim_{n\to\infty} \mu_D(A_k \setminus A_n) \\ &= \lim_{n\to\infty} \mu_D(A_k) - \mu_D(A_n) \\ &= \mu_D(A_k) - \lim_{n\to\infty} \mu_D(A_n)\end{aligned}$$

Hence $\lim_{n\to\infty} \mu_D(A_n) = \mu_D(A)$

Definition 9 :A D-valued (or hyperbolic valued) measure μ_D on the sigma algebra S of subsets of a non empty set X is said to be D-finite measure if $0 \preceq_D \mu_D(X) \prec_D +\infty_D$

In that case (X,S, μ_D) is called a D-finite measures space.

Definition 10: A D-valued (or hyperbolic valued) measure μ_D on the sigma algebra S of subsets of a non empty set X is said to be σ-finite measure if there exist sets A_n ∈S, n =1,2,3,....., such that

$X = \bigcup_{n=1}^{¥} A_n$ and $0 \preceq_D \mu_D(A_n) \prec_D +\infty_D$ for every n.

Definition 11 : Let (X,S, μ_D) be a D-measures space and N_D ={E⊆X:E⊆N for some N∈S with $\mu_D(N)$ =0}. Then (X,S, μ_D) is said to be D-complete measure space if N_D ⊆S. Elements of N_D are called the μ_D -null subsets of X.

IV. D-valued (or hyperbolic valued) signed measures

Definition 12 :Let (X,S) be a measurable space. A hyperbolic valued set function $\mu_D : S \to \overline{D}^{+} \cup \overline{D}^{-}$ is called a D-valued (or hyperbolic valued) signed measure if it has the following properties:

(i) $\mu_D(\phi) = 0$

(ii) μ_D takes at most one of the values $+\infty_D$ or $.-\infty_D$ Whenever $\{E_n\}_{n\geq 1}$ is a sequence of pairwise disjoint sets in S with $E = \bigcup_{n=1}^{\infty} E_n$ then

$$\mu_D(E) = \mu_D\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu_D(E_n)$$

where the equality holds in the sense that every rearrangement of the series $\sum_{n=1}^{\infty} \mu_D(E_n)$ converges to $\mu_D(E)$ if $0 \preceq_D E \prec_D +\infty_D$ and diverges properly to $\mu_D(E)$ otherwise.

Note that the series $\sum_{n=1}^{\infty} \mu_D(E_n)$ is absolutely convergent whenever $0 \preceq_D \mu_D(E) \prec_D +\infty_D$.

A hyperbolic valued signed measure μ_D on (X,S) is said to be D-finite if $0 \preceq_D \mu_D(X) \prec_D +\infty_D$ and σ-finite if there exist sets A_n ∈S, n=1,2,3,....., such that

$X = \bigcup_{n=1}^{¥} A_n$ and $0 \preceq_D \mu_D(A_n) \prec_D +\infty_D$ for every n.

Theorem 1: Let μ_D be a D-valued (or hyperbolic valued) signed measure on (X,S). Then the following hold:

(i)If A,B∈S and A∩B=∅, then $\mu_D(A \cup B) = \mu_D(A) + \mu_D(B)$

(ii) If A∈S with $0 \preceq_D \mu_D(A_n) \prec_D +\infty_D$ and $B \in S$ with B⊆A, then $0 \preceq_D \mu_D(B_n) \prec_D +\infty_D$ and $\mu_D(A \setminus B) = \mu_D(A) + \mu_D(B)$

(iii) μ_{D} is finite iff $0 \preceq_D \mu_D(A) \prec_D +\infty_D$ for all $A \in S$.

Proof: The proof of (i) is obvious

To prove (ii), let A∈S with $0 \preceq_D \mu_D(A) \prec_D +\infty_D$

If B∈S and B⊆A, then A = (A\B) ∪ B, and we have

$\mu_D(A) = \mu_D(A \setminus B) + \mu_D(B)$

Since $0 \preceq_D \mu_D(A) \prec_D +\infty_D$ and μ_D can take at most one of the values $+\infty_D$ or $-\infty_D$ we get

$$0 \preceq_D \mu_D(A \setminus B) \prec_D +\infty_D$$

And $$0 \preceq_D \mu_D(B) \prec_D +\infty_D$$

Further $\mu_D(A \setminus B) = \mu_D(A) - \mu_D(B)$

(iii) follows from (ii)

Theorem 2. Let μ_D be a D valued (or hyperbolic valued) signed measure (X,S) and $\{E_n\}_{n \geq 1}$ be a sequence in S .Then the following hold:

(i) If $E_n \subseteq E_{n+1}$ for every $n \geq 1$ and $E = \bigcup_{n=1}^{\infty} E_n$ then

$\mu_D(E) = \lim_{n \to \infty} \mu_D(E_n)$

(ii) If $E_{n+1} \subseteq E_n$ for every n≥1 and $0 \preceq_D \mu_D(E_n) \preceq_D +\infty_D$ for some n, then for $E = \bigcap_{n=1}^{\infty} E_n$

$\mu_D(E) = \lim_{n \to \infty} \mu_D(E_n)$

Proof. Use Lemma 1 to prove it.

Definition 13: Let μ_D be a D valued (or hyperbolic valued) signed measure on (X,S) . A set $A \in S$ is called a D positive set for μ_D if

$\mu_D(E) \in \overline{D}^{+} \quad \forall E \subseteq A, \quad E \in S$

Similarly, A set $A \in S$ is called a D negative set for μ_D if

$$\mu_D(E) \in \overline{D}^{-} \quad \forall E \subseteq A, \quad E \in S$$

A Set $A \in S$ which is both a positive and a negative set for μ_D is called a μ_D null set.

Theorem 3: Let μ_D be a D valued (or hyperbolic valued) signed measure on S. Then the following hold.

(i) If A is a D positive set for

μ_D and $B \subseteq A$, $B \in S$ then B is also a D positive set for μ_D

(ii) If $\{A_n\}_{n \geq 1}$ is a sequence for D positive set for μ_D

(iii) If $E \in S$ and $\mu_D(E) \in D^{+} \setminus \{0\}$ then there exist a set $A \subseteq E$, $A \in S$ such that A is a D positive set for μ_D and $\mu_D(A) \in \overline{D}^{+} \setminus \{0\}$.

Proof. The proof of (i) is obvious.

(ii) Let $\{A_n\}_{n \geq 1}$ be a sequence of sets which are D positive for μ_D and let $E = \bigcup_{n=1}^{\infty} E_n$

Let $B_1 = A_1$ and $B_n = B_n \setminus \left(\bigcup_{k=1}^{n-1} A_k \right)$, for $n \geq 2$

Then the set B_n n = 1,2,3,… are D positive disjoint and $A = \bigcup_{n=1}^{\infty} B_n$. Let $E \in S$ and $E \subseteq A$ then $E = \bigcap_{n=1}^{\infty} (B_n \cap E)$. Since $B_n \cap E \subseteq A_n$ and A_n is a D positive set for μ_D , $\mu_D(B_n \cap E) \in \overline{D}^{+} \setminus \{0\}$ for every n. Thus $\mu_D(E) = \sum_{n=1}^{\infty} \mu_D(B_n \cap E) \in \overline{D}^{+}$. Hence A is a D- positive set for μ_D .

(iii) Let $E \in S$ and $\mu_D(E) \in \overline{D}^{+} \setminus \{0\}$. Either E itself is a D positive set for μ_D or it contains sets with hyperbolic measure in $D^{-} \setminus \{0\}$. In the earlier case we are through. In the later case, let

$\eta_1 = \eta_{11}e_1 + \eta_{12}e_2$ with smallest $\eta_{11}, \eta_{12} \in N$ such that there exists a set $E_1 \subset E$ with $E_1 \in S$ and $\mu_D(E_1) \prec_D -\frac{1}{\eta_1}$. Note that $\mu_D(E \setminus E_1) \prec_D \infty_D$. Thus we can apply the above argument to $E \setminus E_1$. Proceeding. Inductively , either we will be through after some finite number of steps, or we will have a sequence $\{\eta_k\}_{k \geq 1}$ with two projections from N and set $E_k \in S$, $k \geq 1$ with the properties that $\forall k \geq 1$

$$E_k \subseteq E \setminus \left(\bigcup_{j=1}^{k-1} E_j \right)$$

and $\eta_k = \eta_{k1}e_1 + \eta_{k2}e_2$ with smallest $\eta_{k1}, \eta_{k2} \in N$ such that $\mu_D(E_k) \prec_D -\frac{1}{\eta_k}$

Put ,

$A = E \setminus \left(\bigcup_{k=1}^{\infty} E_k \right)$ Then $E = A \cup \left(\bigcup_{k=1}^{\infty} E_k \right)$ and these sets are pairwise disjoint . Thus

$$\mu_D(E) = \mu_D(A) + \sum_{k=1}^{\infty} \mu_D(E_k)$$

Since $\mu_D(E) \prec_D +\infty_D$ the series on the right hand side of the above equality is absolutely convergent.Hence $\sum_{k=1}^{\infty} 1/\eta_k$ is convergent, and we have $\eta_k \to +\infty_D$ as $k \to \infty$. Also since $\mu_D(E_k) \in D^{-}\ \forall k$ and $\mu_D(E) \in D^{+} \setminus \{0\}$. We have $\mu_D(A) \in D^{+} \setminus \{0\}$ .To complete the proof we show that A is a D positive set . Let $B \subseteq A$ with $B \in S$ and let $\varepsilon \in D^{+} \setminus \{0\}$ be given , choose η_k such that $1/(\eta_k - 1) \prec_D \varepsilon$. Since $\forall B \subseteq E \setminus \left(\bigcup_{j=1}^{k} E_j \right)$ with $B \in S$ we have $\mu_D(B) \succ_D -1/(\eta_k - 1)$ (by the defining properties of the η_k's.). In particular for all $B \subseteq A \subseteq E \setminus \left(\bigcup_{j=1}^{k} E_j \right)$ with $B \in S$,we have $\mu_D(B) \succ_D -1/(\eta_k - 1) \succ_D -\varepsilon$. Since $\varepsilon \in D^{+} \setminus \{0\}$ is arbitrary $\mu_D(B) \in D^{+}$

References

[1] D. Alpay, M. E. Luna-Elizarraras, M. Shapiro: Kolmogorov's axioms for probabilities with values in hyperbolic numbers, Adv. Appl. Clifford Algebras, DOI 10.1007/s00006-016-0706-6, (2016).

[2] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti, P. Zampetti, The Mathematics of Minkowski Space--Time (Birkhäuser Verlag, Basel, 2008)

[3] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, P. Zampetti: Geometry of Minkowski Space--Time, Springer (2011).

[4] R. Kumar, K. Sharma: Hyperbolic valued measures and Fundamental law of probability, Global Journal of Pure and Applied Mathematics, Vol. 13, No. 10 (2017), pp. 7163-7177.

[5] M. E. Luna-Elizarraras, M. Shapiro, D. C. Struppa, A. Vajiac: Bicomplex numbers and their elementary functions, Cubo A Mathematical Journal v. 14, No. 2 (2012), 61--80.

[6] M.E. Luna-Elizarraras, M. Shapiro, D. C. Struppa, A. Vajiac: Bicomplex holomorphic functions: The algebra, geometry and analysis of bicomplex numbers, Frontiers in Mathematics, Birkhäuser Basel, 2015.

[7] G. B. Price: An Introduction to Multicomplex Spaces and Functions, 3rd Edition, Marcel Dekker, New York, 1991.

[8] I. K. Rana: An Introduction to Measure and Integration, 2nd Edition, Narosa Publishing House, 2013.

[9] D. Rochan and M. Shapiro: On algebraic properties of bicomplex and hyperbolic numbers, Anal. Univ. Oradea, Fasc. Math. 11(2004), 71--110.