Fixed Point Theorems in Generalized M-Fuzzy Metric Spaces and its Applications

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Abstract: *In this paper, we define the expansive mapping in G-metric space and we prove some fixed point theorems in generalized M-fuzzy (GM-fuzzy) Metric Space.*

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1. INTRODUCTION

The theory of fuzzy sets has evolved in many directions after investigation of the notion of fuzzy set by Zadeh. Many authors have introduced the concept of fuzzy metric space in different ways. George and Veeramani modified the concept of a fuzzy metric space introduced by Kramosil and Michalek and defined a Hausdorff topology on this fuzzy metric space.

The study of fixed points of a function satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et. al obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades initiated the study of common fixed point theory in generalized metric spaces.

2. PRELIMINARIES

Definition 2.1: A fuzzy Set M in an arbitrary Set X is a function with domain X and values in [0,1].

Definition 2.2: A binary operation $*:[0,1] \times [01] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 \leq a_2 * b_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$ for all $a_1, a_2, b_1, b_2 \in [0,1]$.

Examples of *t*-norm (1)

(1) Minimum t-norm $(*_M)$: $*_M(x,y) = \min\{x,y\}$

(2) Product t-norm $(*_P)$: $*_P(x,y) = x.y$

(3) Lukasiewicz: t-norm $(*_L)$: $*_L(x,y) = \max\{x+y-1,0\}$

Definition 2.3: Let X be a non-empty set and let $G : X \times X \times X \to R^+$, be a function satisfying the following properties:

(G_1) : $G(x,x,y) > 0$ for all $x,y \in X$, with $x \neq y$;

(G_2) : $G(x,y,z) = 0$ if $x = y = z$;

(G_3) : $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z = y$;

(G_4) : $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots\ldots\ldots\ldots$

(Symmetry in all three variables)

(G_5) : $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$

(rectangle in equality)

Then the function G is called Generalized metric or more specifically G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 2.4: The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$satisfying the following conditions, for all x,y,z $\in$ X and t_1, t_2, t>0

1. $M(x,y,0) = 0$;
2. $M(x,y,t) = 1$ if and only if x=y;
3. $M(x,y,t) = M(y,x,t)$;
4. $M(x,y,t_1+t_2) \geq M(x,z,t_1) * M(z,y,t_2)$;
5. $M(x,y,\cdot) : (0, \infty) \to [0,1]$ is continuous.

Then M is called Fuzzy metric on X and (X,M,*) is called fuzzy metric space and M(x,y,t) denotes the degree of nearness between x and y.

Definition 2.5: A 3-tuple $(X,M, *)$ is said to be a Generalized M(GM) – fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions for each $t,s >0$:

(M_1) : $M(x,x,y,t) > 0$ for all $x,y \in X$ with $x \neq y$;

(M_2) : $M(x,x,y,t) \geq M(x,y,z,t)$ for all $x,y,z \in X$ with $y \neq z$;

(M_3) : $M(x,y,z,t) = 1$ if and only if $x = y = z$;

(M_4) : $M(x,y,z,t) = M(P(x,y,z),t)$, where P is a permutation function;

(M_5): $M(x,a,a,t) * M(a,y,z,s) \leq M(x,y,z,t^+ s)$

(the triangle in-equality)

(M_6) : $M(x,y,z, .) : (0,\infty) \to [0,1]$ is continuous.

A GM-fuzzy metric space is said to be symmetric if $M(x,y,y,t) = M(x,x,y,t)$ for all $x,y \in X$ and $t>0$.

Example 2: Let X is a non-empty set and G is the G-metric on X. Denote $a*b = a.b$ for all $a,b \in [0,1]$, For each $t>0$:

$$M(x,y,z,t) = \frac{t}{t + G(x,y,z)}$$

Then $(X,M, *)$ is a GM-fuzzy metric space.

Definition 2.6: Let $(X,M, *)$ be a GM-fuzzy metric space. Then

a) A sequence $\{x_n\}$ in X is said to coverages to x if and only if

$M(x_m,x_n,x,t) \to 1$ as $n \to \infty$, $m\to\infty$, for each $t>0$.

b) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if $M(x_m,x_n,x_l,t) \to 1$ as $m \to \infty$, $l \to \infty$ for each $t > 0$.

c) A GM-fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.7: If $(X,M, *)$ be a GM-fuzzy metric space, then $M(x,y,z,t)$ is non-decreasing with respect to t for all $x,y,z \in X$.

Through out this study we assume that $\lim_{t\to\infty} M(x,y,z,t) = 1$ and that N is the set of all natural numbers and that R^+ is the set of all positive real numbers.

Lemma 2.8: Let $(X,M, *)$ be a GM-fuzzy metric space. Then the following properties are equivalent:

1) $\{x_n\}$ is convergent to x;
2) $M(x_n,x_n,x,t) \to 1$ as $n\to\infty$
3) $M(x_n,x,x,t) \to 1$ as $n\to\infty$
4) $M(x_m,x_n,x,t) \to 1$ as $m,n\to\infty$

Lemma 2.9: Let $(X,M, *)$ be a GM-fuzzy metric space, then the following are equivalent

1) The sequence $\{x_n\}$ is Cauchy;
2) For every $\varepsilon \in (0,1)$ and $t>0$, there exists $k \in N$ such that

   $M(x_n,x_m,x_m,t) > 1 - \varepsilon$ for $n,m \geq k$

Definition 2.10: Let $(X,M, *)$ be a GM-fuzzy metric space. The following conditions are satisfied:

$\lim_{n\to\infty} M(x_n,y_n,z_n,t_n) = M(x,y,z,t)$

Whenever $\lim_{n\to\infty} x_n=x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} M(x,y,z,t_n) = M(x,y,z,t)$; then M is called continuous function on X^3 x $(0,\infty)$.

Lemma 2.11: Let $(X,M, *)$ be a GM-fuzzy metric space. Then M is a continuous function on X^3 x $(0,\infty)$.

Lemma 2.12: Let $(X,M, *)$ be a complete GM- fuzzy metric space and $T : X \to X$ be a mapping satisfies the following conditions for all $x,y,z \in X$ and $t > 0$

$kM(T_x, T_y, T_z,t) \geq M(x,y,z,t)$ where $k \in [0,1)$ ... (2.12.1)

Lemma 2.13: Let $(X,M, *)$ be a complete GM- fuzzy metric space and T: $X\to X$ be a mapping satisfies the following conditions for all $x,y \in X$ and $t > 0$

$kM(T_xT_y,T_y,t) \geq M(x,y,y,t)$

Where $k \in [0,1)$. Then T has a unique fixed point.

Definition 2.14: Let $(X,M, *)$ be a GM-fuzzy metric space and T be a self mapping on X. Then T is called expansive mapping if there exists a constant $a \leq 1$, such that for all $x,y,z \in X$ and $t>0$, we have $M(T_x,T_y,T_z,t) \leq a\, M(x,y,z,t)$.

3. MAIN RESULTS

Theorem 3.1: Let $(X,M,*)$ be a complete GM-fuzzy metric space. If there exists a contant $a \leq 1$ and a onto self mapping T on X, such that for all $x,y,z \in X$ and $t > 0$,

$M(T_x,T_y,T_z,t) \leq a\, M(x,y,z,t)$ ... (3.1.1)

Then T has a unique fixed point.

Proof: Under the assumption, if

$T_x = T_y$, then

$1 = G\,(T_x,T_y,T_y,t) \leq aM(x,y,y,t)$

Which implies that

$M(x,y,y,t) = 1$

$\Rightarrow x = y$. So T is injective and invertible.

Let h be the inverse mapping of T,

Then $M(x,y,z,t) = M(T(hx), T(hy), T(hz),t) \leq aM(hx,hy,hz,t)$

Thus, for all $x,y,z \in X$ and $t > 0$

We have,

$aM(hx,hy,hz,t) \geq M(x,y,z,t)$

Applying, Lemma (2.12), we conclude that inverse mapping h has a unique fixed point $u \in X$; $h(u) = u$

But, $u = T(h(u))$

$= T(u)$

This gives that u is also a fixed point of T.

Suppose there exists another fixed point $v \neq u$ such that $Tv = v$

Then $Tv = v = T(h(v)) = h(T(v))$

So, Tv is another fixed point of h.

By uniqueness, we conclude that $u = Tv = v$, which implies that u is a unique fixed point of T.

Theorem 3.2: Let$(X,M,*)$ be a complete GM-fuzzy metric space. If there exists a constant $c \leq 1$ and a surjective self mapping T on X, such that for all $x,y \in X$ and $t > 0$

$M(Tx,Ty,Ty,t) \leq cM(x,y,y,t)$ ... (3.2.1)

Then T has a unique fixed point.

Proof: Under the assumption, if

$Tx = Ty$, then

$1 = M(Tx,Tx,Ty,t) \leq cM(x,x,y,t)$

Which implies that

$M(x,x,y,t) = 1$

$\Rightarrow x = y$

and hence T is invertible.

Let h be the inverse mapping of T,

So,

$M(x,y,y,t) = M(T(hx), T(hy), T(hy), t) \le cM(hx,hy,hy,t)$

Then for all x,y $\in$ X we have

$cM(hx,hy,hy,t) \ge M(x,y,y,t)$

Applying Lemma (2.12) on the inverse mapping h, and use argument similar to that in proof theorem (3.1), we conclude that T has unique fixed point.

Corollary 3.3: Let $(X,M,*)$ be a complete GM-fuzzy metric space. If there exists a constant $k \le 1$ and surjective self mapping on X, such that for all $x,y,z \in X$ and $t>0$

$M(Tx,Ty,Tz,t) \le kM(x,z,z,t) * M(y,z,z,t)$ ... (3.3.1)

Then T has a unique fixed point

Proof: Follows from theorem (3.2) by taking $z = y$ in condition (3.3.1).

Theorem 3.4: Let $(X,M,*)$ be a complete GM-fuzzy metric space and let $T : X \to X$ be a surjective mapping ssatisfying the following condition for all $x,y,z \in X$ and $t > 0$,

$M(T(x),T(y),T(z),t) \le k \max \{(M(x,z,z,t/2) * M(y,z,z,t/2)), (M(z,y,y,t/2) * M(xx,y,y,t/2)), (M(z,x,x,t/2) * M(y,x,x,t/2))\}$ … (3.4.1)

Where $k \le 1$. Then T has a unique fixed point.

Proof: Condition (3.7.1) implies that T is injective and therefore invertible.

Let h be the inverse mapping of T.

By condition (3.7.1) for all $x,y,z \in X$, $t>0$

We have,

$M(x,y,z,t)=M(T(hx),T(hy),T(hz),t) \le k \max \{(G(hx,hz,hz,t/2) * M(hy,hz,hz,t/2)), (M(hz,hy,hy,t/2) * M(hx,hy,hy,t/2)), (M(hz,hx,hx,t/2) * M(hy,hx,hx,t/2))\}$ ... (3.7.2)

But

by (M_4), we have

$$\text{Max} \left\{ \begin{array}{l} (M(hx,hz,hz,t/2) * M(hy,hz,hz,t/2)), \\ (M(hz,hy,hy,t/2) * M(hx,hy,hy,t/2)), \\ (M(hz,hx,hx,t/2) * M(hy,hx,hx,t/2)) \end{array} \right\} \le M(hx,hy,hz,t) \quad ...(3.7.3)$$

Thus Equation (3.7.2) implies that

$kM(hx,hy,hz,t) \ge M(x,y,z,t)$ ... (3.7.4)

Applying, Theorem (3.1) on (3.7.4)

We conclude that the inverse mapping h has a unique fixed point $u \in X$

Such that $h(u) = u$. But $u = T(h(u)) = T(u)$,

Which shows that u is also a fixed point of T.

To show u is unique fixed point, we can use the same argument in theorem (3.4).

Theorem 3.8: Let $(X,M,*)$ be a complete non symmetric GM-fuzzy metric space and let $T : X \to X$ be a surjective mapping satisfying the following condition for all $x,y \in X$, $t > 0$

$M(T(x), T(y), T(y),t) \le k \max \{M(x,y,y,t), M(y,x,x,t)\}$ ... (3.8.1)

When $k \le 1$. Then T has a unique fixed point.

Proof:

Since

Max $\{M(x,y,y,t), M(y,x,x,t)\} \le M(x,y,y,t)$, then from (3.8.1), we deduce that

$M(T(x), T(y), T(y),t) \le k\, M(x,y,y,t)$ for all $x,y \in X$, $t>0$ ... (3.8.2)

From (3.8.2), it is clear that Theorem (3.5) implies that T has a unique fixed point.

Corollary 3.9:

Let $(X,M, *)$ be a complete non-symmetric GM-fuzzy metric space, and let $T{:}X\to X$ be a surjective mapping satisfying the following condition for all $x,y,z \in X$, $t>0$

$M(T(x), T(y), T(z),t) \le k \max \{(M(x,y,y,t/2) * M(y,x,x,t/2)), (M(x,z,z,t/2) * M(z,x,x,t/2)), (M(z,y,y,t/2) * (M(y,z,z,t/2))\}$

When $k \le 1$. Then T has a unique fixed point.

Proof: Follows from the theorem (3.8) by taking $z = y$.

Corollary 3.10: Let $(X,M, *)$ be a complete GM-fuzzy metric space and let $T{:}X\to X$ be a surjective mapping satisfying the following condition for all $x,y,z \in X$, $t>0$

$M(T(x), T(y), T(z),t) \le k\{M(x,Tx,Tx,t/2) * M(Tx,y,z,t/2)\}$ ... (3.10.1)

Where $k \le 1$. Then T has a unique fixed point.

Proof: From (M_4), we have

$M(x,Tx,Tx,t/2) * M(Tx,y,z,t/2) \le M(x,y,z,t)$

Then condition (3.10.1) becomes

$M(T(x), T(y), T(z),t) \le k\, M(x,y,z,t)$ for all $x,y,z \in X$ and the proof follows from (3.4).

Theorem 3.11: Let $(X,M, *)$ be a complete GM-fuzzy metric space and T: $X\to X$ be an onto and continuous mapping satisfying the followings condition for all $x \in X$ and $t>0$

$M(T(x), T^2(x), T^3(x), t) \le a\, M(x,Tx, T^2x,t)$ ... (3.11.1)

Where a ≤ 1. Then T has a fixed point.

Proof: Let $x_0 \in X$, since T is onto then there exists an element x_1 satisfying $x_1 \in T^{-1}(x_0)$. By the same argument we can pick up $x_n \in T^{-1}(x_{n-1})$ where $n= 2,3,4,5,\ldots\ldots$

Let $x_n \ne x_{n-1}$, then there is a sequence $\{x_n\}$ with $x_n \ne x_{n-1}$ and $T(x_n) = x_{n-1}$

Then (3.11.1) implies that

$$M(x_{n-1}, x_{n-2},x_{n-3},t)=M(Tx_n, T^2x_n,T^3x_n,t) \le aM(x_n,Tx_n, T^2x_n,t) = a\, M(x_n,x_{n-1},x_{n-2},t) \quad \text{... (3.11.2)}$$

And therefore

$$M(x_n,x_{n-1},x_{n-2},t) \ge \frac{1}{a} M(x_{n-1},x_{n-2},x_{n-3},t)$$

Let $q = \dfrac{1}{a}$, then $q \ge 1$

We can easily show that the sequence $\{x_n\}$ is Cauchy and by completeness of $(X,M,*)$, the sequence $\{x_n\}$ converges to a point $u \in X$

Since T is continuous, then

$T(x_n) = x_{n-1} \to T(u)$ as $n \to \infty$

Hence, $T(u) = u$, which shows that u is a fixed point of T.

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