Conjugate Secondary Normal Matrices with Conjugate Secondary Normal Submatrices

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Abstract:

In studying, the reduction of a complex *n* x *n* matrix *A* to its Hessenbery form by the Arnoldi algorithm, T.Huckle **[2]** discovered that an irreducible Hessenbery normal matrix with a normal leading principal *m* x *m* submatrix, where *1<m<n*, actually is tridiagonal. We prove a similar assertion for the con-s-normal matrices, which play the same role in the theory of s-unitary congruences as the conventional s-normal matrices in the theory of s-unitary similarities.

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1. Introduction

Let C_{nxn} be the space of *nxn* complex matrices of order *n*. For $A \in C_{n \times n}$, let A^T, $\overline{A}$, A^*, A^s, $A^\theta \left(= \overline{A}^s \right)$ and A^{-1} denote the transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose and inverse of matrix *A* respectively. The conjugate secondary transpose of *A* satisfies the following properties such as $\left(A^\theta \right)^\theta = A$, $\left(A + B \right)^\theta = A^\theta + B^\theta$, $\left(AB \right)^\theta = B^\theta A^\theta$. etc

Definition 1

A matrix $A \in C_{n \times n}$ is said to be normal if $AA^* = A^* A$.

Definition 2

A Matrix $A \in C_{n \times n}$ is said to be conjugate normal (con-normal) if $AA^* = \overline{A^* A}$.

Definition 3

A matrix $A \in C_{n \times n}$ is said to be secondary normal (*s*-normal) if $AA^\theta = A^\theta A$.

Definition 4

A matrix $A \in C_{n \times n}$ is said to be unitary if $AA^* = A^* A = I$.

Definition 5

A matrix $A \in C_{n \times n}$ is said to be *s*-unitary if $AA^\theta = A^\theta A = I$.

Definition 6 [3]

A matrix $A \in C_{n\times n}$ is said to be a conjugate secondary normal matrix (con-s-normal) if $AA^{\theta} = \overline{A^{\theta}A}$ where $A^{\theta} = \overline{A}^{s}$. **. . . (1)**

2. Let $A \in M_n(C)\ (n \geq 3)$ be a block matrix of the form

$$A = \begin{pmatrix} B & C \\ D^{\theta} & E \end{pmatrix}, \qquad \textbf{. . . (2)}$$

where $B \in M_m(C)\ (1 < m < n)$, whereas *X* and *D* are *m* x *(n-m)* matrices. Assume that *A* is s-normal but neither s-hermitian nor s-skew hermitian. The case where *B* is s-normal is unusual, and, in this case, the structure of *A* is quite specific. For instance, in studying the reduction of *A* to its Hessenberg form *H* by the Arnoldi algorithm, if the matrix *H* is irreducible and its leading principal submatrix H_m *(1<m<n)* is normal, then, *H* actually is a tridiagonal matrix. In **[1]**, this fact was stated and proved as a purely matrix-theoretic theorem without any reference to the Arnoldi algorithm.

Now, instead of s-unitary similarities, consider s-unitary congruences, that is, transformations of the form

$$A \to Q^{S}AQ,\ Q^{\theta}Q = I.$$

Slightly modifying the standard reduction to Hessenberg form by plane rotations or Householder reflections, one can easily show that every complex matrix can also be brought to a Hessenberg matrix by a sequence of elementary s-unitary congruences. We apply such a sequence to a con-s-normal matrix *A*, that is, to a matrix satisfying the relation **(1).**

In particular, s-unitary congruences preserve the property of being a con-s-normal matrix.

Assume that a con-s-normal matrix *A* is reduced to an irreducible Hessenberg matrix whose leading principal submatrix of order *m (1<m<n)* also is con-s-normal. Our aim is to prove the following assertion, which is an analogue of the Huckle theorem.

Theorem 1

Let $A \in M_n(C)\ (n \geq 3)$ be an irreducible con-s-normal matrix in Hessenberg form. (For definiteness, assume that A is an upper Hessenberg matrix.) If its leading principal submatrix of order m (1<m<n) also is con-s-normal, then the matrix A actually is tridiagonal.

Proof

Let *A* be partitioned as in **(2)**. Then, in view of the condition

$$BB^{\theta} = \overline{B^{\theta}B} \qquad \textbf{. . . (3)}$$

By **(1)** amounts to the three matrix relations

$$CC^{\theta} = \overline{DD^{\theta}} \qquad \textbf{. . . (4)}$$

$$BD + CE^{\theta} = \overline{B^{\theta}C} + \overline{DE}, \qquad \textbf{. . . (5)}$$

$$D^{\theta}D + EE^{\theta} = \overline{C^{\theta}C} + \overline{E^{\theta}E} \qquad \textbf{. . . (6)}$$

Since A is an irreducible Hessenberg matrix, the only nonzero entry of the block D occurs in position $(m,1)$. We will show that X has the same structure.

Note that for every i $(1 \le i \le n)$, the 2-norm of the i^{th} row of A is equal to the 2-norm of its i^{th} column. This follows from the equality of the diagonal entries $\{AA^{\theta}\}_{ii}$ and $\{\overline{A^{\theta}A}\}_{ii}$. The same fact is valid for the submatrix B as well. Assuming that $1 \le i \le m-1$, we find that

$$\sum_{j=1}^{n} |a_{ji}|^2 = \sum_{j=1}^{m} |a_{ji}|^2 = \sum_{j=1}^{m} |a_{ij}|^2 = \sum_{j=1}^{n} |a_{ij}|^2$$

Whence $a_{ij} = 0,\ i = 1,\dots,m-1, \quad j = m+1,\dots,n$

Thus, the right-hand portions of the first m-1 rows of A are zero vectors. On the contrary, the zero subvectors of rows $m+1,\dots,n$ are their left-hand portions correspoinding to the positions $1 \le j \le m$. In particular, this means that

$$\{AA^{\theta}\}_{m-1,j} = 0, \qquad j = m+2,\dots,n$$

It follows that $$\{\overline{A^{\theta}A}\}_{m-1,j} = a_{m,m-1}\overline{a}_{mj} = 0, \qquad j = m+2,\dots,n$$

Since A is irreducible, we have $a_{m,m-1} \neq 0$, whence $a_{mj} = 0, \qquad j = m+2,\dots,n$

Thus, we have shown that $a_{m,m+1}$ is the only nonzero entry in the block C. Now the equality of the 2-norms of the m[th] row and column in A implies that

$$|a_{m,m-1}|^2 + |a_{m,m+1}|^2 = \sum_{j=1}^{m-1} |a_{jm}|^2 + |a_{m+1,m}|^2 \qquad \textbf{. . . (7)}$$

A similar equality for the submatrix B yields

$$|a_{m,m-1}|^2 = \sum_{j=1}^{m-1} |a_{jm}|^2 \qquad \textbf{. . . (8)}$$

Comparing **(7)** and **(8)**, we conclude that

$$|a_{m,m+1}| = |a_{m+1,m}| \qquad \textbf{. . . (9)}$$

It follows that $$D^{\theta}D = \overline{C^{\theta}C} \qquad \textbf{. . . (10)}$$

(the only nonzero entry of each of these matrices occurs in position (1,1) and is equal to $|a_{m+1,m}|^2 = |a_{m,m+1}|^2$). By substituting **(10)** into **(6)**, we obtain that the submatrix E also is con-s-normal.

Now, assuming that *m<n-1*, we show that the leading principal submatrix B_{m+1} is con-s-normal as well. Indeed, the equality $\{B_{m+1}B^{\theta}_{m+1}\}_{m+1,m+1} = \overline{\{B^{\theta}_{m+1}B_{m+1}\}}_{m+1,m+1}$ is implied by **(9)**, whereas for the other pairs *(i , j)* we have

$$\{B_{m+1}B^{\theta}_{m+1}\}_{ij} = \{AA^{\theta}\}_{ij} = \overline{\{A^{\theta}A\}}_{ij} = \overline{\{B^{\theta}_{m+1}B_{m+1}\}}_{ij}$$

Thus, the fact that the submatrix *B* is con-s-normal implies that B_{m+1} has the same property and, in addition, its last column has tridiagonal structure. Applying the same argument to B_{m+1}, then to B_{m+2}, and so on, we conclude that all the columns in *A* with indices greater than m have tridiagonal form.

It remains to verify that the submatrix *B* is tridiagonal. Observe that only the last two entries in column *m+1* of B_{m+1} are nonzero; hence we have

$$\{B^{\theta}_{m+1}B_{m+1}\}_{m+1,j} = 0, \qquad j = 1,\ldots,m-2$$

It follows that $b_{m+1,m}\overline{b}_{jm} = \{B_{m+1}B^{\theta}_{m+1}\}_{m+1,j} = 0, \; j = 1,\ldots,m-2$

Whence $b_{jm} = 0, \qquad j = 1,\ldots,m-2$

Thus, the fact that B_{m+1} is con-s-normal implies that the last column in *B* has tridiagonal structure. Show that the leading principal submatrix B_{m-1} also is con-s-normal. This will allow us to apply the same argument as above to the pairs (B, B_{m-1}), (B_{m-1}, B_{m-2}), and so on. Ultimately, this will prove that *B* is tridiagonal.

The relations $\{BB^{\theta}\}_{mm} = \overline{\{B^{\theta}B\}}_{mm}$ and $\{BB^{\theta}\}_{m-1.,-1} = \overline{\{B^{\theta}B\}}_{m-1,m-1}$

Imply that $|a_{m,m-1}| = |a_{m-1,m}|$ and $|a_{m-1,m-2}|^2 = \sum_{j=1}^{m-2} |a_{j,m-1}|^2,$

That is, $\{B_{m-1}B^{\theta}_{m-1}\}_{m-1,m-1} = \overline{\{B^{\theta}_{m-1}B_{m-1}\}}_{m-1,m-1}$

Since *B* is a Hessenberg matrix and since its last column has tridiagonal structure, we conclude that the equalities

$$\{B_{m-1}B^{\theta}_{m-1}\}_{ij} = \{BB^{\theta}\}_{ij} = \overline{\{B^{\theta}B\}}_{ij} = \overline{\{B^{\theta}_{m-1}B_{m-1}\}}_{ij}$$

hold for the other pairs *(i,j)* as well. This shows that B_{m-1} is con-s-normal which completes the proof of the theorem.

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