Derivation of Some Results Connecting Fluid Dynamics and Complex Analysis

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ABSTRACT

*Some results focusing on the applications of the interrelationship between fluid dynamics and complex analysis have been derived in the paper***.**

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1 INTRODUCTION, DEFINITIONS AND NOTATIONS.

In order to determine the characteristics of different kinds of fluid flows, complex analysis may act as an essential tool for them. In fact analytic function are mostly are characterized by their flows and therefore they are regarded as equivalent to incompressible and irrotational fluid flows.We may clear our notion with the introduction of complex potentials. If the velocity functions $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations, the term complex potential is defined as

$$f(z) = u(x, y) + iv(x, y).$$

Now it is natural to note that $u = const.$ and $v = const.$ correspond to equipotential curve and streamline respectively.Several researchers {cf. [4] and [5]} have investigated a stronginter-connection between fluid dynamics and pure complex analysis covering the fields of works on potential fluid flows and complex potential. The classical and standard theories of fluid dynamics as well as complex analysis have not been explained in this paper in detail as those are available in {cf.[1], [2], [3], [6] and [7]}. In this paper our target is to further investigate some relationship between analytic functions and potential fluid flows. Also, in the paper we discuss about the nature of the composition of two potential fluid flows in terms of their growth indicators.

Now, let us define a function as follows:

Let $\Psi: [0, \infty) \to (0, \infty)$ be a non-decreasing unbounded function, satisfying the following two conditions:

$$(i)\ \lim_{r\to\infty}\frac{\log^{[p]}(r)}{\log^{[q]}[\Psi(r)]}=1$$

and

$$(ii)\ \lim_{r\to\infty}\frac{\log^{[q]}(\alpha r)}{\log^{[q]}\Psi[(r)]}=1,$$

for some α>1 and p, q are any two posotive integers.

We use the notations as $M(r,f) = Max\,|f(z)|$ on the circle $|z| = r$ and

$log^{[k]}x = \log(log^{[k-1]}x)$for $= 1,2,3, \ldots, log^{[0]}x = x.$

With the help of the function Ψ, the classical definitions of several growth indicators of analytic functions especially entire functions can be reformulated in the following manner:

Definition1. The Ψ −order $\rho_{f,\Psi}$and lowerΨ −order $\lambda_{f,\Psi}$ of an entire function f

$$\rho_{f,\Psi}=\limsup_{r\to\infty}\frac{\log^{[2]}\mathrm{M(r,f)}}{\log[\Psi(r)]}$$

and

$$\lambda_{f,\Psi}=\liminf_{r\to\infty}\frac{\log^{[2]}\mathrm{M(r,f)}}{\log[\Psi(r)]}.$$

If$\rho_{f,\Psi} < \infty$ then f is of finite Ψ −order.Also $\rho_{f,\Psi} = 0$ means that f is of Ψ −order zero.

Definition2. TheΨ − hyper order$\bar{\rho}_{f,\Psi}$ and the Ψ −lower hyper order $\bar{\lambda}_{f,\Psi}$of an entire function $f(z)$ are defined as follows:

$$\bar{\rho}_{f,\Psi}=\limsup_{r\to\infty}\frac{\log^{[3]}\mathrm{M(r,f)}}{\log[\Psi(r)]}$$

and

$$\bar{\lambda}_{f,\Psi}=\liminf_{r\to\infty}\frac{\log^{[3]}\mathrm{M(r,f)}}{\log[\Psi(r)]}.$$

Definition3. The generalized Ψ −order $\rho^{(k)}_{f,\Psi}$and generalized Ψ −lower order $\lambda^{(k)}_{f,\Psi}$of an entire function $f(z)$ are defined by

$$\rho^{(k)}_{f,\Psi}=\limsup_{r\to\infty}\frac{\log^{[k]}M(r,f)}{\log[\Psi(r)]}$$

and

$$\lambda^{(k)}_{f,\Psi}=\liminf_{r\to\infty}\frac{\log^{[k]}M(r,f)}{\log[\Psi(r)]}$$

where $k = 1,2,3 \ldots$..

Definition4. Let $f(z)$ be an entire function of Ψ**-**order zero. The quantities $\rho^{*}_{f,\Psi}$ and $\lambda^{*}_{f,\Psi}$are respectively defined as

$$\rho^{*}_{f,\Psi} = \limsup_{r\to\infty} \frac{\log^{[2]}\mathrm{M(r,f)}}{\log^{[2]}[\Psi(\mathrm{r})]}$$

and

$$\lambda^{*}_{f,\Psi} = \liminf_{r\to\infty} \frac{\log^{[2]}M(r,f)}{\log^{[2]}[\Psi(\mathrm{r})]}.$$

Similarly, the quantities $\rho^{*}_{f,\Psi}$, $\lambda^{*}_{f,\Psi}$, $\rho^{*(k)}_{f,\Psi}$and $\lambda^{*(k)}_{f,\Psi}$ for $k = 1,2,3$ ....may also be defined.

In the present paper our main aim is to further establish some interrelationship between potential fluid flows and analytic functions the nature of the composition of two potential fluid flows in terms of their growth indicators is also studied here.

2 LEMMA.

In this section we present a lemma which will be needed in the sequel.

Lemma 1. Let the complex potential fluid flow $f(z) = u(x,y) + iv(x,y)$be defined in the region. {y>0}.If $f(z)$satisfiesthe followingproperties :

(i) $f(z)$is continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f\acute{(}z)$ is parallel to the x-axis when $y = 0$ and
(iii) $f\acute{(}z)$is uniformly bounded in $\{y > 0\}$

then the Ψ −order and Ψ −lower order of $f(z)$ is zero.

Proof. The proof of some portion of Lemma 1 has been carried out as in [4].

Since $f(z) = u(x,y) + iv(x,y)$,

We have on the x-axis $u_y = 0$.

Therefore, using Cauchy-Riemann equations, we get that

$$v_y = 0.$$

Thus, $v(x,y)$ reduces to a constant, sayc, on the x-axis.

By reflection it extends to an entire function also the reflection formula shows that is bounded in C. Hence, $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$are bounded harmonic functions on C. Thus, by Liouville's theorem, both of them are constants.

Therefore we may write for real a, b, c, d

$$u(x,y) = ax + by + c.$$

and$v(x,y)$ takes the form

$$v(x,y) = ay - bx + d.$$

Hence $f(z)$ will take the form

$f(z) = u(x,y) + iv(x,y) = (a - ib)z + B$, where a, b are real and B is complex.

i.e, $f(z) = Az + B$, where $A \text{ and } B$ are complex constants.

Thus, the Ψ −order of f is

$$\rho_{f,\Psi} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[\Psi(r)]}$$

$$= \limsup_{r\to\infty} \frac{\log^{[2]}(|A|r + B)}{\log[\Psi(r)]} = 0.$$

Similarly, the Ψ −lower order $\lambda_{f,\Psi}$ of f is zero.

Remark 1. Under the identical conditions of Lemma 1, the Ψ −hyper order (Ψ −hyper lower order) and generalized Ψ −order (the generalized Ψ −lower order) of the complex potential fluid flow $f(z) = u(x,y) + iv(x,y)$ are all zero.

3 THEOREMS.

In this section we present our main results.

Theorem 1. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$ and $g'(z)$ are both uniformly bounded in $\{y > 0\}$.

Also let $0 < \lambda^*_{fog,\Psi} \leq \rho^*_{fog,\Psi} < \infty$ and $0 < \lambda^*_{f,\Psi} \leq \rho^*_{f,\Psi} < \infty$.

Then for any positive number A,

$$\frac{\lambda^*_{fog,\Psi}}{A\rho^*_{f,\Psi}} \leq \liminf_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,f)} \leq \frac{\lambda^*_{fog,\Psi}}{A\lambda^*_{f,\Psi}} \leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(r^A,f)} \leq \frac{\rho^*_{fog,\Psi}}{A\lambda^*_{f,\Psi}}.$$

Proof. In view of Lemma 1, $\rho_{f,\Psi}$ and $\rho_{g,\Psi}$ will be both zero. Therefore by Definition 4, $\rho^*_{f,\Psi}$, $\lambda^*_{f,\Psi}$, and $\rho^*_{g,\Psi}$, $\lambda^*_{g,\Psi}$ are all defined.

Now, we have for arbitrary positive ϵ and for all sufficiently large values of r,

$$\log^{[2]} M(r,fog) \geq \left(\lambda^*_{fog,\Psi} - \varepsilon\right)\log^{[2]}[\Psi(r)] \tag{1}$$

and $\log^{[2]} M(r^A,fog) \leq A\left(\rho^*_{f,\Psi} + \varepsilon\right)\log^{[2]}[\Psi(r)]$. (2)

Now from (1) and (2), it follows that for all sufficiently large values of r,

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \geq \frac{\lambda^*_{fog,\Psi} - \varepsilon}{A\left(\rho^*_{f,\Psi} + \varepsilon\right)}$$

As ϵ (>0) is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{log^{[2]} M(r, fog)}{log^{[2]} M(r^A, f)} \geq \frac{\lambda^*_{fog,\Psi}}{A\rho^*_{f,\Psi}}. \tag{3}$$

Again for a sequence of values of r, tending to infinity,

$$log^{[2]} M(r, fog) \leq \left(\lambda^*_{fog,\Psi} + \varepsilon\right) log^{[2]}[\Psi(r)] \tag{4}$$

and for all sufficiently large values of r,

$$\log^{[2]} M(r^A, f) \geq A\left(\lambda^*_{f,\Psi} - \varepsilon\right) \log^{[2]}[\Psi(r)]. \tag{5}$$

Combining (4) and (5), we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \leq \frac{\lambda^*_{fog,\Psi} + \varepsilon}{A\left(\lambda^*_{f,\Psi} - \varepsilon\right)}.$$

Since ϵ (>0) is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \leq \frac{\lambda^*_{fog,\Psi}}{A\lambda^*_{f,\Psi}} \ . \tag{6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[2]} M(r^A, f) \leq A\left(\lambda^*_{f,\Psi} + \varepsilon\right) \log^{[2]}[\Psi(r)]. \tag{7}$$

Now from (1) and (7), we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \geq \frac{\lambda^*_{fog,\Psi} - \varepsilon}{A\left(\lambda^*_{f,\Psi} + \varepsilon\right)}.$$

As ϵ (>0) is arbitrary, we get that

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \geq \frac{\lambda^*_{fog,\Psi}}{A\lambda^*_{f,\Psi}}. \tag{8}$$

Also for all sufficiently large values of r,

$$\log^{[2]} M(r, fog) \leq \left(\rho^*_{fog,\Psi} + \varepsilon\right) \log^{[2]}[\Psi(r)] \tag{9}$$

From (5) and (9) it follows for all sufficiently large values of r,

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \leq \frac{\rho^*_{fog,\Psi} + \varepsilon}{A\left(\lambda^*_{f,\Psi} - \varepsilon\right)}.$$

Since ϵ (>0) is arbitrary, we obtain from above that

$$\limsup_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,f)}\le\frac{\rho^*_{fog,\Psi}}{A\lambda^*_{f,\Psi}}\,. \qquad (10)$$

Thus the theorem follows from (3), (6), (8) and (10).

Remark 2. If we take the condition $0<\lambda^*_{g,\Psi}\le\rho^*_{g,\Psi}<\infty$ instead of $0<\lambda^*_{f,\Psi}\le\rho^*_{f,\Psi}<\infty$and the other conditions remain the same, Theorem 1 is still valid with the right factor$g$ of the composition fog in place of the left factor f i.e, for any positive constant A,

$$\frac{\lambda^*_{fog,\Psi}}{A\rho^*_{g,\Psi}}\le\liminf_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,g)}\le\frac{\lambda^*_{fog,\Psi}}{A\lambda^*_{g,\Psi}}\le\limsup_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,g)}\le\frac{\rho^*_{fog,\Psi}}{A\lambda^*_{g,\Psi}}$$

holds.

Theorem 2. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y>0\}$ satisfying the following properties :

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region$\{y\ge 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y=0$ and
(iii) $f'(z)$and$g'(z)$ are both uniformly bounded in $\{y>0\}$.

Also let $0<\lambda^*_{fog,\Psi}\le\rho^*_{fog,\Psi}<\infty$and $0<\rho^*_{f,\Psi}<\infty$.

Then for any positive number A,

$$\liminf_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,f)}\le\frac{\rho^*_{fog,\Psi}}{A\rho^*_{f,\Psi}}\le\limsup_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,f)}.$$

Proof. By Lemma 1, $\rho_{f,\Psi}$and $\rho_{g,\Psi}$ will be both zero. Therefore in view of Definition 4, $\rho^*_{f,\Psi}$, $\lambda^*_{f,\Psi}$and$\rho^*_{g,\Psi}$, $\lambda^*_{g,\Psi}$ are all defined.

From the definition of Ψ −order, we get for a sequence of values r tending to infinity,

$$\log^{[2]}M(r^A,f)\ge A\left(\rho^*_{f,\Psi}-\varepsilon\right)\log^{[2]}[\Psi(r)]. \qquad (11)$$

Now from (9) and (11), it follows for a sequence of values of $r$tending to infinity,

$$\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,f)}\le\frac{\rho^*_{fog,\Psi}+\varepsilon}{A\left(\rho^*_{f,\Psi}-\varepsilon\right)}.$$

As ϵ (>0) is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[2]}M(r,fog)}{\log^{[2]}M(r^A,f)}\le\frac{\rho^*_{fog,\Psi}}{A\rho^*_{f,\Psi}}\,. \qquad (12)$$

Again for a sequence of values of r tending to infinity,

$$\log^{[2]}M(r,fog)\ge\left(\rho^*_{fog,\Psi}-\varepsilon\right)\log^{[2]}[\Psi(r)]. \qquad (13)$$

So combining (2) and (13), we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \geq \frac{\rho^*_{fog,\Psi} - \varepsilon}{A(\rho^*_{f,\Psi} + \varepsilon)}.$$

Since ϵ (>0) is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \geq \frac{\rho^*_{fog,\Psi}}{A\rho^*_{f,\Psi}}. \quad (14)$$

Thus the theorem follows from (12) and (14).

Remark 3. Parallel investigations may be done for the right factor g of the composition $f \circ g$i.e, for any positive constant A,

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)} \leq \frac{\rho^*_{fog,\Psi}}{A\rho^*_{g,\Psi}} \leq \limsup_{r\to\infty} \frac{log^{[2]} M(r, fog)}{log^{[2]} M(r^A, f)}.$$

Combining Theorem 1 and Theorem 2 we get the following theorem:

Theorem 3. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region$\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$and$g'(z)$ are both uniformly bounded in $\{y > 0\}$.
Also let $0 < \lambda^*_{fog,\Psi} \leq \rho^*_{fog,\Psi} < \infty$ and $0 < \lambda^*_{f,\Psi} \leq \rho^*_{f,\Psi} < \infty$.

Then for any positive number A,

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, f_0 g)}{\log^{[2]} M(r^A, f)} \leq min\left\{\frac{\lambda^*_{fog,\Psi}}{A\lambda^*_{f,\Psi}}, \frac{\rho^*_{fog,\Psi}}{A\rho^*_{f,\Psi}}\right\} \leq max\left\{\frac{\lambda^*_{fog,\Psi}}{A\lambda^*_{f,\Psi}}, \frac{\rho^*_{fog,\Psi}}{A\rho^*_{f,\Psi}}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)}.$$

In view of Remark 1 and Remark 2 we obtain a parallel result to Theorem 3.

Theorem 4. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region$\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$and$g'(z)$ are both uniformly bounded in $\{y > 0\}$.
Also let $0 < \lambda^*_{fog,\Psi} \leq \rho^*_{fog,\Psi} < \infty$and $0 < \lambda^*_{g,\Psi} \leq \rho^*_{g,\Psi} < \infty$.

Then for any positive number A,

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, f_0 g)}{\log^{[2]} M(r^A, g)} \leq min\left\{\frac{\lambda^*_{fog,\Psi}}{A\lambda^*_{g,\Psi}}, \frac{\rho^*_{fog,\Psi}}{A\rho^*_{g,\Psi}}\right\} \leq max\left\{\frac{\lambda^*_{fog,\Psi}}{A\lambda^*_{g,\Psi}}, \frac{\rho^*_{fog,\Psi}}{A\rho^*_{g,\Psi}}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, g)}.$$

In view of Lemma 1 and Remark 1, we may obtain identical investigations of Theorem 1, Theorem 2, Theorem 3 and Theorem 4 as follows and therefore their proofs are omitted.

Theorem 5. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$ and $g'(z)$ are both uniformly bounded in $\{y > 0\}$.
Also let $0 < \bar{\lambda}^*_{f \circ g,\Psi} \leq \bar{\rho}^*_{f \circ g,\Psi} < \infty$ and $0 < \bar{\lambda}^*_{f,\Psi} \leq \bar{\rho}^*_{f,\Psi} < \infty$.
Then for any positive number A,

$$\frac{\bar{\lambda}^*_{f \circ g,\Psi}}{A\bar{\rho}^*_{f,\Psi}} \leq \liminf_{r\to\infty} \frac{\log^{[3]} M(r, f \circ g)}{\log^{[3]} M(r^A, f)} \leq \frac{\bar{\lambda}^*_{f \circ g,\Psi}}{A\bar{\lambda}^*_{f,\Psi}} \leq \limsup_{r\to\infty} \frac{\log^{[3]} M(r, f \circ g)}{\log^{[3]} M(r^A, f)} \leq \frac{\bar{\rho}^*_{f \circ g,\Psi}}{A\bar{\lambda}^*_{f,\Psi}}.$$

Theorem 6. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$ and $g'(z)$ are both uniformly bounded in $\{y > 0\}$.
Also let $0 < \bar{\lambda}^*_{f \circ g,\Psi} \leq \bar{\rho}^*_{f \circ g,\Psi} < \infty$ and $0 < \bar{\rho}^*_f < \infty$.
Then for any positive number A,

$$\liminf_{r\to\infty} \frac{log^{[3]} M(r, f \circ g)}{log^{[3]} M(r^A, f)} \leq \frac{\bar{\rho}^*_{f \circ g,\Psi}}{A\bar{\rho}^*_{f,\Psi}} \leq \limsup_{r\to\infty} \frac{log^{[3]} M(r, f \circ g)}{log^{[3]} M(r^A, f)}.$$

Theorem 7. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$ and $g'(z)$ are both uniformly bounded in $\{y > 0\}$.
Also let $0 < \bar{\lambda}^*_{f \circ g,\Psi} \leq \bar{\rho}^*_{f \circ g,\Psi} < \infty$ and $0 < \bar{\lambda}^*_{f,\Psi} \leq \bar{\rho}^*_{f,\Psi} < \infty$.
Then for any positive number A,

$$\liminf_{r\to\infty} \frac{\log^{[3]} M(r, f \circ g)}{\log^{[3]} M(r^A, f)} \leq min\left\{\frac{\bar{\lambda}^*_{f \circ g,\Psi}}{A\bar{\lambda}^*_{f,\Psi}}, \frac{\bar{\rho}^*_{f \circ g,\Psi}}{A\bar{\rho}^*_{f,\Psi}}\right\} \leq max\left\{\frac{\bar{\lambda}^*_{f \circ g,\Psi}}{A\bar{\lambda}^*_{f,\Psi}}, \frac{\bar{\rho}^*_{f \circ g,\Psi}}{A\bar{\rho}^*_{f,\Psi}}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[3]} M(r, f \circ g)}{\log^{[3]} M(r^A, f)}.$$

Theorem 8. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$ and $g'(z)$ are both uniformly bounded in $\{y > 0\}$.
Also let $0 < \bar{\lambda}^*_{f \circ g,\Psi} \leq \bar{\rho}^*_{f \circ g,\Psi} < \infty$ and $0 < \bar{\lambda}^*_{g,\Psi} \leq \bar{\rho}^*_{g,\Psi} < \infty$.
Then for any positive number A,

$$\liminf_{r\to\infty} \frac{\log^{[3]} M(r, f \circ g)}{\log^{[3]} M(r^A, g)} \leq min\left\{\frac{\bar{\lambda}^*_{f \circ g,\Psi}}{A\bar{\lambda}^*_{g,\Psi}}, \frac{\bar{\rho}^*_{f \circ g,\Psi}}{A\bar{\rho}^*_{g,\Psi}}\right\} \leq max\left\{\frac{\bar{\lambda}^*_{f \circ g,\Psi}}{A\bar{\lambda}^*_{g,\Psi}}, \frac{\bar{\rho}^*_{f \circ g,\Psi}}{A\bar{\rho}^*_{g,\Psi}}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[3]} M(r, f \circ g)}{\log^{[3]} M(r^A, g)}.$$

Theorem 9. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and

(iii) $f'(z)$and$g'(z)$ are both uniformly bounded in $\{y > 0\}$.

Also let $0 < \lambda^{*(k)}_{fog,\Psi} \leq \rho^{*(k)}_{fog,\Psi} < \infty$ and $0 < \lambda^{*(k)}_{f,\Psi} \leq \rho^{*(k)}_{f,\Psi} < \infty$ for $k = 2,3,\ldots$ Ψ

Then for any positive number A,

$$\frac{\lambda^{*(k)}_{fog,\Psi}}{A^{\rho^{*(k)}_{f,\Psi}}} \leq \liminf_{r\to\infty} \frac{\log^{[k]} M(r, f\circ g)}{\log^{[k]} M(r^A, f)} \leq \frac{\lambda^{*(k)}_{fog,\Psi}}{A^{\lambda^{*(k)}_{f,\Psi}}} \leq \limsup_{r\to\infty} \frac{\log^{[k]} M(r, f\circ g)}{\log^{[k]} M(r^A, g)} \leq \frac{\rho^{*(k)}_{fog,\Psi}}{A^{\lambda^{*(k)}_{f,\Psi}}}.$$

Theorem 10. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$and$g'(z)$ are both uniformly bounded in $\{y > 0\}$.

Also let $0 < \lambda^{*(k)}_{fog,\Psi} \leq \rho^{*(k)}_{fog,\Psi} < \infty$ and $0 < \rho^{*(k)}_{f,\Psi} < \infty$ for $k = 2,3,\ldots$

Then for any positive number A,

$$\liminf_{r\to\infty} \frac{\log^{[k]} M(r, f\circ g)}{\log^{[k]} M(r^A, f)} \leq \frac{\rho^{*(k)}_{fog,\Psi}}{A^{\rho^{*(k)}_{f,\Psi}}} \leq \limsup_{r\to\infty} \frac{\log^{[k]} M(r, f\circ g)}{\log^{[k]} M(r^A, f)}.$$

Theorem 11. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region$\{y \geq 0\}$,
(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and
(iii) $f'(z)$and$g'(z)$ are both uniformly bounded in $\{y > 0\}$.

Also let $0 < \lambda^{*(k)}_{fog,\Psi} \leq \rho^{*(k)}_{fog,\Psi} < \infty$ and $0 < \lambda^{*(k)}_{f,\Psi} \leq \rho^{*(k)}_{f,\Psi} < \infty$ for $k = 2,3,\ldots$

Then for any positive number A,

$$\liminf_{r\to\infty} \frac{\log^{[k]} M(r, f\circ g)}{\log^{[k]} M(r^A, f)} \leq min\left\{\frac{\lambda^{*(k)}_{fog,\Psi}}{A^{\lambda^{*(k)}_{f,\Psi}}}, \frac{\rho^{*(k)}_{fog,\Psi}}{A^{\rho^{*(k)}_{f,\Psi}}}\right\} \leq max\left\{\frac{\lambda^{*(k)}_{fog,\Psi}}{A^{\lambda^{*(k)}_{f,\Psi}}}, \frac{\rho^{*(k)}_{fog,\Psi}}{A^{\rho^{*(k)}_{f,\Psi}}}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[k]} M(r, f\circ g)}{\log^{[k]} M(r^A, f)}.$$

Theorem 12. Let $f(z)$ and $g(z)$ be two complex potential fluid flows defined in the region $\{y > 0\}$ satisfying the following properties:

(i) $f(z)$ and $g(z)$ are both continuously differentiable in the region$\{y \geq 0\}$,

(ii) $f'(z)$ and $g'(z)$ are both parallel to the x –axis when $y = 0$ and

(iii) $f'(z)$and$g'(z)$ are both uniformly bounded in $\{y > 0\}$.

Also let $0 < \lambda^{*(k)}_{fog,\Psi} \leq \rho^{*(k)}_{fog,\Psi} < \infty$ and $0 < \lambda^{*(k)}_{f,\Psi} \leq \rho^{*(k)}_{f,\Psi} < \infty$for $k = 2,3,\ldots$

Then for any positive number A,

$$\liminf_{r\to\infty}\frac{\log^{[k]}M(r,f\circ g)}{\log^{[k]}M(r^A,g)}\le min\left\{\frac{\lambda^{*(k)}_{fog,\Psi}}{A^{\lambda^{*(k)}_{f,\Psi}}},\frac{\rho^{*(k)}_{fog,\Psi}}{A^{\rho^{*(k)}_{f,\Psi}}}\right\}\le max\left\{\frac{\lambda^{*(k)}_{fog,\Psi}}{A^{\lambda^{*(k)}_{f,\Psi}}},\frac{\rho^{*(k)}_{fog,\Psi}}{A^{\rho^{*(k)}_{f,\Psi}}}\right\}\le \limsup_{r\to\infty}\frac{\log^{[k]}M(r,f\circ g)}{\log^{[k]}M(r^A,g)}.$$

CONCLUSION AND FUTURE PROSPECT

The most recent mathematical tool to develop the theory of complex numbers is bi-complex analysis. The treatment focusing the intimate connection between complex potential fluid flow and analytic functions as established in the above theorems may also be applied in case of functions of bi-complex variables and so can be regarded as a virgin area of research in this field.

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