A Note on Generalized Skew Derivations on Rings

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Abstract:- *In this article we investigate the structure of a ring R with involution of second kind admitting a generalized skew-derivation G satisfying one of the following:*

(i) $G([x,x^*])+[x,x^*]\in Z(R)$

(ii) $G(x\circ x^*)\in Z(R)$

(iii) $G([x,x^*])\mp x\circ x^*\in Z(R)$

(iv) $G(x\circ x^*)\mp x\circ x^*\in Z(R)$

(v) $G(x\circ x^*)\mp [x,x^*]\in Z(R)$

for all $x\in R$.

Keywords :- *Prime rings, *-prime rings, skew derivations, generalized skew derivations.*

I. INTRODUCTION

Throughout the paper R will represent an associative ring with center Z(R). For any $x,y\in R$ the symbol $[x,y]$ will denote the commutator $xy-yx$; while the symbol $x\circ y$ will stand for anti-commutator $xy+yx$. R is prime if $aRb=0$ implies $a=0$ or $b=0$. An additive map $*:R\rightarrow R$ is called an involution if * is an anti-automorphism of order 2; that is $(x^*)^*=x$ for all $x\in R$. R is *-prime if $aRb^*=0$ implies $a=0$ or $b=0$. An element x in a ring R with involution * is said to be hermitian and skew-hermitian elements of R will be denoted by H(R) and S(R), respectively. The involution is said to be of the firest kind if $Z(R)\subseteq H(R)$, otherwise it is said to be of second kind. In the later case $Z(R)\cap S(R)\neq(0)$. An additive mapping $d:R\rightarrow R$ is said to be a derivation if $d(xy)=d(x)y+xd(y)$ for all $x,y\in R$. An additive map $F:R\rightarrow R$ is a generalized derivation if their exists a derivation d such that $F(xy)=F(x)y+xd(y)$ for all $x,y\in R$. All derivations are generalized derivations.

Let R be an associative ring and α be an automorphism of R. An additive mapping $D:R\rightarrow R$ is called a skew-derivation of R if $D(xy)=D(x)y+\alpha(x)D(y)$ for all $x,y\in R$ and α is called an associated automorphism of D. An additive mapping $G:R\rightarrow R$ is said to be a generalized skew-derivation of R if there exists a skew-derivation D of R with associated automorphism α such that $G(xy)=G(x)y+\alpha(x)D(y)$ for all $x,y\in R$. The definition of generalized skew-derivation is a unified notion of skew-derivation and generalized derivation, which are considered as classical additive mappings of non-associative algebras. The behaviour of these has been investigated by many researchers from various views, see [1-5]. In [6, Theorem 2], Daif and Bell proved that if R is a semiprime witha nonzero ideal I and d is a derivation of R such that $d([x,y]=[x,y]$ for all $x,y\in I$, then $I\subseteq Z(R)$. In particular if R is a prime ring, then R must be commutative. Recently in [7] Filippis and Huang studied the situation $(F([x,y]))^n=[x,y]$ for all $x,y\in I$, where I is a nonzero ideal in a prime ring R, F is a generalized derivation of R and $n\geq1$, a fixed integer. In this case they conclude that either R is commutative or $n=1$: $d=0$ and $F(x)=x$ for all $x\in R$.

Motivated by the aforementioned results in this paper we prove some theorems for a generalized skew-derivation of a ring with involution of second kind.

II. MAIN RESULT PAGE LAYOUT

Fact 1: Let R be a prime ring of characteristic not 2 with involution * of second kind. If R is prime and $S(R)\cap Z(R)\neq(0)$, then $D(h)=0$ for all $h\in H(R)\cap Z(R)$ implies that $D(z)=0$ for all $z\in Z(R)$. Indeed, if $D(h)=0$ for all $h\in H(R)\cap Z(R)$, replacing h by k^2 where $k\in S(R)\cap Z(R)$, then we have $D(k)k=0$ for all $k\in S(R)\cap Z(R)$ since α is an automorphism. As conclusion, we get $D(z)=0$ for all $z\in Z(R)$.

Theorem 2.1 Let R be a prime ring of characteristic not 2 with involution * of second kind. If R admits a generalized skew-derivation G associated with a skew-derivation D with an automorphism $\alpha:R\to R$ such that $G([x,x^*])+[x,x^*]\in Z(R)$ for all $x\in R$, then either R is commutative or $[\alpha([x,y])D(x),x]=0$ for all $x\in R$.

Proof. Assume that

$$G([x,x^*])+[x,x^*]\in Z(R) \quad \text{for all } x,y\in R. \tag{2.1}$$

Replacing x by x+y in (2.1), we get

$$G([x,y^*])+G([y,x^*])+[x,y^*]+[y,x^*]\in Z(R) \quad \text{for all } x,y\in R. \tag{2.2}$$

Substituting y* for y, we find that

$$G([x,y])+G([y^*,x^*])+[x,y]+[y^*,x^*]\in Z(R) \quad \text{for all } x,y\in R. \tag{2.3}$$

Replacing y by yh in (2.3), where $h\in Z(R)\cap H(R)\setminus\{0\}$, we have

$$(G([x,y])+G([y^*,x^*])+[x,y]+[y^*,x^*])h+\alpha([x,y])D(h)+\alpha([y^*,x^*])D(h)\in Z(R). \tag{2.4}$$

By equation (2.3), we have

$$\alpha([x,y])D(h)+\alpha([y^*,x^*])D(h)\in Z(R) \quad \text{for all } x,y\in R. \tag{2.5}$$

This implies that

$$[\alpha([x,y])D(h)+\alpha([y^*,x^*])D(h),r]=0 \quad \text{for all } r,x,y\in R.$$

This gives

$$[(\alpha([x,y])+\alpha([y^*,x^*]))D(h),r]=0 \quad \text{for all } r,x,y\in R.$$

This implies

$$[\alpha([x,y])D(h)+\alpha([y^*,x^*]),r]D(h)=0 \quad \text{for all } r,x,y\in R.$$

Therefore, either $[\alpha([x,y])D(h)+\alpha([y^*,x^*]),r]=0$ or $D(h)=0$ for $h\in Z(R)\cap H(R)\setminus\{0\}$.

Suppose that

$$[\alpha([x,y]))+\alpha([y^*,x^*]),r]=0 \quad \text{for all } r,x,y\in R. \tag{2.6}$$

Substituting ys for y, where $s\in S(R)\cap Z(R)\setminus\{0\}$, we obtain

$$[\alpha([x,y])-\alpha([y^*,x^*]),r]\alpha(s)=0 \quad \text{for all } r,x,y\in R.$$

Replacing r by rr', we get

$$[\alpha([x,y])-\alpha([y^*,x^*]),r]R\alpha(s)=0 \quad \text{for all } r,x,y\in R.$$

Using primeness of R, we get

$$[\alpha([x,y])-\alpha([y^*,x^*]),r]=0 \quad \text{for all } r,x,y\in R. \tag{2.7}$$

Now adding (2.6) and (2.7) and using the fact that R is not of characteristic 2, we have

$$[\alpha([x,y]),r]=0 \quad \text{for all } r,x,y\in R.$$

Substituting yx for y, we obtain

$\alpha([x,y])[r,\alpha(x)]=0$ for all $r,x,y\in R$.

Again replacing r by $r\alpha(y)$ we get $\alpha([x,y])R\alpha([x,y])=0$. Using primeness of R we get $\alpha([x,y])=0$. This implies that $[x,y]=0$ for all $x,y\in R$. Hence R is commutative.

On the other hand if $D(h)=0$ for all $h\in Z(R)\cap H(R)\backslash\{0\}$, by Fact 1 we have $D(z)=0$ for all $z\in Z(R)$. Now replacing y by ys in (2.3), where $s\in Z(R)\cap S(R)\backslash\{0\}$, we obtain

$$(G([x,y])s-G([y^*,x^*]s)+[x,y]s-[y^*,x^*]s)\in Z(R).$$

That is

$$(G([x,y])-G([y^*,x^*])s+[x,y]s-[y^*,x^*]s)\in Z(R). \quad (2.8)$$

Multiplying (2.3) by s from right and adding (2.8), we get

$$2(G([x,y])+[x,y])s\in Z(R).$$

Since R is not of characteristic 2, we have

$$(G([x,y])+[x,y])s\in Z(R) \quad \text{for all } x,y\in R. \quad (2.9)$$

This gives that

$$(G([x,y])+[x,y])\in Z(R) \quad \text{for all } x,y\in R \quad (2.10)$$

and hence

$$[(G([x,y])+[x,y],r]=0 \quad \text{for all } x,y,r\in R. \quad (2.11)$$

Replacing y by yx in (2.11), we get

$$[(G([x,y])x+\alpha([x,y]))D(x)+[x,y]x,r]=0 \quad \text{for all } x,y,r\in R.$$

Next replacing r by x and using (2.11), we obtain

$$[\alpha([x,y])D(x),r]=0 \quad \text{for all } x,y,r\in R.$$

Theorem 2.2 Let R be a prime ring of characteristic not 2 with involution * of second kind. If R admits a generalized skew-derivation G associated with a skew-derivation D with an automorphism $\alpha:R\rightarrow R$ such that $G(x\circ x^*)\in Z(R)$ for all $x\in R$, then either R is commutative or $[\alpha(x^2)D(x),x]=0$ for all $x\in R$.

Proof. Suppose that $G\neq 0$ and

$$G(x\circ x^*)\in Z(R) \quad \text{for all } x,y\in R. \quad (2.12)$$

Replacing x by x+y in (2.12), we get

$$G(x\circ y^*)+G(y\circ x^*)\in Z(R) \quad \text{for all } x,y\in R. \quad (2.13)$$

Substituting y* for y, we find that

$$G(x\circ y)+G(y^*\circ x^*)\in Z(R) \quad \text{for all } x,y\in R. \quad (2.14)$$

Again replacing y by yh in (2.14), where $h\in Z(R)\cap H(R)\backslash\{0\}$ and using (2.14), we have

$$[(\alpha(x\circ y)+\alpha(y^*\circ x^*))D(h),r]=0 \quad \text{for all } x,y\in R. \tag{2.15}$$

This implies that

$$[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]D(h)=0 \quad \text{for all } x,y,r\in R.$$

Since R is prime, either $D(h)=0$ or $[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]=0$.

Suppose that

$$[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]=0 \quad \text{for all } r,x,y\in R. \tag{2.16}$$

Replacing y by ys, where $s\in Z(R)\cap S(R)\backslash\{0\}$, we get

$$[\alpha(x\circ y)\alpha(s)-\alpha(y^*\circ x^*)\alpha(s),r]=0 \quad \text{for all } r,x,y\in R. \tag{2.17}$$

Multiplying (2.16) by $\alpha(s)$ from right and adding (2.17), we get

$$2[\alpha(x\circ y)\alpha(s),r]=0 \quad \text{for all } r,x,y\in R.$$

Since R is not of characteristic 2, we get

$$[\alpha(x\circ y)\alpha(s),r]=0 \quad \text{for all } r,x,y\in R.$$

This gives

$$[\alpha(x\circ y)\alpha(s),r]\alpha(s)=0 \quad \text{for all } r,x,y\in R.$$

Primeness of R, yields that either $[\alpha(x\circ y),r]=0$ or $\alpha(s)=0$. Since $\alpha(s)\neq 0$, we have $[\alpha(x\circ y),r]=0$.

Replacing y by yx, we get

$$[\alpha(x\circ y)\alpha(x),r]=0 \quad \text{for all } r,x,y\in R.$$

This gives

$$\alpha(x\circ y)[\alpha(x),r]=0 \quad \text{for all } r,x,y\in R.$$

Replacing r by $r\alpha(z)$, we get

$$\alpha(x\circ y)r[\alpha(x),\alpha(z)]=0 \quad \text{for all } r,x,y,z\in R.$$

Using primeness of R, we get either $\alpha(x\circ y)=0$ or $[\alpha(x),\alpha(z)]=0$. In each case R is commutative.

Now if $D(h)=0$, by Fact 1 we conclude that $D(z)$ for all $z\in Z(R)$. Substituting y by ys in (2.14), where $s\in Z(R)\cap S(R)\backslash\{0\}$, we obtain

$$(G(x\circ y)-G(y^*\circ x^*))s\in Z(R) \quad \text{for all } x,y\in R. \tag{2.18}$$

Multiplying (2.14) by s from right and adding (2.18), we get

$$2(G(x\circ y))s\in Z(R) \quad \text{for all } x,y\in R.$$

Since R is not characteristic 2, we have

$$(G(x\circ y))s\in Z(R) \quad \text{for all } x,y\in R.$$

This gives that

$$G(x\circ y)\in Z(R) \quad \text{for all } x,y\in R. \tag{2.19}$$

That is

$$[G(x\circ y),r]=0 \quad \text{for all } x,y\in R. \tag{2.20}$$

Replacing y by yx in (2.20), we obtain

$$[G(x\circ y)x+\alpha(x\circ y)D(x),r]=0 \quad \text{for all } x,y\in R.$$

Substituting x instead of r and using (2.20), we have

$$[\alpha([x,y])D(x),x]=0 \quad \text{for all } x,y\in R.$$

Now replacing y by yx, we obtain

$$[\alpha(2x^2)D(x),x]=0 \quad \text{for all } x,y\in R.$$

Since R is not of characteristic 2, we have

$$[\alpha(x^2)D(x),x]=0 \quad \text{for all } x\in R.$$

Theorem 2.3 Let R be a prime ring of characteristic not 2 with involution * of second kind. If R admits a generalized skew-derivation G associated with a skew-derivation D with an automorphism $\alpha:R\rightarrow R$ such that $G([x,x^*])\mp x\circ x^*\in Z(R)$ for all $x\in R$, then either R is commutative or $[\alpha([x,y])D(x),x]=0$ for all $x\in R$.

Proof. Let

$$G([x,x^*])-x\circ x^*\in Z(R) \quad \text{for all } x,y\in R.$$

Linearization gives

$$G([x,y])+G([y^*,x^*])-x\circ y-y^*\circ x^*\in Z(R) \quad \text{for all } x,y\in R. \tag{2.21}$$

Replacing y by yh, where $h\in Z(R)\cap H(R)\backslash\{0\}$, we get

$$(\alpha([x,y])+\alpha([y^*,x^*]))D(h)\in Z(R) \quad \text{for all } x,y\in R.$$

This implies that

$$[\alpha([x,y])+\alpha([y^*,x^*]),r]D(h)=0. \quad \text{for all } x,y\in R \tag{2.22}$$

Using primeness of R, we have either $[\alpha([x,y])+\alpha([y^*,x^*]),r]=0$ or $D(h)=0$.

Suppose that

$$[\alpha([x,y]))+\alpha([y^*,x^*]),r]=0 \quad \text{for all } r,x,y\in R. \tag{2.23}$$

Now, replacing y by ys, where $s\in Z(R)\cap S(R)\backslash\{0\}$, we get

$$[\alpha([x,y]s)-\alpha([y^*,x^*]s),r]=0 \quad \text{for all } s,x,y\in R.$$

i.e.

$[(\alpha([x,y])-\alpha([y^*,x^*]))\alpha(s),r]=0$ for all $s,x,y\in R$.

This implies that

$[\alpha([x,y])-\alpha([y^*,x^*]),r]\alpha(s)=0$ for all $s,x,y\in R$. (2.24)

Using primeness of R, we get

$[\alpha([x,y])-\alpha([y^*,x^*]),r]=0$ for all $r,x,y\in R$. (2.25)

Now adding (2.23) and (2.25), we get

$2[r,\alpha([x,y])]=0$ for all $r,x,y\in R$.

Since R is not of characteristic 2, we have

$[r,\alpha([x,y])]=0$ for all $r,x,y\in R$. (2.26)

Arguing in the similar manner as in Theorem 2.1, R is commutative.

On the other hand if D(h)=0 by Fact 1 we conclude that D(z)=0 for all $z\in Z(R)$.

Substituting ys for y in (2.21), where $s\in Z(R)\cap S(R)\backslash\{0\}$, we obtain

$G([x,y])s-G([y^*,x^*]s-(x\circ y)s+(y^*\circ x^*)s\in Z(R)$ for all $x,y,s\in R$. (2.27)

Multiplying (2.21) by s from right and adding (2.27), we get

$2\,(G([x,y])-(x\circ y)s)\in Z(R)$ for all $x,y,s\in R$.

Since R is not of characteristic 2, we get

$(G([x,y])-(x\circ y))s\in Z(R)$ for all $x,y,s\in R$

This gives that

$(G([x,y])-(x\circ y))\in Z(R)$ for all $x,y\in R$.

This implies that

$[(G([x,y])-(x\circ y),r]=0$ for all $x,y,r\in R$. (2.28)

Substituting yx for y, we get

$[G([x,y])x+\alpha([x,y])D(x)-(x\circ y)x,r]=0$ for all $x,y,r\in R$.

Replacing r by x and using (2.28), we obtain

$[\alpha([x,y])D(x),x]=0$ for all $x,y,r\in R$.

Theorem 2.4 Let R be a prime ring of characteristic not 2 with involution * of second kind. If R admits a generalized skew-derivation G associated with a skew-derivation D with an automorphism $\alpha:R\rightarrow R$ such that $G(x\circ x^*)\mp x\circ x^*\in Z(R)$ for all $x\in R$, then either R is commutative or $[\alpha(x^2)D(x),x]=0$ for all $x\in R$.

Proof. By hypothesis

$$G(x\circ x^*)-x\circ x^*\in Z(R) \quad \text{for all } x,y\in R. \tag{2.29}$$

Linearization gives

$$G(x\circ y^*)+G(y\circ x^*)-x\circ y^*-y\circ x^*\in Z(R) \quad \text{for all } x,y\in R.$$

Replacing y by y*, we get

$$G(x\circ y)+G(y^*\circ x^*)-x\circ y-y^*\circ x^*\in Z(R) \quad \text{for all } x,y\in R. \tag{2.30}$$

Next replacing y by yh, where $h\in Z(R)\cap H(R)\backslash\{0\}$ and using (2.30), we get

$$(\alpha(x\circ y)+\alpha(y^*\circ x^*))D(h)\in Z(R) \quad \text{for all } x,y\in R.$$

This implies

$$[(\alpha(x\circ y)+\alpha(y^*\circ x^*))D(h),r]=0 \quad \text{for all } r,x,y\in R,$$

i.e.

$$[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]D(h)=0 \quad \text{for all } r,x,y\in R.$$

Using primeness of R, we get either $[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]=0$ or $D(h)=0$.

Suppose that

$$[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]=0 \quad \text{for all } r,x,y\in R, \tag{2.31}$$

Replacing y by yc, where $c\in Z(R)\cap S(R)\backslash\{0\}$, we get

$$[\alpha([x,y]c)-\alpha([y^*,x^*]c),r]=0 \quad \text{for all } r,c,x,y\in R.$$

i.e.

$$[(\alpha([x,y])-\alpha([y^*,x^*]))\alpha(c),r]=0 \quad \text{for all } r,c,x,y\in R. \tag{2.32}$$

Multiplying (2.31) by $\alpha(c)$ from right, we get

$$[\alpha(x\circ y)\alpha(c)+\alpha(y^*\circ x^*)\alpha(c),r]=0 \quad \text{for all } r,c,x,y\in R. \tag{2.33}$$

Adding (2.32) and (2.33), we have

$$2[\alpha(x\circ y)\alpha(c),r]=0 \quad \text{for all } r,x,y\in R.$$

Since R is not of characteristic 2, we get

$$[\alpha(x\circ y)\alpha(c),r]=0 \quad \text{for all } c,r,x,y\in R.$$

This implies that

$$[\alpha(x\circ y),r]\alpha(c)=0 \quad \text{for all } c,r,x,y\in R.$$

Using primeness of R, we get either $\alpha(x\circ y)=0$ or $\alpha(c)=0$. By hypothesis $\alpha(c)\neq 0$ yields that $x\circ y=0$ for all $x,y\in R$. Hence R is commutative.

Now if $D(h)=0$, by Fact 1 $D(z)=0$ for all $z\in Z(R)$. Substituting ys for y in (2.30), where $s\in Z(R)\cap S(R)\backslash\{0\}$, we obtain

$$(G(x\circ y)s-G(y^*\circ x^*)s-(x\circ y)s+(y^*\circ x^*)s)\in Z(R) \quad \text{for all } x,y,s\in R. \quad (2.34)$$

Multiplying (2.30) by s from right and adding (2.34), we get

$$(G(x\circ y)-(x\circ y))s\in Z(R) \quad \text{for all } x,y,s\in R.$$

This gives that

$$(G(x\circ y)-(x\circ y))\in Z(R) \quad \text{for all } x,y\in R.$$

This implies that

$$[(G(x\circ y)-(x\circ y),r]=0 \quad \text{for all } x,y,r\in R. \quad (2.35)$$

Replacing y by yx, we obtain

$$[(G(x\circ y)-(x\circ y))x+\alpha(x\circ y)D(x),r]=0 \quad \text{for all } x,y,r\in R.$$

Substituting x instead of r and using (2.35), we get

$$[\alpha(x\circ y)D(x),r]=0 \quad \text{for all } x,y\in R.$$

Now replacing by x, we have

$$[\alpha(x^2)D(x),x]=0 \quad \text{for all } x,y\in R.$$

Theorem 2.5 Let R be a prime ring of characteristic not 2 with involution * of second kind. If R admits a generalized skew-derivation G associated with a skew-derivation D with an automorphism $\alpha:R\rightarrow R$ such that $G(x\circ x^*)\mp[x,x^*]\in Z(R)$ for all $x\in R$, then either R is commutative or $[\alpha(x^2)D(x),x]=0$ for all $x\in R$.

Proof. By hypothesis

$$G(x\circ x^*)-[x,x^*]\in Z(R) \quad \text{for all } x,y\in R. \quad (2.36)$$

Linearization gives

$$G(x\circ y^*)+G(y\circ x^*)-[x,y^*]-[y,x^*]\in Z(R) \quad \text{for all } x,y\in R.$$

Replacing y by y*, we get

$$G(x\circ y)+G(y^*\circ x^*)-[x,y]-[y^*,x^*]\in Z(R) \quad \text{for all } x,y\in R. \quad (2.37)$$

Replacing y by yh, where $h\in Z(R)\cap H(R)\setminus\{0\}$ and using (2.37), we get

$$(\alpha(x\circ y)+\alpha(y^*\circ x^*))D(h)\in Z(R).$$

This gives that

$$[(\alpha(x\circ y)+\alpha(y^*\circ x^*))D(h),r]=0 \quad \text{for all } r,x,y\in R,$$

i.e.

$$[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]D(h)=0 \quad \text{for all } r,x,y\in R.$$

Using primeness of R, we get either $[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]=0$ or $D(h)=0$.

Suppose that

$$[\alpha(x\circ y)+\alpha(y^*\circ x^*),r]=0 \qquad \text{for all } r,x,y\in R, \qquad (2.38)$$

Replacing y by yc, where $c\in Z(R)\cap S(R)\setminus\{0\}$, we get

$$[\alpha([x,y]c)-\alpha([y^*,x^*]c),r]=0 \qquad \text{for all } r,c,x,y\in R.$$

i.e.

$$[\alpha([x,y])\alpha(c)-\alpha([y^*,x^*])\alpha(c),r]=0 \qquad \text{for all } r,c,x,y\in R. \qquad (2.39)$$

Multiplying (2.38) by $\alpha(c)$ from right and adding (2.39), we get we get

$$2[\alpha(x\circ y)\alpha(c),r]=0 \qquad \text{for all } c,r,x,y\in R.$$

Since R is not of characteristic 2, we get

$$[\alpha(x\circ y),r]\alpha(c)=0 \qquad \text{for all } c,r,x,y\in R.$$

Using primeness of R, we get either $\alpha(x\circ y)=0$ or $\alpha(c)=0$. By hypothesis $\alpha(c)\neq 0$ yields that $x\circ y=0$ for all $x,y\in R$. Hence R is commutative.

On the other hand if $D(h)=0$, by Fact 1 $D(z)=0$ for all $z\in Z(R)$. Substituting ys for y in (2.37), where $s\in Z(R)\cap S(R)\setminus\{0\}$, we obtain

$$(G(x\circ y)s-G(y^*\circ x^*)s-[x,y]s+[y^*,x^*]s)\in Z(R) \qquad \text{for all } x,y,s\in R. \qquad (2.40)$$

Multiplying (2.37) by s from right and adding (2.40), we get

$$2(G(x\circ y)-[x,y])s\in Z(R) \qquad \text{for all } x,y,s\in R.$$

This gives that

$$(G(x\circ y)-[x,y])\in Z(R) \qquad \text{for all } x,y\in R, \qquad (2.41)$$

and hence

$$[G(x\circ y)-[x,y],r]=0 \qquad \text{for all } x,y,r\in R. \qquad (2.35)$$

Replacing y by yx in (2.42), we get

$$[(G(x\circ y)-[x,y])x+\alpha(x\circ y)D(x),r]=0 \qquad \text{for all } x,y,r\in R.$$

Substituting x instead of r and using (2.42), we get

$$[\alpha(x\circ y)D(x),x]=0 \qquad \text{for all } x,y\in R.$$

Now replacing y by x, we have

$$[\alpha(2x^2)D(x),x]=0 \qquad \text{for all } x,y\in R.$$

Since R is not of characteristic 2, we have

$$[\alpha(x^2)D(x),x]=0 \qquad \text{for all } x\in R.$$

The following example demonstrates that in the hypothesis of the above Theorems R to be a prime and * to be involution of second kind are essential.

Example 2.6 Let R= $\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} a,b,c \in Z, ring\ of\ integers\right\}$. Define maps G,D,α :R→R by G $\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $D\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$, α $\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$*S= $\begin{pmatrix} 0 & -b \\ 0 & a \end{pmatrix}$.

Then G is a generalized skew derivation associated with a skew derivation D and an automorphism α on R with an involution * of first kind satisfying: (i) G([x,x*])+[x,x*] ∈Z(R), (ii) G(x∘x*) ∈ Z(R), (iii) G([x,x*]) $\mp$ x∘x* ∈ Z(R), (iv) G(x∘x*) $\mp$ x∘x* ∈ Z(R), (v) G(x∘x*) $\mp$ [x,x*] ∈ Z(R) for all x ∈ R. However, R is not commutative.

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