Domination of Graph of Mobius Function

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Abstract – *The Domination theory is an important branch of Graph theory that has wide range of applications to various branches of Science and Technology. A subset D of the vertex set V of the graph G (V, E) is said to be a Dominating set if every vertex in V-D is dominated by at least one vertex of D. A dominating set D in which no two vertices of it are adjacent is called an Independent Dominating Set. In this paper, we determine the Independent dominating set, domination number and some results on domination for the graph of Mobius function for '0', $G(\mu_n^{(0)})$, which has the vertex set as the set of first n natural numbers and any two vertices a, b are adjacent if the value of Mobius function $\mu(ab) = 0$.*

Key words – *Mobius function, Mobius graph, Independent dominating set, Domination number, Relatively primes, Adjacent.*

I. INTRODUCTION

The theory of domination was introduced by C. Berge [2], and O. Ore [14] in 1962. It is very useful concept in the Graph theory and a lot of work is carrying out on the domination theory. A number of graph theorists Hedetniemi [10], Cockayne [7], Laskar [15], Arumugam and many other have contributed significant work in the domination numbers and other related topics. Cockayne and Hedetniemi [7] gave a comprehensive survey on dominating sets in 1977. In a graph G (V, E), V is the set of vertices and E is the set of edges. From [4], [8], [12], a subset D of the vertex set V is said to be dominating set if every vertex in V-D is adjacent to at least one vertex in D. The minimum cardinality of the dominating set is called the domination number. It is denoted by γ(G). A dominating set D in which no two vertices of it are adjacent is called an Independent dominating set of G. The minimum cardinality of an independent dominating set is called Independent domination number. It is denoted by γ_i(G)

Many branches of Mathematics such as Group theory, Ring theory, Number theory, Topology etc. are related to the Graph theory. Particularly, in [6] the Arithmetic Cayley Graphs are used by Madhavi, Chalapathi, which are defined on the symmetry of the group theory. Anderson, Badawi [1], Eswara Rao [9], Bharathi [3] were used a commutative ring as the vertex set of a graph. Nathanson [13] was the pioneer of use of the Number theory in graph theory. Also Cadogan [5] used the Mobius function to find the coefficients in the counting series for unlabelled connected graphs. Vasumathi [19] in 1994 defined a new graph by using the Mobius function called Mobius graph. And Srimitra[16], [17] worked on this graph of Mobius function for 0 and proved some basic results like adjacency of two vertices, necessary and sufficient condition, degree of vertices etc.

From [11], [18], a number d is said to be divides another number n, whenever n=cd for some integer c. It can be written as d/n. For the integers a, b, there is one and only one number d with (i) d>0 (ii) d/a, d/b (iii) if e/a, e/b implies e/d, then the number d is called the greatest common divisor (gcd) of a, b. And it is denoted by (a, b). The numbers a, b are said to be relatively primes if (a, b) =1. For a positive integer n, the Mobius function, $\mu(n)$ is defined as $\mu(1) = 1$ and if $n > 1$, we write, $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$, then $\mu(n) = \begin{cases} (-1)^k, & if\ \alpha_1 = \alpha_2 = \cdots = \alpha_k = 1 \\ 0, & otherwise \end{cases}$

From [16], a graph of Mobius function for 0 is a simple graph denoted by $G(\mu_n^{(0)})$ with vertex set as the set of first n natural numbers and two vertices a, b are adjacent if the Mobius function value of ab is 0 i.e., $\mu(ab) = 0$. In this paper, we mainly concentrate on the domination parameters of the graph of Mobius function for 0 such as domination number and independent dominating set.

II. MAIN THEOREMS

The following theorem says that the dominating set of $G(\mu_n^{(0)})$ is a singleton set and hence it is an Independent dominating set because, the graph $G(\mu_n^{(0)})$ is a simple graph.

Theorem 1:- In a graph $G(\mu_n^{(0)})$, if u be any vertex, then $\{u\}$ is an Independent dominating set if and only if $\mu(u) = 0$.

Proof: - Consider the graph of Mobius function for '0' with n vertices, $G(\mu_n^{(0)})$

Let us suppose that the set $D = \{u\}$ is an independent dominating set of $G\left(\mu_n^{(0)}\right)$
Then the set D dominates the set of vertices $V - D$
That is the vertex u is adjacent with all the remaining vertices in $G\left(\mu_n^{(0)}\right)$
Which implies that, let v be any other vertex, then there is an edge between the vertices $u,\ v$ so that $\mu(uv) = 0$
If $\gcd(u, v) \neq 1$, then the value of $\mu(u)$ can be any one of 0, 1, -1 but if
$\gcd(u, v) = 1$, then the value of $\mu(u)$ must be 0.
Here, u is adjacent with both type of vertices such that $\gcd(u, v) \neq 1$ and $\gcd(u, v) = 1$, then $\mu(u)$ cannot be 1 and -1.
Therefore, $\mu(u) = 0$
Conversely, Let us suppose that u be any vertex of $G\left(\mu_n^{(0)}\right)$ such that $\mu(u) = 0$
Let w be any vertex other than u of $G\left(\mu_n^{(0)}\right)$
Since, $\mu(u) = 0$, then $\mu(uw) = 0$
So, there is an edge between the vertices u, w
Therefore, the vertices u, w are adjacent in $G\left(\mu_n^{(0)}\right)$
Since, w is an arbitrary vertex, then u is adjacent to all other vertices of $G\left(\mu_n^{(0)}\right)$
That is the vertex dominates all the other vertices.
Therefore, the set $D = \{u\}$ is the dominating set and since it has only one element,
So, $\{u\}$ is the independent dominating set of $G\left(\mu_n^{(0)}\right)$
Hence, for a vertex u of $G\left(\mu_n^{(0)}\right)$, the set $\{u\}$ is an independent dominating set if and only if $\mu(u) = 0$.
■

Corollary 2:- In a graph $G\left(\mu_n^{(0)}\right)$, if $\{u\}$ is an Independent dominating set then the set $\{ku\}$ is also an Independent Dominating Set for a positive integer 'k' and $ku \leq n$.
Proof: - Let us consider the graph $G\left(\mu_n^{(0)}\right)$ with n vertices
Let $\{u\}$ be any vertex of $G\left(\mu_n^{(0)}\right)$ and $\{u\}$ is an Independent dominating set
Then by Theorem -1, $\mu(u) = 0$
For a positive integer k so that $ku \leq n$,
Since, $\mu(u) = 0$, then $\mu(ku) = 0$
Also since, $ku \leq n$, which implies that ku is also a vertex of the graph $G\left(\mu_n^{(0)}\right)$ and $\mu(ku) = 0$
Again by the Theorem -1, the set $\{ku\}$ is also an Independent Dominating Set.
Hence, if $\{u\}$ is an Independent Dominating Set, then $\{ku\}$ is also an Independent Dominating Set of $G\left(\mu_n^{(0)}\right)$ for a positive integer k and $ku \leq n$. ■

The graph $G\left(\mu_{10}^{(0)}\right)$ has the Independent dominating sets $\{4\}$, $\{9\}$ and then $\{2.4\} = \{8\}$ is also an independent dominating set, which is shown in Figure 1.

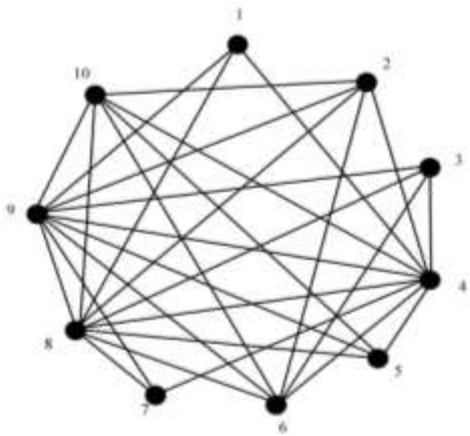

Fig. 1: The graph of Mobius function for 0 with 10 vertices, $G\left(\mu_{10}^{(0)}\right)$

Now we calculate the domination number for $G\left(\mu_n^{(0)}\right)$ for every positive integer n, which is $'n'$ for $n \leq 3$ and is 1 for $n > 3$

Theorem 3:- The domination number of $G(\mu_n^{(0)})$ is $\gamma\left(G(\mu_n^{(0)})\right) = \begin{cases} n, & if\ n \leq 3 \\ 1, & if\ n > 3 \end{cases}$

Proof:- Let $G(\mu_n^{(0)})$ be a graph of Mobius function for 0.

Case 1:- Let $n \leq 3$

Let u, v be any two vertices of $G(\mu_n^{(0)}), n \leq 3$
Then, $u, v \leq 3$
For $2 \leq u, v \leq 3$, then $\mu(u) = \mu(v) = -1 \neq 0$
And for $u = 1$ or $v = 1$, then $\mu(u) = \mu(v) = 1$
then, $\mu(uv) = 1$ or $\mu(uv) = -1$
So, $\mu(uv) \neq 0$
Therefore, there is no edge between u and v
That is, there is no edge in the graph $G(\mu_n^{(0)}), n \leq 3$
The graphs $G(\mu_n^{(0)}), n \leq 3$ are null graphs
Therefore, all the vertices of the graphs $G(\mu_n^{(0)}), n \leq 3$ are isolated vertices.
Then, the dominating set of these graphs is the set of all vertices.
That is, the domination number is the number of vertices in the graph.
Hence, $\gamma\left(G(\mu_n^{(0)})\right) = n$ for $n \leq 3$

Case 2:- Let $n > 3$

Let u be any vertex of $G(\mu_n^{(0)}), n > 3$ such that $\mu(u) = 0$
Let v be any vertex other than u of $G(\mu_n^{(0)}), n > 3$
Since $\mu(u) = 0$, then $\mu(uv) = 0$
So, there is an edge between u and v.
Then, u, v are adjacent in $G(\mu_n^{(0)}), n > 3$
Since, v is an arbitrary vertex,
then, u is adjacent with all other vertices of $G(\mu_n^{(0)}), n > 3$
i.e., the vertex u dominates all other vertices.
Therefore, the dominating set, D=$\{u\}$ and
The domination number, $\gamma\left(G(\mu_n^{(0)})\right) = 1$ for $n > 3$

From the above two cases, the domination number of $G(\mu_n^{(0)})$ is given by

$$\gamma\left(G(\mu_n^{(0)})\right) = \begin{cases} n, & if\ n \leq 3 \\ 1, & if\ n > 3 \end{cases} \quad \blacksquare$$

In Figure 2 shown below, the graph $G(\mu_6^{(0)})$ has the dominating set D_1= {4}, thus the domination number $\gamma\left(G(\mu_6^{(0)})\right) = 1$. And in the Figure3, the graph $G(\mu_3^{(0)})$ has the dominating set D_2 = {1, 2, 3}, thus the domination number $\gamma\left(G(\mu_6^{(0)})\right) = 3$.

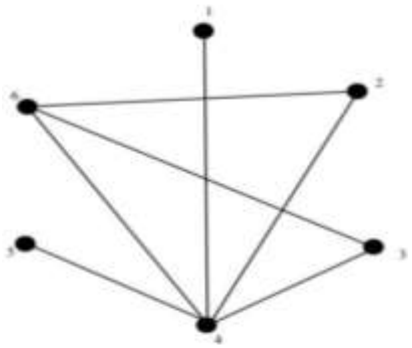

Fig. 2: The graph of Mobius function for 0 with 6 vertices, $G(\mu_6^{(0)})$

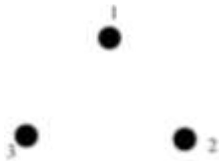

Fig. 3: The graph of Mobius function for 0 with 3 vertices, $G(\mu_3^{(0)})$

Now we find the condition for Independent Dominating set based on the adjacency of two vertices.

Theorem 4:- If k, l are relatively primes, $1 \le k, l \le n$ and if they are adjacent in $G(\mu_n^{(0)})$, then either $\{k\}$ or $\{l\}$ is an Independent dominating set.

Proof: - Let $G(\mu_n^{(0)})$ be a graph.
Let k, l are relatively prime and $1 \le k, l \le n$
Then, $\gcd(k, l) = 1$
Also k, l are adjacent in $G(\mu_n^{(0)})$
Implies, $\mu(kl) = 0$
Implies, $kl = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ and at least one α_i is greater than 1, $1 \le i \le t$, where p_i's are primes.
Let $\alpha_j > 1$ for some $j, j \le t$ such that $p_j^{\alpha_j}$ is a factor of kl.
Since, $\gcd(k, l) = 1$
Assume, $k = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ and
$l = q_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$ where $q_i^{\beta_i}$'s are coincide with $p_{r+1}^{\alpha_{r+1}} p_{r+2}^{\alpha_{r+2}} \ldots p_t^{\alpha_t}$ in some order and $r + s = t$
Implies, $p_j^{\alpha_j}$ is a factor of either k or l, where $\alpha_j > 1$
If $p_j^{\alpha_j}$ is a factor of k, where $\alpha_j > 1$
Implies, $\mu(k) = 0$
i.e., $\{k\}$ is an Independent dominating set.
Also, if $p_j^{\alpha_j}$ is a factor of l, where $\alpha_j > 1$
Implies, $\mu(l) = 0$
i.e., $\{l\}$ is an Independent dominating set.
Hence, the relatively primes $k, l, 1 \le k, l \le n$ are adjacent in $G(\mu_n^{(0)})$, then either $\{k\}$ or $\{l\}$ is an Independent dominating set. ■ .

Corollary 5:- If $k < k + 1 \le n$ and $k, k + 1$ are adjacent in $G(\mu_n^{(0)})$ then either $\{k\}$ or $\{k + 1\}$ or both are Independent dominating sets.

Proof: - Let $G(\mu_n^{(0)})$ be a graph with n vertices.

Case 1: - Let $k, k + 1$ are adjacent in $G(\mu_n^{(0)})$ where, $k < k + 1 \le n$
Implies, $k, k + 1$ are two consecutive positive integers and $\gcd(k, k + 1) = 1$
i.e., $k, k + 1$ are relatively prime and they are adjacent in $G(\mu_n^{(0)})$
Then, by the Theorem-4, we have, either $\{k\}$ or $\{k + 1\}$ is an Independent dominating set.
Here in this case, if $\mu(k) \ne 0$, then $\mu(k + 1) = 0$ else if $\mu(k + 1) \ne 0$, then $\mu(k) = 0$.

Case 2: - Let $k, k + 1$ are adjacent in $G(\mu_n^{(0)})$ such that $\mu(k) = \mu(k + 1) = 0$, where $k + 1 \le n$
Since $\mu(k) = \mu(k + 1) = 0$, then by theorem-1, both {k} and {k+1} are Independent Dominating sets.

From the above two cases, we can say that, if the vertices k, k+1 are adjacent in $G(\mu_n^{(0)})$, then either {k} or {k+1} or both are Independent Dominating sets. ■

In the graph $G(\mu_{12}^{(0)})$, the two vertices 5, 12 are relatively primes i.e., gcd(5, 12)=1 and they are adjacent, then clearly {12} is an Independent Dominating set. Also, k=3, k+1=4 are adjacent in $G(\mu_{12}^{(0)})$ then {k+1=4} is an Independent Dominating set. It is shown in Figure 4. Also we can observe from this graph that the vertices k=8, k+1=9 are adjacent and both {k=8}, {k+1=9} are Independent Dominating sets.

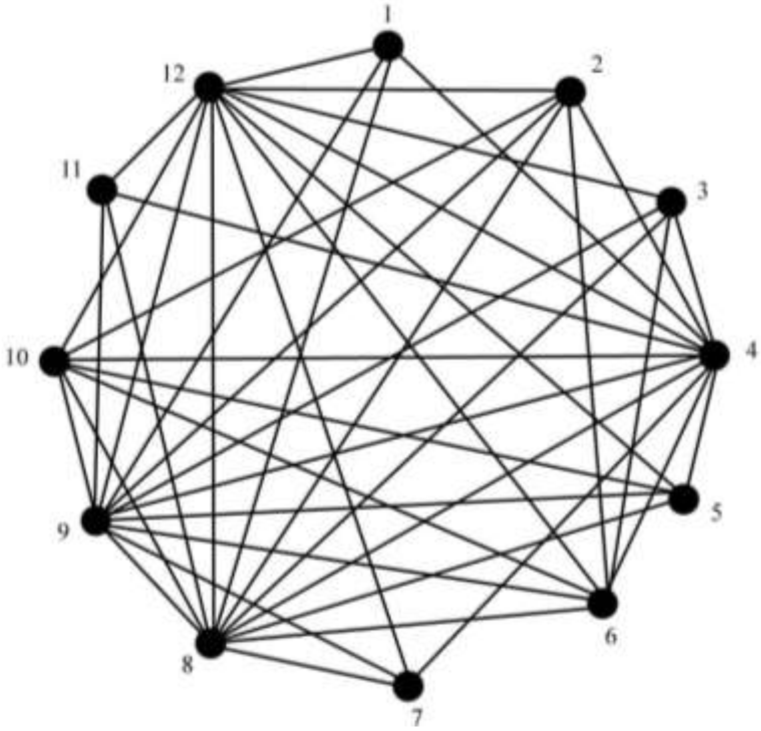

Fig. 4: The graph of Mobius function for 0 with 12 vertices, $G\left(\mu_{12}^{(0)}\right)$

III. CONCLUSION

In this paper, we used the graph of Mobius function for 0, $G\left(\mu_n^{(0)}\right)$. For this graph, here first we concluded that the Independent Dominating set is a singleton set, the element of this set has the Mobius function value 0 and hence each multiple of that element forms an Independent Dominating set. Next, we calculated the domination number of this graph $G\left(\mu_n^{(0)}\right)$ for every positive integer n and it is observed that the domination number of this graph is n for $n \leq 3$ and is 1 for n>3. Finally established a condition that if any two vertices, which are relatively prime, are adjacent, then, one of them forms an Independent Dominating set. Hence if the consecutive two vertices are adjacent, then any one or both of them form Independent Dominating sets.

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