Common Fixed Point Theorem for Mappings in Quasi Metric Space

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Abstract: *In this paper we study common fixed point theorem in complete quasi metric space for mappings satisfying certain rational inequality.*

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1. Introduction

In this paper new fixed point theorem in complete quasi metric space has been proved for mappings satisfying some new type of rational contraction conditions.

Definition 1.1. Let X be a non-empty set and let $d : X \times X \to R$ be a function satisfying the conditions

(d_1) $d(x, y) \geq 0$

(d_2) $d(x, y) = 0 \Leftrightarrow x = y$

(d_3) $d(x, y) = d(y, x)$

(d_4) $d(x, y) \leq d(x, z) + d(z, y)$

$\forall x, y, z \in X$

If *d* satisfies the above conditions then it is called a metric on *X* and (*X*, *d*) is called a metric space.

Definition 1.2. Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the conditions

(d_1) $x = y \Leftrightarrow d(x, y) = 0 \ \forall x, y \in X$

(d_2) $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in X$

If *d* satisfies the above conditions then it is called quasi metric on *X* and (*X*, *d*) is called a quasi-metric space.

Definition 1.3. A sequence $\{x_n\}$ in metric space (*X*, *d*) is called Cauchy sequence if for each $\varepsilon > 0$ there exist a positive integer n_0 such that $m, n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$.

Definition 1.4. A complete metric space is a metric space in which every Cauchy sequence is convergent.

Definition 1.5. A sequence $\{x_n\}$ converges to *z* if $\lim_{n \to \infty} d(x_n, z) = 0$. In this case *z* is called the limit of $\{x_n\}$ and we write $x_n \to z$.

2. Main Results

Theorem 2.1. Let S and T be two continuous self mappings defined on a complete quasi metric space (X, d) satisfying

$$d(Sx,Ty)+\alpha\frac{[d(Sx,Ty)+d(x,y)]}{[1+d(x,Ty)d(Sx,x)]}+\beta\frac{[1+d(x,STy)+d(Sx,TSx)]d(Sx,x)}{1+d(x,TSx)} \leq \gamma\frac{[d(STy,y)d(Ty,y)]}{[d(x,y)+d(x,Ty)+d(Ty,y)]}+\delta d(x,y) \quad \ldots(1)$$

for all x, y in X, where $\gamma+\delta<1+2\alpha+\beta$, $1+\alpha+\beta<\frac{\gamma}{2}, 1+2\alpha>\delta, \alpha>0, \beta>0, \gamma>0, \delta>0$. Then S and T have a unique common fixed point.

Proof. Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X such that $Sx_n = x_{n-1}$ and $Tx_{n+1} = x_n$ also $TSx_n = x_{n-2}$ and $STx_{n+1} = x_{n-1}$ for $n = 1, 2, 3, \ldots$. Then by (1)

$$d(Sx_n,Tx_{n+1})+\alpha\frac{[d(Sx_n,Tx_{n+1})+d(x_n,x_{n+1})]}{[1+d(x_n,Tx_{n+1})d(Sx_n,x_n)]}+\beta\frac{[1+d(x_n,STx_{n+1})+d(Sx_n,TSx_n)]d(Sx_n,x_n)}{1+d(x_n,TSx_n)} \leq \gamma\frac{[d(STx_{n+1},x_{n+1})d(Tx_{n+1},x_{n+1})]}{[d(x_n,x_{n+1})+d(x_n,Tx_{n+1})+d(Tx_{n+1},x_{n+1})]}+\delta d(x_n,x_{n+1})$$

or

$$d(x_{n-1},x_n)+\alpha\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{[1+d(x_n,x_n)d(x_{n-1},x_n)]}+\beta\frac{[1+d(x_n,x_{n-1})+d(x_{n-1},x_{n-2})]d(x_{n-1},x_n)}{1+d(x_n,x_{n-2})} \leq \gamma\frac{[d(x_{n-1},x_{n+1})d(x_n,x_{n+1})]}{[d(x_n,x_{n+1})+d(x_n,x_n)+d(x_n,x_{n+1})]}+\delta d(x_n,x_{n+1})$$

or

$$d(x_{n-1},x_n)+\alpha[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+\beta\frac{[1+d(x_n,x_{n-2})]d(x_{n-1},x_n)}{1+d(x_n,x_{n-2})} \leq \gamma\frac{[d(x_{n-1},x_{n+1})d(x_n,x_{n+1})]}{[2d(x_n,x_{n+1})]}+\delta d(x_n,x_{n+1})$$

or

$$d(x_{n-1}, x_n) + \alpha\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right] + \beta d(x_{n-1}, x_n) \le \frac{\gamma}{2} d(x_{n-1}, x_{n+1}) + \delta d(x_n, x_{n+1})$$

or

$$d(x_{n-1}, x_n) + \alpha\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right] + \beta d(x_{n-1}, x_n) \le \frac{\gamma}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right] + \delta d(x_n, x_{n+1})$$

or $$d(x_n, x_{n+1})(\alpha - \frac{\gamma}{2} - \delta) \le (\frac{\gamma}{2} - \alpha - \beta - 1) d(x_{n-1}, x_n)$$

or $$d(x_n, x_{n+1}) \le \frac{\left(\frac{\gamma}{2} - \alpha - \beta - 1\right)}{\left(\alpha - \frac{\gamma}{2} - \delta\right)} d(x_{n-1}, x_n)$$

or $d(x_n, x_{n+1}) \le h d(x_{n-1}, x_n)$

Where $h = \frac{\left(\frac{\gamma}{2} - \alpha - \beta - 1\right)}{\left(\alpha - \frac{\gamma}{2} - \delta\right)}$ and $0 < h < 1$ in view of $\gamma + \delta < 1 + 2\alpha + \beta, \quad 1 + \alpha + \beta < \frac{\gamma}{2}$, $\alpha > 0, \beta > 0, \gamma > 0, \delta > 0$.

Similarly $d(x_{n-1}, x_n) \le h d(x_{n-2}, x_{n-1})$ and so on.

Hence $d(x_n, x_{n+1}) \le h^n d(x_0, x_1) \to 0$ as $n \to \infty$ since $0 < h < 1$.

This proves that $\{x_n\}$ is a Cauchy's sequence in X which is Complete, so it converges to a point say *u* in X.

Since *S* is continuous we have $Su = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} x_{n-1} = u$. Thus *u* is a fixed point of *S*. Similarly as *T* is continuous we can show *Tu* = *u*. Thus *u* is the common fixed point of *S* and *T*.

Suppose *v* is another fixed point of S and T. Then we have

$$d(Su,Tv)+\alpha\frac{[d(Su,Tv)+d(u,v)]}{[1+d(u,Tv)d(Su,u)]}+\beta\frac{[1+d(u,STv)+d(Su,TSu)]d(Su,u)}{1+d(u,TSu)}$$
$$\leq\gamma\frac{[d(STv,v)d(Tv,v)]}{[d(u,v)+d(u,Tv)+d(Tv,v)]}+\delta d(u,v)$$

or $$d(u,v)+\alpha\frac{[d(u,v)+d(u,v)]}{[1+d(u,v)d(u,u)]}+\beta\frac{[1+d(u,v)+d(u,u)]d(u,u)}{1+d(u,u)}$$
$$\leq\gamma\frac{[d(v,v)d(v,v)]}{[d(u,v)+d(u,v)+d(v,v)]}+\delta d(u,v)$$

or $d(u,v)+\alpha[d(u,v)+d(u,v)]\leq\delta d(u,v)$

or $d(u,v)(1+2\alpha-\delta)\leq 0$

$\Rightarrow d(u,v)=0$ since $1+2\alpha>\delta$

$\Rightarrow u=v$

This completes the proof.

Theorem 2.2. Let **.** Let $\{T_k\}$ be a sequence of continuous self mappings defined on a complete quasi metric space (X, d) satisfying

$$d(T_ix,T_jy)+\alpha\frac{[d(T_ix,T_jy)+d(x,y)]}{[1+d(x,T_jy)d(T_ix,x)]}+\beta\frac{[1+d(x,T_iT_jy)+d(T_ix,T_jT_ix)]d(T_ix,x)}{1+d(x,T_jT_ix)}$$
$$\leq\gamma\frac{[d(T_iT_jy,y)d(T_jy,y)]}{[d(x,y)+d(x,T_jy)+d(T_jy,y)]}+\delta d(x,y)\quad \ldots(2)$$

for all x, y in X, where $\gamma+\delta<1+2\alpha+\beta,\ \ 1+\alpha+\beta<\frac{\gamma}{2},1+2\alpha>\delta,\alpha>0,\ \beta>0,\gamma>0,\delta>0$. Then $\{T_k\}$ have a unique common fixed point.

Proof. Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X such that $T_ix_n=x_{n-1}$ and $T_jx_{n+1}=x_n$ also $T_jT_ix_n=x_{n-2}$ and $T_iT_jx_{n+1}=x_{n-1}$ for n = 1, 2, 3, … . Then by (2)

$$d(T_ix_n,T_jx_{n+1})+\alpha\frac{[d(T_ix_n,T_jx_{n+1})+d(x_n,x_{n+1})]}{[1+d(x_n,T_jx_{n+1})d(T_ix_n,x_n)]}+\beta\frac{[1+d(x_n,T_iT_jx_{n+1})+d(T_ix_n,T_jT_ix_n)]d(T_ix_n,x_n)}{1+d(x_n,T_jT_ix_n)}$$
$$\leq\gamma\frac{[d(T_iT_jx_{n+1},x_{n+1})d(T_jx_{n+1},x_{n+1})]}{[d(x_n,x_{n+1})+d(x_n,T_jx_{n+1})+d(T_jx_{n+1},x_{n+1})]}+\delta d(x_n,x_{n+1})$$

or

$$d(x_{n-1}, x_n) + \alpha \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{[1 + d(x_n, x_n) d(x_{n-1}, x_n)]} + \beta \frac{[1 + d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2})] d(x_{n-1}, x_n)}{1 + d(x_n, x_{n-2})}$$

$$\leq \gamma \frac{[d(x_{n-1}, x_{n+1}) d(x_n, x_{n+1})]}{[d(x_n, x_{n+1}) + d(x_n, x_n) + d(x_n, x_{n+1})]} + \delta d(x_n, x_{n+1})$$

or

$$d(x_{n-1}, x_n) + \alpha [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \beta \frac{[1 + d(x_n, x_{n-2})] d(x_{n-1}, x_n)}{1 + d(x_n, x_{n-2})}$$

$$\leq \gamma \frac{[d(x_{n-1}, x_{n+1}) d(x_n, x_{n+1})]}{[2 d(x_n, x_{n+1})]} + \delta d(x_n, x_{n+1})$$

or

$$d(x_{n-1}, x_n) + \alpha [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \beta d(x_{n-1}, x_n) \leq \frac{\gamma}{2} d(x_{n-1}, x_{n+1}) + \delta d(x_n, x_{n+1})$$

or

$$d(x_{n-1}, x_n) + \alpha [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \beta d(x_{n-1}, x_n) \leq \frac{\gamma}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \delta d(x_n, x_{n+1})$$

or $$d(x_n, x_{n+1})(\alpha - \frac{\gamma}{2} - \delta) \leq (\frac{\gamma}{2} - \alpha - \beta - 1) d(x_{n-1}, x_n)$$

or $$d(x_n, x_{n+1}) \leq \frac{\left(\frac{\gamma}{2} - \alpha - \beta - 1\right)}{\left(\alpha - \frac{\gamma}{2} - \delta\right)} d(x_{n-1}, x_n)$$

or $d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n)$

Where $h = \dfrac{\left(\frac{\gamma}{2} - \alpha - \beta - 1\right)}{\left(\alpha - \frac{\gamma}{2} - \delta\right)}$ and $0 < h < 1$ in view of $\gamma + \delta < 1 + 2\alpha + \beta, \quad 1 + \alpha + \beta < \frac{\gamma}{2}$,

$\alpha > 0, \beta > 0, \gamma > 0, \delta > 0$.

Similarly $d(x_{n-1}, x_n) \le h d(x_{n-2}, x_{n-1})$ and so on.

Hence $d(x_n, x_{n+1}) \le h^n d(x_0, x_1) \to 0$ as $n \to \infty$ since $0 < h < 1$.

This proves that $\{x_n\}$ is a Cauchy's sequence in X which is Complete, so it converges to a point say *u* in X.

Since T_i is continuous we have $T_i u = \lim_{n\to\infty} T_i x_n = \lim_{n\to\infty} x_{n-1} = u$. Thus *u* is a fixed point of T_i . Similarly as T_j is continuous we can show $T_j u = u$. Thus *u* is the common fixed point of T_i and T_j .

Suppose *v* is another fixed point of T_i and T_j . Then we have

$$d(T_i u, T_j v) + \alpha \frac{\left[d(T_i u, T_j v) + d(u, v)\right]}{[1 + d(u, T_j v) d(T_i u, u)]} + \beta \frac{\left[1 + d(u, T_i T_j v) + d(T_i u, T_j T_i u)\right] d(T_i u, u)}{1 + d(u, T_j T_i u)}$$

$$\le \gamma \frac{\left[d(T_i T_j v, v) d(T_j v, v)\right]}{\left[d(u, v) + d(u, T_j v) + d(T_j v, v)\right]} + \delta d(u, v)$$

or $\quad d(u, v) + \alpha \dfrac{\left[d(u, v) + d(u, v)\right]}{[1 + d(u, v) d(u, u)]} + \beta \dfrac{\left[1 + d(u, v) + d(u, u)\right] d(u, u)}{1 + d(u, u)}$

$$\le \gamma \frac{\left[d(v, v) d(v, v)\right]}{\left[d(u, v) + d(u, v) + d(v, v)\right]} + \delta d(u, v)$$

or $\quad d(u, v) + \alpha\left[d(u, v) + d(u, v)\right] \le \delta d(u, v)$

or $\quad d(u, v)(1 + 2\alpha - \delta) \le 0$

$\Rightarrow d(u, v) = 0 \quad$ since $1 + 2\alpha > \delta$

$\Rightarrow u = v$

This completes the proof.

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