

Fixed Point Theorems for $(\psi - \varphi - \lambda)$ Contractions in S_b – Metric Spaces

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Abstract:

In this paper we introduce $(\psi - \varphi - \lambda)$ contraction in an S_b – metric space and prove fixed point theorems for $(\psi - \varphi - \lambda)$ contraction in S_b – metric spaces. We also obtain an application of our result. Further an example in support of our result is given. Our results improve those of G.N.V Kishore et al.

Keywords: S_b – metric space, fixed point, $(\psi - \varphi)$ contraction, $(\psi - \varphi - \lambda)$ contraction

MSC: 54H25, 47H10.

1. Introduction

In 1992, polish mathematician Stephen Banach established the most remarkable fixed point theorem which is popularly known as Banach contraction principle, and widely applied in different areas of mathematics and applications.

Banach contraction principle is considered to be the initial result of the study of fixed point theory in metric spaces. Several mathematicians around the world introduced different generalizations of metric spaces [11,20,21]. In 2006, Z. Mustafa and B. Sims [18] introduced the concept of G-metric space and proved some fixed point theorems in G-metric space. Several researchers worked on G-Metric Space and proved some fixed point theorems in G-Metric Space [5,6,10,19]. B.D. Dhage [3] introduced the notion D-metric space. Many mathematicians worked on D-metric space [12,13]. In 2007 S. Sedghi, N. Shobe and H. Zhan [12] introduced D^* -metrc space, many researchers proved fixed point theorems in D^* -metrc spaces [18] and also in G_b Metric Spaces [1,2].

In 2012, S. Sedghi et al. [14] introduced the notion of S-metric space which is a generalization of G-metric space and D^* -metrc space. Several researchers proved some fixed point results in S-metric spaces [7,8,9,15,16]. Recently, Sedghi.S et al. [17] defined S_b -Metric space using S-metric space [17] and proved some fixed point theorems.

In this paper we introduce $(\psi - \varphi - \lambda)$ contraction in S_b -Metric Space and prove fixed point theorems for $(\psi - \varphi - \lambda)$ contractions in S_b -Metric Spaces. We obtain results of G.N.V. Kishore et al. [4] as corollaries.

2. Preliminaries

2.1 Definition (S.Sedghi et al. [14])

Let X be a non-empty set. An S-Metric on X is a function $S : X^3 \rightarrow [0, \infty)$ that satisfies the following conditions for each $x, y, z, a \in X$

(2.1.1) $S(x, y, z) = 0$ if and only if $x = y = z$

(2.1.2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The function S is called S-Metric on X and the pair (X, S) is called an S-Metric Space.

2.2 Observations : (S. Sedghi et al. [14])

Let (X, S) be an S-Metric space. Then

(2.2.1) $S(x, x, y) = S(y, y, x)$

(2.2.2) $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$ and

(2.2.3) $S(x, y, y) \leq S(x, x, y)$.

2.3 Definition (S.Sedghi et al. [17])

Let X be a non-empty set and $b \geq 1$ be a real number. Suppose that a mapping $S_b : X^3 \rightarrow [0, \infty)$ satisfies the following properties:

(2.3.1) $0 < S_b(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$

(2.3.2) $S_b(x, y, z) = 0$ if and only if $x = y = z$

(2.3.3) $S_b(x, y, z) \leq b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$ for all $x, y, z, a \in X$.

Then the function S_b is called an S_b -Metric on X and the pair (X, S_b) is called an S_b -Metric Space.

2.4 Remark [17]

It should be noted that the class of S_b -Metric Spaces is effectively larger than that of S-Metric Spaces. Indeed each S-Metric Space is an S_b -Metric Space with $b = 1$.

2.5 Examples (S.Sedghi et al. [17])

Example 2.5.1:

Let (X, S) be an S-Metric Space and $S_*(x, y, z) = S(x, y, z)^p$ where $p > 1$ is a real number. Note that S_* is an S_b -Metric with $b = 2^{2(p-1)}$. Also (X, S_*) is not necessarily an S-Metric Space.

Example 2.5.2:

Let $X = [0, 1]$. Define $S : X \times X \times X \rightarrow R^+$ by $S_b(x, y, z) = (|y + z - 2x| + |y - z|)^2$, then (X, S_b) is a S_b -Metric space $b = 2$.

Solution: we show that $S_b(x, y, z) = (|y + z - 2x| + |y - z|)^2$ is a S_b -Metric space.

$$\begin{aligned} \text{(i)} \quad S_b(x, y, z) &= 0 \text{ if and only if } x = y = z \\ \text{If } S_b(x, y, z) = 0 &\Leftrightarrow (|y + z - 2x| + |y - z|)^2 = 0 \\ &\Leftrightarrow |y + z - 2x| = |y - z| = 0 \\ &\Leftrightarrow x = y = z \end{aligned}$$

If $x = y = z$ then $S_b(x, y, z) = S_b(x, x, x) = 0$

(ii) Clearly $S_b(x, y, z) = 0$ if $x \neq y \neq z \neq x$

(iii) $S_b(x, y, z) \leq b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$

$$\begin{aligned} \text{R.H.S} &= 2((|x + a - 2x| + |x - a|)^2 + (|y + a - 2y| + |y - a|)^2 + (|z + a - 2z| + |z - a|)^2) \\ &= 2((2|x - a|)^2 + (2|y - a|)^2 + (2|z - a|)^2) \\ &= (2)(4)((|x - a|)^2 + (|y - a|)^2 + (|z - a|)^2) \\ &= 8((|x - a|)^2 + (|y - a|)^2 + (|z - a|)^2) \end{aligned}$$

$$\begin{aligned} \text{L. H. S} &= S_b(x, y, z) = (|y + z - 2x| + |y - z|)^2 \\ &= (|(y - a) + (z - a) - 2(x - a)| + |(y - a) - (z - a)|)^2 \\ &= (2|y - a| - 2|x - a|)^2 \\ &= 4(|y - a| - |x - a|)^2 \\ &\leq \text{R.H.S} \end{aligned}$$

$\therefore S_b(x, y, z)$ is a S_b -Metric space with $b = 2$

Note b cannot be less than 2

Take $x = 0, y = \frac{1}{2}, z = 1, a = \frac{1}{2}$

$$\text{L.H.S} = S_b(x, y, z) = (|y + z - 2x| + |y - z|)^2$$

$$\begin{aligned} &= \left(\left| \left(\frac{1}{2} \right) + 1 - (2)(0) \right| + \left| \left(\frac{1}{2} \right) - 1 \right| \right)^2 \\ &= \left| \left(\frac{3}{2} \right) + \left(\frac{1}{2} \right) \right|^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{R.H.S} &= b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)) \\ &= b((4|x - a|)^2 + (4|y - a|)^2 + (4|z - a|)^2) \\ &= (b)(4) \left(\left| 1 - \left(\frac{1}{2} \right) \right|^2 + 0^2 + \left| 1 - \left(\frac{1}{2} \right) \right|^2 \right) \\ &= (b)(4) \left(\left(\frac{1}{4} \right) + \left(\frac{1}{4} \right) \right) \\ &= (b)(4) \left(\frac{1}{2} \right) \\ &= 2b \end{aligned}$$

$$\therefore 4 \leq 2b$$

$$\therefore b \geq 2$$

2.6 Lemma (S.Sedghi et al. [17])

In an S_b -Metric Space, we have $S_b(x, x, y) \leq bS_b(y, y, x)$ and $S_b(y, y, x) \leq bS_b(x, x, y)$.

2.7 Lemma (S.Sedghi et al. [17])

In an S_b -Metric Space, we have

$$S_b(x, x, z) \leq 2bS_b(x, x, y) + b^2S_b(y, y, z).$$

2.8 Definition (S.Sedghi et al. [17])

If (X, S_b) is an S_b -Metric Space, a sequence $\{x_n\}$ in X is said to be S_b - Cauchy sequence, if for each $\epsilon > 0$ there exists $n_0 \in N$ such that $S_b(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$.

2.9 Definition (S.Sedghi et al. [17])

A sequence $\{x_n\}$ is said to be S_b -convergent to a point x if for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n, x_n, x) < \epsilon$ or $S_b(x, x, x_n) < \epsilon$ for all $n \geq n_0$ and denote $\lim_{n \rightarrow \infty} x_n = x$.

2.10 Definition (S.Sedghi et al. [17])

An S_b -Metric Space (X, S_b) is called complete if every S_b - Cauchy sequence is S_b -convergent in X .

2.11 Lemma (S.Sedghi et al. [17])

If (X, S_b) is an S_b -Metric Space with $b \geq 1$ and suppose that $\{x_n\}$ is S_b -convergent to x , then we have

$$\begin{aligned} (i) \quad \frac{1}{2b}S_b(y, x, x) &\leq \liminf_{n \rightarrow \infty} S_b(y, y, x_n) \\ &\leq \limsup_{n \rightarrow \infty} S_b(y, y, x_n) \\ &\leq 2bS_b(y, y, x) \end{aligned}$$

$$\text{and (ii)} \quad \begin{aligned} \frac{1}{b^2}S_b(x, x, y) &\leq \liminf_{n \rightarrow \infty} S_b(x_n, x_n, y) \\ &\leq \limsup_{n \rightarrow \infty} S_b(x_n, x_n, y) \\ &\leq b^2S_b(y, y, x) \end{aligned}$$

for all $y \in X$.

In particular if $x = y$, then we have $\lim_{n \rightarrow \infty} S_b(x_n, x_n, y) = 0$

Notation (K.P.R.Sastry, K.K.M.Sarma, P.Krishna Kumari [7])

Let $\Psi = \{\psi | \psi: [0, \infty) \rightarrow [0, \infty) \text{ where } \psi \text{ is continuous, increasing, } \psi(t) = 0 \text{ if } t = 0 \text{ and } \psi(t) < t \text{ if } t > 0\}$.

Theorem (K.P.R.Sastry, K.K.M.Sarma, P.Krishna Kumari [7])

Let (X, S) be a complete S-Metric space and $\psi \in \Psi$. Suppose $T: X \rightarrow X$ is a ψ – contraction. That is $S(Tx, Ty, Tz) \leq \psi(\max\{S(x, y, z), S(x, Tx, Tx), S(y, Ty, Ty), S(z, Tz, Tz)\}) \forall x, y, z \in X$. Then T has a unique fixed point.

3. Main Results

First we introduce the notion of $(\psi - \varphi - \lambda)$ contraction in S_b -Metric Spaces.

3.1 Definition

Let (X, S_b) be a S_b -Metric Space, $\psi, \varphi \in \Psi$ and $0 < \lambda \leq 1/2$. Let $f: X \rightarrow X$ be a mapping. We say that f is a $(\psi - \varphi - \lambda)$ contraction if

$$\begin{aligned} \psi(3b^3S_b(fx, fx, fy)) &\leq \frac{\psi\left(\max\left\{S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), S_b(x, x, fy), S_b(y, y, fx)\right\}\right)}{\varphi\left(\max\left\{S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \lambda(S_b(x, x, fy) + S_b(y, y, fx))\right\}\right)} - \end{aligned}$$

for all $x, y \in X$. (3.1.1)

Now we state and prove our main result.

3.2 Theorem: Let (X, S_b) be a S_b -Metric Space, $\psi, \varphi \in \Psi$. Let $f: X \rightarrow X$ be a mapping and $0 < \lambda \leq 1/2$. Suppose f is a $(\psi - \varphi - \lambda)$ contraction (i.e,

$$\begin{aligned} \psi(3b^3S_b(fx, fx, fy)) &\leq \frac{\psi\left(\max\left\{S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), S_b(x, x, fy), S_b(y, y, fx)\right\}\right)}{\varphi\left(\max\left\{S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \lambda(S_b(x, x, fy) + S_b(y, y, fx))\right\}\right)} - \end{aligned}$$

for all $x, y \in X$). Then f has a unique fixed point.

Proof: Let $x_0 \in X$. Define $x_{n+1} = fx_n$ for $n = 0, 1, 2, \dots$

Case(i) If $x_n = x_{n+1}$ for some n , then x_n is a fixed point of f .

Case(ii) Suppose $x_n \neq x_{n+1}$ for all n

Let $a_n = S_b(x_n, x_n, x_{n+1})$ for $n = 0, 1, 2, \dots$

we observe that $a_{n+1} < \left(\frac{2}{b(3b-1)}\right) a_n$ for $n = 0, 1, 2, \dots$

Now $\psi(3b^3 a_{n+1}) = \psi(3b^3 S_b(x_{n+1}, x_{n+1}, x_{n+2}))$

$$\begin{aligned} &= \psi(3b^3 S_b(fx_n, fx_n, fx_{n+1})) \\ &\leq \varphi \left(\max \left\{ \begin{array}{l} S_b(x_n, x_n, x_{n+1}), S_b(x_n, x_n, fx_n), S_b(x_{n+1}, x_{n+1}, fx_{n+1}), \\ S_b(x_n, x_n, fx_{n+1}), S_b(x_{n+1}, x_{n+1}, fx_n) \end{array} \right\} \right) - \\ &\leq \varphi \left(\max \left\{ \begin{array}{l} S_b(x_n, x_n, x_{n+1}), \lambda(S_b(x_n, x_n, fx_n) + S_b(x_{n+1}, x_{n+1}, fx_{n+1})), \\ \lambda(S_b(x_n, x_n, fx_{n+1}) + S_b(x_{n+1}, x_{n+1}, fx_n)) \end{array} \right\} \right) \\ &\leq \varphi \left(\max \left\{ \begin{array}{l} S_b(x_n, x_n, x_{n+1}), S_b(x_n, x_n, x_{n+2}), S_b(x_{n+1}, x_{n+1}, x_{n+2}), \\ S_b(x_n, x_n, x_{n+2}), S_b(x_{n+1}, x_{n+1}, x_{n+1}) \end{array} \right\} \right) - \\ &\leq \varphi \left(\max \left\{ \begin{array}{l} S_b(x_n, x_n, x_{n+1}), \lambda(S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_{n+2})), \\ \lambda(S_b(x_n, x_n, x_{n+2}) + S_b(x_{n+1}, x_{n+1}, x_{n+1})) \end{array} \right\} \right) \\ &= \varphi \left(\max \left\{ \begin{array}{l} S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, x_{n+2}), \\ S_b(x_n, x_n, x_{n+2}), 0 \end{array} \right\} \right) - \\ &= \varphi \left(\max \left\{ \begin{array}{l} S_b(x_n, x_n, x_{n+1}), \lambda(S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_{n+2})), \\ \lambda(S_b(x_n, x_n, x_{n+2}) + 0) \end{array} \right\} \right) \\ &= \varphi(\max \{a_n, a_{n+1}, S_b(x_n, x_n, x_{n+2})\}) - \\ &= \varphi(\max \{a_n, \lambda(a_n + a_{n+1}), \lambda(S_b(x_n, x_n, x_{n+2}))\}) \\ &= \varphi(\max \{a_n, a_{n+1}, 2b(S_b(x_n, x_n, x_{n+1}) + b^2 S_b(x_{n+1}, x_{n+1}, x_{n+2}))\}) - \\ &\quad \varphi(\max \{a_n, \lambda(a_n + a_{n+1}), \lambda(S_b(x_n, x_n, x_{n+2}))\}) \\ &\leq \varphi(\max \{a_n, a_{n+1}, 2ba_n + b^2 a_{n+1}\}) - \\ &< \varphi(2ba_n + b^2 a_{n+1}) \end{aligned}$$

$\therefore \psi(3b^3 a_{n+1}) < \psi(2ba_n + b^2 a_{n+1})$

$\therefore 3b^3 a_{n+1} < 2ba_n + b^2 a_{n+1}$

$$\therefore a_{n+1} < \frac{2}{b(3b-1)} a_n$$

Therefore $a_{m+i} < \left(\frac{2}{b(3b-1)}\right)^i a_m$ for $i = 1, 2, \dots$

$$\therefore a_{m+i} < \beta^i a_m \text{ where } \beta = \frac{2}{b(3b-1)} < 1$$

$\therefore a_n \rightarrow 0$ as $n \rightarrow \infty$ and further $\sum a_n$ is convergent.

Consequently, the sequence $\{s_n = a_1 + a_2 + \dots + a_n\}$ of partial sums is a Cauchy sequence.

To show that $\{f^n(x_0)\}$ is Cauchy

we show that $S_b(x_m, x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$

By lemma (2.7)

$$\begin{aligned} S_b(x_m, x_m, x_{m+2}) &\leq 2bS_b(x_m, x_m, x_{m+1}) + b^2 S_b(x_{m+1}, x_{m+1}, x_{m+2}) \\ &= 2ba_m + b^2 a_{m+1} \end{aligned} \tag{3.2.2}$$

Now we show for $k \geq 2$

$$S_b(x_m, x_m, x_{m+k}) \leq 2b \sum_{i=0}^{k-2} b^{2i} a_{m+i} + b^{2(k-1)} a_{m+k-1} \tag{3.2.3}$$

This is true for $k = 2$ and for any $m = 0, 1, 2, \dots$

$$\begin{aligned} \text{Now } S_b(x_m, x_m, x_{m+(k+1)}) &\leq 2bS_b(x_m, x_m, x_{m+1}) + b^2 S_b(x_{m+1}, x_{m+1}, x_{m+(k+1)}) \\ &= 2ba_m + b^2 S_b(x_{m+1}, x_{m+1}, x_{m+1+k}) \\ &\leq 2ba_m + b^2 [2b \sum_{i=0}^{k-2} b^{2i} a_{m+1+i} + b^{2(k-1)} a_{m+k}] \\ &= 2ba_m + 2b \sum_{i=0}^{k-2} b^{2i+2} a_{m+1+i} + b^{2k} a_{m+k} \\ &= 2ba_m + 2b \sum_{j=1}^{k-1} b^{2j} a_{m+j} + b^{2k} a_{m+k} \text{ where } i+1 = j \\ &= 2b \sum_{j=0}^{k-1} b^{2j} a_{m+j} + b^{2k} a_{m+k} \text{ for } k+1 \text{ and } m = 0, 1, 2, \dots \end{aligned}$$

Thus (3.2.2) holds for $k \geq 2$ and $m = 0, 1, 2, \dots$

Suppose $m < n$, write $n = m + k$, then we show that

$$S_b(x_m, x_m, x_{m+k}) \leq 2b \sum_{i=0}^{k-2} b^{2i} a_{m+i} + b^{2(k-1)} a_{m+k-1} \tag{3.2.4}$$

$$\text{We have } a_{n+1} \leq \frac{2}{b(3b-1)} a_n = \beta a_n, a_{m+i} < \beta^i a_m$$

$$\text{and } b^{2i} \beta^i = (b^2 \beta)^i = \left(b^2 \frac{2}{b(3b-1)}\right)^i = \left(\frac{2b}{3b-1}\right)^i$$

Therefore from (3.2.3)

$$\begin{aligned} S_b(x_m, x_m, x_{m+k}) &\leq 2b \sum_{i=0}^{k-2} b^{2i} \beta^i a_m + b^{2(k-1)} a_{m+k-1} \\ &\leq 2ba_m \sum_{i=0}^{k-2} \left(\frac{2b}{3b-1}\right)^i a_m + b^{2(k-1)} \beta^{k-1} a_m \\ &= 2ba_m \sum_{i=0}^{k-2} (r)^i + (b^2 \beta)^{k-1} a_m \text{ where } r = \frac{2b}{3b-1} < 1 \\ &= 2ba_m \sum_{i=0}^{k-2} (r)^i + (r)^{k-1} a_m \\ &< 2ba_m \sum_{i=0}^{k-2} (r)^i + 2b(r)^{k-1} a_m \end{aligned}$$

$$= 2ba_m(\sum_{i=0}^{k-1} r^i) \\ < 2ba_m \frac{1}{1-r} \rightarrow 0 \text{ as } m \rightarrow \infty$$

$\therefore S_b(x_m, x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$

Therefore $\{x_n\}$ is a Cauchy sequence.

Suppose $x_n \rightarrow x$

Now we show that x is a fixed point of f

$$\begin{aligned} \text{We have } \psi(3b^3S_b(fx, fx, x_{n+1})) &= \psi(3b^3S_b(fx, fx, fx_n)) \\ &\leq \varphi \left(\max \left\{ \begin{array}{l} S_b(x, x, x_n), S_b(x, x, fx), S_b(x_n, x_n, fx_n), \\ S_b(x, x, fx_n), S_b(x_n, x_n, fx) \end{array} \right\} \right) - \\ &\quad \varphi \left(\max \left\{ \begin{array}{l} S_b(x, x, x_n), \lambda(S_b(x, x, fx) + S_b(x_n, x_n, fx_n)), \\ \lambda(S_b(x, x, fx_n) + S_b(x_n, x_n, fx)) \end{array} \right\} \right) \\ &\quad \psi \left(\max \left\{ \begin{array}{l} S_b(x, x, x), S_b(x, x, fx), S_b(x, x, x), \\ S_b(x, x, fx), S_b(x, x, fx) \end{array} \right\} \right) - \\ &\quad \varphi \left(\max \left\{ \begin{array}{l} 0, \lambda(S_b(x, x, fx) + S_b(x, x, x)), \\ \lambda(S_b(x, x, x) + S_b(x, x, fx)) \end{array} \right\} \right) \\ \text{On letting } n \rightarrow \infty, \text{ we get } \psi(3b^3S_b(fx, fx, x)) &\leq \varphi \left(\max \left\{ \begin{array}{l} 0, \lambda(S_b(x, x, fx) + S_b(x, x, x)), \\ \lambda(S_b(x, x, x) + S_b(x, x, fx)) \end{array} \right\} \right) \\ &= \frac{\psi(\max \{0, S_b(x, x, fx), 0, 0, S_b(x, x, fx)\}) -}{\varphi(\max \{0, \lambda(S_b(x, x, fx) + 0), \lambda(0 + S_b(x, x, fx))\})} \\ &= \psi(S_b(x, x, fx)) - \varphi(\lambda(S_b(x, x, fx))) \\ &\leq \psi(bS_b(fx, fx, x)) - \varphi(\lambda(S_b(x, x, fx))) \quad (\text{by Lemma (2.6)}) \\ &< \psi(bS_b(fx, fx, x)) \text{ if } x \neq fx, \text{ a contradiction since } b > 1 \end{aligned}$$

$\therefore fx = x$

$\therefore x$ is a fixed point of f .

Now we show that fixed point is unique

Let x, y be fixed points of f . Then

$$\begin{aligned} \psi(3b^3S_b(fx, fx, fy)) &\leq \varphi \left(\max \left\{ \begin{array}{l} S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ S_b(x, x, fy), S_b(y, y, fx) \end{array} \right\} \right) - \\ &\quad \varphi \left(\max \left\{ \begin{array}{l} S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \end{array} \right\} \right) \\ &= \frac{\psi(\max \{S_b(x, x, y), 0, 0, S_b(x, x, y), S_b(y, y, x)\}) -}{\varphi(\max \{S_b(x, x, y), \lambda(S_b(x, x, x) + S_b(y, y, y)), \\ \lambda(S_b(x, x, y) + S_b(y, y, x))\})} \\ &= \frac{\psi(\max \{S_b(x, x, y), S_b(y, y, x)\}) -}{\varphi(\max \{S_b(x, x, y), 0, \lambda(S_b(x, x, y) + S_b(y, y, x))\})} \\ &< \psi(\max \{S_b(x, x, y), S_b(y, y, x)\}) \text{ if } x \neq y \\ &\leq \psi(\max \{S_b(x, x, y), bS_b(x, x, y)\}) \quad (\text{by Lemma (2.6)}) \end{aligned}$$

$\therefore \psi(3b^3S_b(x, x, y)) < \psi(bS_b(x, x, y))$, a contradiction since $b > 1$

$\therefore x = y$

3.3 Definition

Let (X, S_b) be a S_b -Metric Space, $\psi, \varphi \in \Psi$. Let $f: X \rightarrow X$ be a mapping. We say that f is a $(\psi - \varphi - \lambda)$ contraction of first-kind if

$$\psi(3b^3S_b(fx, fx, fy)) \leq \frac{\psi(\max \{S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx))\}) -}{\varphi(\max \{S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx))\})} \quad (3.3.1)$$

and we say that f is a generalized $(\psi - \varphi - \lambda)$ contraction of second-kind if

$$\psi(3b^3S_b(fx, fx, fy)) \leq \frac{\psi(\max \{S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx))\}) -}{\varphi(\max \{S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx))\})} \quad (3.3.2)$$

3.4 Corollary: Suppose (3.3.1) holds for $f, \varphi, \psi, \lambda$ then f has a unique fixed point.

Proof: We observe that

$$\begin{aligned} \max \left\{ \begin{array}{l} S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \end{array} \right\} &\leq \max \left\{ \begin{array}{l} S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \end{array} \right\} \\ \therefore \varphi \left(\max \left\{ \begin{array}{l} S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \end{array} \right\} \right) &\leq \varphi \left(\max \left\{ \begin{array}{l} S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \end{array} \right\} \right) \end{aligned}$$

Consequently

$$\psi \left(\max \left\{ \begin{array}{l} S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ S_b(x, x, fy), S_b(y, y, fx) \end{array} \right\} \right) - \varphi \left(\max \left\{ \begin{array}{l} S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \\ \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \end{array} \right\} \right) \leq$$

$$\psi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right\} \right) - \varphi \left(\max \left\{ S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \right\} \right)$$

so that (3.3.1) holds \Rightarrow (3.2.1) holds

Hence by theorem (3.2), f has a unique fixed point.

3.5 Corollary: Suppose (3.3.2) holds for $f, \varphi, \psi, \lambda$ then f has a unique fixed point.

Proof: We observe that

$$\max \left\{ S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \right. \\ \left. \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \right\} \leq \max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \\ \left. S_b(x, x, fy), S_b(y, y, fx) \right\}$$

and hence

$$\psi \left(\max \left\{ S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \right. \right. \\ \left. \left. \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \right\} \right) \leq \psi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \right. \\ \left. \left. S_b(x, x, fy), S_b(y, y, fx) \right\} \right)$$

so that (3.3.2) holds \Rightarrow (3.3.1) holds

$\therefore f$ has a unique fixed point by corollary 3.4

3.6 Corollary: Suppose f satisfies the following conditions

$$\psi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \right. \\ \left. \left. S_b(x, x, fy), S_b(y, y, fx) \right\} \right) - \\ \psi(3b^3 S_b(fx, fx, fy)) \leq \varphi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \right. \\ \left. \left. S_b(x, x, fy), S_b(y, y, fx) \right\} \right) \quad (3.6.1)$$

Then f has a unique fixed point.

Proof: Since

$$\varphi \left(\max \left\{ S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \right. \right. \\ \left. \left. \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \right\} \right) \leq \varphi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \right. \\ \left. \left. S_b(x, x, fy), S_b(y, y, fx) \right\} \right)$$

It follows that

$$\psi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \right. \\ \left. \left. S_b(x, x, fy), S_b(y, y, fx) \right\} \right) - \varphi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \right. \\ \left. \left. S_b(x, x, fy), S_b(y, y, fx) \right\} \right) \leq \\ \psi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \right. \\ \left. \left. S_b(x, x, fy), S_b(y, y, fx) \right\} \right) - \varphi \left(\max \left\{ S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \right. \right. \\ \left. \left. \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \right\} \right)$$

Hence (3.6.1) \Rightarrow (3.2.1). So the results follows from Theorem 3.2

Now we show that theorem 3.4 of G.N.V..Kishore, K.P.R.Rao et al. [4] follows as a corollary of our results.

4. Fixed point results for $(\psi - \varphi - \lambda)$ contractions in partially ordered S_b -Metric Spaces

4.1 Definition:

Let (X, S_b) be a S_b -Metric Space and \leq be a partial order on X . Then (X, S_b, \leq) is called a partially ordered S_b -Metric Space. Suppose (X, S_b, \leq) is a partially ordered S_b -Metric Space and $f: X \rightarrow X$ be a self map. We say that

- (i) f is non-decreasing if $x \leq y \Rightarrow fx \leq fy$ and
- (ii) f is non-increasing if $x \leq y \Rightarrow fx \geq fy$

4.2 Definition:

Let (X, S_b, \leq) be a partially ordered S_b -Metric Space and $\psi, \varphi \in \Psi$, $0 < \lambda \leq 1/2$. Let $f: X \rightarrow X$ be a mapping. We say that f is a $(\psi - \varphi - \lambda)$ contraction on (X, S_b, \leq) if

$$\psi \left(\max \left\{ S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy), \right. \right. \\ \left. \left. S_b(x, x, fy), S_b(y, y, fx) \right\} \right) - \\ \psi(3b^3 S_b(fx, fx, fy)) \leq \varphi \left(\max \left\{ S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \right. \right. \\ \left. \left. \lambda(S_b(x, x, fy) + S_b(y, y, fx)) \right\} \right)$$

whenever x, y are comparable.

4.3 Theorem: Let (X, S_b, \leq) be a partially Ordered Complete S_b -Metric Space and Let $f: X \rightarrow X$ be non-decreasing $(\psi - \varphi - \lambda)$ contraction. Suppose there exists $x_0 \in X$ such that $x_0 \leq fx_0$. Define the sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for $n = 0, 1, 2, \dots$ Then $\{x_n\}$ is a Cauchy sequence.

Suppose (X, S_b, \leq) has the following property

P: $\{y_n\}$ is an increasing sequence in X and $y_n \rightarrow y$ implies $y_n \leq y$ for all n . Then f has a fixed point. Infact if $x_n \rightarrow x$, then x is a fixed point of f . Further no two fixed points of f are comparable.

Proof: Let $x_0 \in X$. Define $x_{n+1} = fx_n$ for $n = 0, 1, 2, \dots$

Since $x_0 \leq fx_0 = x_1$, we have $x_0 \leq x_1$.

Now by induction, since f is non decreasing $x_n \leq x_{n+1}$ for $n = 0, 1, 2, \dots$

Case(i): If $x_n = x_{n+1}$ for some n , then x_n is a fixed point of f .

Case(ii): Suppose $x_n \neq x_{n+1} \forall n$

let $a_n = S_b(x_n, x_n, x_{n+1})$ for $n = 0, 1, 2, \dots$

First we show that $a_{n+1} < \frac{2}{b(3b-1)}a_n$ for $n = 0, 1, 2, \dots$

We have that $\psi(3b^3a_{n+1}) = \psi(3b^3S_b(x_{n+1}, x_{n+1}, x_{n+2}))$

$$\begin{aligned} &= \psi(3b^3S_b(fx_n, fx_n, fx_{n+1})) \\ &\leq \psi\left(\max\left\{\begin{array}{l} S_b(x_n, x_n, x_{n+1}), S_b(x_n, x_n, fx_n), S_b(x_{n+1}, x_{n+1}, fx_{n+1}), \\ S_b(x_n, x_n, fx_{n+1}), S_b(x_{n+1}, x_{n+1}, fx_n) \end{array}\right\}\right) - \\ &\quad \varphi\left(\max\left\{\begin{array}{l} S_b(x_n, x_n, x_{n+1}), \lambda(S_b(x_n, x_n, fx_n) + S_b(x_{n+1}, x_{n+1}, fx_{n+1})), \\ \lambda(S_b(x_n, x_n, fx_{n+1}) + S_b(x_{n+1}, x_{n+1}, fx_n)) \end{array}\right\}\right) \\ &\quad (\text{since } x_n \text{ and } x_{n+1} \text{ are comparable}) \\ &= \psi\left(\max\left\{\begin{array}{l} S_b(x_n, x_n, x_{n+1}), S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, x_{n+2}), \\ S_b(x_n, x_n, x_{n+2}), S_b(x_{n+1}, x_{n+1}, x_{n+1}) \end{array}\right\}\right) - \\ &\quad \varphi\left(\max\left\{\begin{array}{l} S_b(x_n, x_n, x_{n+1}), \lambda(S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_{n+2})), \\ \lambda(S_b(x_n, x_n, x_{n+2}) + S_b(x_{n+1}, x_{n+1}, x_{n+1})) \end{array}\right\}\right) \\ &= \psi\left(\max\left\{\begin{array}{l} S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, x_{n+2}), \\ S_b(x_n, x_n, x_{n+2}), 0 \end{array}\right\}\right) - \\ &\quad \varphi\left(\max\left\{\begin{array}{l} S_b(x_n, x_n, x_{n+1}), \lambda(S_b(x_n, x_n, x_{n+1}) + S_b(x_{n+1}, x_{n+1}, x_{n+2})), \\ \lambda(S_b(x_n, x_n, x_{n+2}) + 0) \end{array}\right\}\right) \\ &\leq \psi\left(\max\left\{a_n, a_{n+1}, 2b(S_b(x_n, x_n, x_{n+1}) + b^2S_b(x_{n+1}, x_{n+1}, x_{n+2}))\right\}\right) - (\text{by lemma 2.7}) \\ &\quad \varphi\left(\max\left\{a_n, \lambda(a_n + a_{n+1}), \lambda(S_b(x_n, x_n, x_{n+2}))\right\}\right) \\ &< \psi(2ba_n + b^2a_{n+1}) \quad (\text{since } \varphi(\max\{a_n, \lambda(a_n + a_{n+1}), \lambda(S_b(x_n, x_n, x_{n+2}))\}) > 0) \end{aligned}$$

$\therefore \psi(3b^3a_{n+1}) < \psi(2ba_n + b^2a_{n+1})$

$\therefore 3b^3a_{n+1} < 2ba_n + b^2a_{n+1}$

$\therefore a_{n+1} < \frac{2}{b(3b-1)}a_n$

Therefore $a_{m+i} < \left[\frac{2}{b(3b-1)}\right]^i a_m$ for $i = 1, 2, \dots$

$\therefore a_{m+i} < \beta^i a_m$ where $\beta = \frac{2}{b(3b-1)} < 1$

$\therefore a_n \rightarrow 0$ as $n \rightarrow \infty$ and further $\sum a_n$ is convergent.

Consequently, the sequence $\{s_n = a_1 + a_2 + \dots + a_n\}$ of partial sums is a Cauchy sequence.

Now we show that $\{f^n(x_0)\}$ is Cauchy

For this, we show that $S_b(x_m, x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$

By lemma (2.7)

$$\begin{aligned} S_b(x_m, x_m, x_{m+2}) &\leq 2bS_b(x_m, x_m, x_{m+1}) + b^2S_b(x_{m+1}, x_{m+1}, x_{m+2}) \\ &= 2ba_m + b^2a_{m+1} \end{aligned} \tag{4.3.1}$$

Now we show for $k \geq 2$

$$S_b(x_m, x_m, x_{m+k}) \leq 2b \sum_{i=0}^{k-2} b^{2i} a_{m+i} + b^{2(k-1)} a_{m+k-1} \tag{4.3.2}$$

This is true for $k = 2$ and for any $m = 0, 1, 2, \dots$

$$\begin{aligned} \text{Now } S_b(x_m, x_m, x_{m+(k+1)}) &\leq 2bS_b(x_m, x_m, x_{m+1}) + b^2S_b(x_{m+1}, x_{m+1}, x_{m+(k+1)}) \\ &= 2ba_m + b^2S_b(x_{m+1}, x_{m+1}, x_{m+1+k}) \\ &\leq 2ba_m + b^2[2b \sum_{i=0}^{k-2} b^{2i} a_{m+1+i} + b^{2(k-1)} a_{m+k}] \\ &= 2ba_m + 2b \sum_{i=0}^{k-2} b^{2i+2} a_{m+1+i} + b^{2k} a_{m+k} \\ &= 2ba_m + 2b \sum_{j=1}^{k-1} b^{2j} a_{m+j} + b^{2k} a_{m+k} \text{ where } i+1 = j \\ &= 2b \sum_{j=0}^{k-1} b^{2j} a_{m+j} + b^{2k} a_{m+k} \text{ for } k+1 \text{ and } m = 0, 1, 2, \dots \end{aligned}$$

Thus (4.1.2) holds for $k \geq 2$ and $m = 0, 1, 2, \dots$

Suppose $m < n$, write $n = m + k$, then we show that

$$S_b(x_m, x_m, x_{m+k}) \leq 2b \sum_{i=0}^{k-2} b^{2i} a_{m+i} + b^{2(k-1)} a_{m+k-1} \tag{4.3.3}$$

We have $a_{n+1} \leq \frac{2}{b(3b-1)}a_n = \beta a_n \therefore a_{m+i} < \beta^i a_m$

And $b^{2i}\beta^i = (b^2\beta)^i = \left(b^2 \frac{2}{b(3b-1)}\right)^i = \left(\frac{2b}{3b-1}\right)^i$

Therefore from (4.1.3)

$$S_b(x_m, x_m, x_{m+k}) \leq 2b \sum_{i=0}^{k-2} b^{2i} \beta^i a_m + b^{2(k-1)} a_{m+k-1}$$

$$\begin{aligned}
 &\leq 2ba_m \sum_{i=0}^{k-2} \left(\frac{2b}{3b-1}\right)^i + b^{2(k-1)}\beta^{k-1}a_m \\
 &= 2ba_m \sum_{i=0}^{k-2}(r)^i + (b^2\beta)^{k-1}a_m \text{ where } r = \frac{2b}{3b-1} < 1 \\
 &= 2ba_m \sum_{i=0}^{k-2}(r)^i + (r)^{k-1}a_m \\
 &< 2ba_m \sum_{i=0}^{k-2}(r)^i + 2b(r)^{k-1}a_m \\
 &= 2ba_m \left(\sum_{i=0}^{k-1} r^i\right) \\
 &< 2ba_m \frac{1}{1-r} \rightarrow 0 \text{ as } m \rightarrow \infty
 \end{aligned}$$

$\therefore S_b(x_m, x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$

Therefore $\{x_n\}$ is a Cauchy sequence.

Now we show that x is a fixed point of f

We have $\{x_n\}$ is a Cauchy sequence

Suppose $x_n \rightarrow x$. By property P, $x_n \leq x \forall n$

We have $\psi(3b^3 S_b(fx, fx, x_{n+1})) = \psi(3b^3 S_b(fx, fx, f x_n))$

$$\begin{aligned}
 &\leq \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, x_n), S_b(x, x, fx), S_b(x_n, x_n, fx_n),))}{S_b(x, x, fx_n), S_b(x_n, x_n, fx_n)} \right\} \right) - \\
 &\quad \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, x_n), \lambda(S_b(x, x, fx) + S_b(x_n, x_n, fx_n),))}{\lambda(S_b(x, x, fx_n) + S_b(x_n, x_n, fx_n))} \right\} \right) \\
 \psi(3b^3 S_b(fx, fx, x)) &\leq \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, x), S_b(x, x, fx), S_b(x, x, x),))}{S_b(x, x, x), S_b(x, x, fx)} \right\} \right) - \\
 &\quad \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, x), \lambda(S_b(x, x, fx) + S_b(x, x, x),))}{\lambda(S_b(x, x, x) + S_b(x, x, fx))} \right\} \right) \\
 &= \varphi \left(\max \left\{ \frac{\psi(0, S_b(x, x, fx),))}{\lambda(S_b(x, x, fx) + S_b(x, x, fx))} \right\} \right) - \\
 &= \psi(S_b(x, x, fx)) - \varphi(S_b(x, x, fx)) \\
 \therefore \psi(3b^3 S_b(fx, fx, x)) &\leq \psi(bS_b(fx, fx, x)) - \varphi(\lambda(S_b(x, x, fx))) \text{ (by lemma 2.6)} \\
 &\quad \left(\text{since } \psi \text{ is non decreasing, since } \varphi(\lambda(S_b(x, x, fx))) > 0 \right)
 \end{aligned}$$

$\therefore \psi(3b^3 S_b(fx, fx, x)) < \psi(bS_b(fx, fx, x))$ if $x \neq fx$,

$\therefore 3b^3 S_b(fx, fx, x) < bS_b(fx, fx, x)$

$\therefore 3b^3 < b$, a contradiction, since $b > 1$

$\therefore fx = x$

$\therefore x$ is a fixed point of f .

Now we show that no two fixed points are comparable

Let x and y be fixed points of f

Suppose x and y are comparable, and $x \leq y$ and $x < y$ then from definition 4.2 we have that

$$\begin{aligned}
 \psi(3b^3 S_b(fx, fx, fy)) &\leq \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy),))}{S_b(x, x, fy), S_b(y, y, fx)} \right\} \right) - \\
 &\quad \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy),))}{\lambda(S_b(x, x, fx) + S_b(y, y, fx))} \right\} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{or } \psi(3b^3 S_b(x, x, y)) &\leq \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, y), S_b(x, x, x), S_b(y, y, y),))}{S_b(x, x, y), S_b(y, y, x)} \right\} \right) - \\
 &\quad \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, y), \lambda(S_b(x, x, x) + S_b(y, y, y),))}{\lambda(S_b(x, x, y) + S_b(y, y, x))} \right\} \right) \\
 &= \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, y), S_b(y, y, x),))}{\lambda(S_b(x, x, y) + S_b(y, y, x))} \right\} \right) - \\
 &= \varphi \left(\max \left\{ \frac{\psi(S_b(x, x, y), \lambda(S_b(x, x, y) + S_b(y, y, x),))}{\lambda(S_b(x, x, y) + S_b(y, y, x))} \right\} \right) \\
 &< \psi(\max \{S_b(x, x, y), S_b(y, y, x)\}) \text{ since } x < y \\
 &\leq \psi(\max \{S_b(x, x, y), bS_b(x, x, y)\}) \text{ (by Lemma 2.6)}
 \end{aligned}$$

$\therefore \psi(3b^3 S_b(x, x, y)) < \psi(bS_b(x, x, y))$ since $x < y$

$\therefore 3b^3 < b$, which is not true, since $b > 1$

Therefore x cannot be less than y

Similarly we can show that y cannot be less than x

$\therefore x = y$

It follows that no two fixed points of f are comparable.

5. Application

5.1 Theorem: Let (X, S_b) be a complete S_b -Metric Space. U be an open subset of X and \bar{U} be a closed subset of X such that $U \subseteq \bar{U}$. Suppose that $H: \bar{U} \times [0,1] \rightarrow X$ is an operator such that the following conditions are satisfied

- (i) $x \neq H(x, \lambda)$ for each $x \in \partial U$ and for any $\lambda \in [0,1]$ (Here ∂U denotes the boundary of U in X).
- (ii) $\psi(3b^3 S_b(H(x, \lambda), H(x, \lambda), H(y, \lambda))) \leq \psi(S_b(x, x, y)) - \varphi(S_b(x, x, y)) \forall x, y \in \bar{U}$ and $\lambda \in [0,1]$ where $\psi: [0, \infty) \rightarrow [0, \infty)$ is continuous non-decreasing and $\varphi: [0, \infty) \rightarrow [0, \infty)$ is continuous with $\varphi(t) > 0$ for $t > 0$
- (iii) There exists $M \geq 0$ such that $S_b(H(x, \lambda), H(x, \lambda), H(x, \mu)) \leq M|\lambda - \mu|$ for every $x \in \bar{U}$ and $\lambda, \mu \in [0,1]$.

Then $H(\cdot, \lambda)$ has a fixed point for some $\lambda \in [0,1]$, if and only if $H(\cdot, \mu)$ has a fixed point for every $\mu \in [0,1]$.

Proof: From (iii) it follows that H is continuous in the second variable. From (ii) it follows that H is continuous in the first Variable. We have that

$$\begin{aligned} \psi(3b^3 S_b(H(x, \lambda), H(x, \lambda), H(x_n, \lambda))) &\leq \psi(S_b(x, x, x_n)) - \varphi(S_b(x, x, x_n)) \\ &\leq \psi(S_b(x, x, x_n)) \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

If $x_n \rightarrow x$ then $\psi(S_b(H(x, \lambda), H(x, \lambda), H(x_n, \lambda))) \rightarrow 0$ as $n \rightarrow \infty$

$$\therefore S_b(H(x, \lambda), H(x, \lambda), H(x, \lambda)) = 0$$

$$\begin{aligned} \text{Now } S_b(H(x, \lambda), H(x, \lambda), H(y, \mu)) &\leq 2b S_b(H(x, \lambda), H(x, \lambda), H(y, \lambda)) + \\ &\quad b^2 S_b(H(y, \lambda), H(y, \lambda), H(y, \mu)) \\ &\leq 2b S_b(H(x, \lambda), H(x, \lambda), H(y, \lambda)) + b^2 M|\lambda - \mu| \\ &\leq 2b S_b(x, x, y) + b^2 M|\lambda - \mu| \text{ (from (i))} \\ &\rightarrow 0 \text{ as } (y, \mu) \rightarrow (x, \lambda) \end{aligned}$$

Hence H is a continous function on $\bar{U} \times [0,1]$. Also

$$\begin{aligned} \psi(S_b(H(x, \lambda), H(x, \lambda), H(y, \lambda))) &\leq \psi(S_b(x, x, y)) - \varphi(S_b(x, x, y)) \\ &< \psi(S_b(x, x, y)) \text{ if } y \neq x \\ \Rightarrow S_b(H(x, \lambda), H(x, \lambda), H(y, \lambda)) &< S_b(x, x, y) \end{aligned} \tag{5.1.1}$$

Consider the set $A = \{\lambda \in [0,1] : x = H(x, \lambda) \text{ for some } x \in U\}$

Suppose λ is a limit point of A . Then there exists a $\{\lambda_n\}$ in A such that $\lambda_n \rightarrow \lambda$. Then there exist a sequence $\{x_n\}$ in X such that $x_n = H(x_n, \lambda_n)$

Now we show that $\{x_n\}$ is a S_b -Cauchy sequence in (X, S_b)

Suppose that $\{x_n\}$ is not S_b -Cauchy sequence.

So there exists $\epsilon > 0$ and monotonically increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k >$

$$m_k, S_b(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon \tag{5.1.2}$$

$$\text{and } S_b(x_{m_k}, x_{m_k}, x_{n_{k-1}}) < \epsilon \tag{5.1.3}$$

From (5.1.2) and (5.1.3), we obtain $\epsilon \leq S_b(x_{m_k}, x_{m_k}, x_{n_k})$

$$\leq 2b S_b(x_{m_k}, x_{m_k}, x_{m_{k+1}}) + b^2 S_b(x_{m_{k+1}}, x_{m_{k+1}}, x_{n_k})$$

Multiplying the above inequality with (3b), applying \square on both sides, and letting $k \rightarrow \infty$ we have that

$$\psi(3b\epsilon) \leq \lim_{k \rightarrow \infty} \psi(3b^3 S_b(x_{m_{k+1}}, x_{m_{k+1}}, x_{n_k})) \tag{5.1.4}$$

Suppose $|\lambda - \lambda_0| < \epsilon$ and $x \in \overline{S_b(x_0, r)}$, $x \neq x_0$

$$\begin{aligned} \text{Then } \psi(3b^3 S_b(H(x, \lambda_0), H(x, \lambda_0), H(x_0, \lambda_0))) &\leq \psi(S_b(x, x, x_0)) - \varphi(S_b(x, x, x_0)) \\ &< \psi(S_b(x, x, x_0)) \text{ (since } x \neq x_0) \end{aligned}$$

$$\therefore 3b^3 S_b(H(x, \lambda_0), H(x, \lambda_0), H(x_0, \lambda_0)) < S_b(x, x, x_0) \leq r$$

$$\text{But } S_b(H(x, \lambda), H(x, \lambda), H(x_0, \lambda)) \leq 2b S_b(H(x, \lambda), H(x, \lambda), H(x_0, \lambda)) +$$

$$b^2 S_b(H(x, \lambda_0), H(x, \lambda_0), H(x_0, \lambda_0))$$

$$\leq 2bM|\lambda - \lambda_0| + \left(\frac{b^2r}{3b^3}\right)$$

$$\leq 2bM\epsilon + \left(\frac{r}{3b}\right)$$

$$\leq 2bM\epsilon \left(\frac{3b-1}{6b^2M}\right) + \frac{r}{3b}$$

$$= \frac{r(3b-1)}{3b} 4r = r$$

$$\therefore S_b(H(x, \lambda), H(x, \lambda), x_0) \leq S_b(H(x, \lambda), H(x, \lambda), H(x_0, \lambda_0)) \leq r$$

$$\therefore H(x, \lambda) \in \overline{S_b(x_0, r)}$$

Thus for any λ , with $|\lambda - \lambda_0| < \epsilon$

and $\lambda \in [0,1]$, it follows that $F: \overline{S_b(x_0, r)} \rightarrow \overline{S_b(x_0, r)}$

defined by $F(x) = H(x, \lambda)$ satisfies all the hypothesis of the main theorem (4.2)

Hence F has a fixed point.

i.e., $Fx = x$ for some $x \in \overline{S_b(x_0, r)} \subseteq U$

$$\therefore H(x, \lambda) = Fx = x$$

$$\therefore \lambda \in A$$

$$\text{Thus } |\lambda - \lambda_0| < \epsilon \Rightarrow \lambda \in A$$

But

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi(3b^3 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})) &= \lim_{n \rightarrow \infty} \psi\left(3b^3 S_b(H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{n_k}, \lambda_{n_k}))\right) \\ &\leq \lim_{n \rightarrow \infty} \psi(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})) - \varphi(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})) \\ &< \lim_{n \rightarrow \infty} \psi(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})) \end{aligned}$$

It follows that

$$\lim_{n \rightarrow \infty} (3b^3 - 1) S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) \leq 0$$

$$\text{Thus } \lim_{n \rightarrow \infty} S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) = 0$$

Hence from (5.1.4) and by the def of \square , we have that $\epsilon \leq 0$

which is a contradiction

Hence $\{x_n\}$ is an S_b -Cauchy sequence in (X, S_b) and by the completeness of (X, S_b) , there exists $\alpha \in U$ with

$$\lim_{n \rightarrow \infty} x_n = \alpha$$

Suppose $\lambda_n \rightarrow \lambda$ then $(x_n, \lambda_n) \rightarrow (\alpha, \lambda)$

so that $H(x_n, \lambda_n) \rightarrow H(\alpha, \lambda)$ (since H is continuous)

$$\text{But } H(x_n, \lambda_n) = x_n \rightarrow \alpha$$

$$\therefore \alpha = H(\alpha, \lambda)$$

$$\therefore \alpha \in A$$

Hence A is closed.

Consider the set $A = \{\lambda \in [0,1] : x = H(x, \lambda) \text{ for some } x \in U\}$

Now we show that A is open

Let $\lambda_0 \in A$, then there exists $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$

$x_0 \in U$, then there exists $r > 0$ such that $S_b(x, x_0) \leq r \Rightarrow x \in U$

$$\text{Choose } \epsilon \text{ such that } 0 < \epsilon < \frac{r(3b-1)}{6b^2 M}$$

Then λ_0 is an interior point of A

Hence A is open. Consequently A is both closed and open

$$\therefore \text{either } A = \emptyset \text{ or } A = [0,1]$$

Now suppose $H(\cdot, \lambda)$ has a fixed point for some $\lambda \in [0,1]$, then $A \neq \emptyset$

$$\therefore A = [0,1]$$

$$\therefore H(\cdot, \mu) \text{ has a fixed point for all } \mu \in [0,1]$$

The following example is in support of the main theorem 3.2

5.2 Example

Let $= [0,1]$. Define $S: X \times X \times X \rightarrow R^+$ by $S_b(x, y, z) = (|y + z - 2x| + |y - z|)^2$, then (X, S_b) is a S_b -Metric Space with $b = 2$. Define $f: X \times X \rightarrow R$ by $f(x) = kx$ (where $0 < k \leq \frac{1}{4\sqrt{3}}$). Also define $\psi, \varphi: R^+ \rightarrow R$ by $\psi(t) = t$ and $\varphi(t) = \frac{t}{2}$. Let $0 < \lambda \leq \frac{1}{2}$ then f is a $(\psi - \varphi - \lambda)$ contraction on (X, S_b) and f has a unique fixed point.

Solution: From Def. 3.1

$$\psi(3b^3 S_b(fx, fx, fy)) \leq \frac{\psi(\max\{S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy)\}) - \varphi(\max\{S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \lambda(S_b(x, x, fy) + S_b(y, y, fx))\})}{\lambda(S_b(x, x, fy) + S_b(y, y, fx))} \text{ for all } x, y \in X.$$

Since $b = 2$, we have

$$\begin{aligned} \psi(3b^3 S_b(fx, fx, fy)) &= \psi((3)(8)S_b(fx, fx, fy)) \\ &\leq \frac{\psi(\max\{S_b(x, x, y), S_b(x, x, fx), S_b(y, y, fy)\}) - \varphi(\max\{S_b(x, x, y), \lambda(S_b(x, x, fx) + S_b(y, y, fy)), \lambda(S_b(x, x, fy) + S_b(y, y, fx))\})}{\lambda(S_b(x, x, fy) + S_b(y, y, fx))} \end{aligned}$$

$$\text{Consider } \psi((3)(8)S_b(fx, fx, fy)) = (3)(8)S_b(fx, fx, fy)$$

$$\begin{aligned} &= (3)(8)S_b(kx, kx, ky) \\ &= (3)(8)(|kx + ky - 2kx| + |kx - ky|)^2 \\ &= (3)(8)(4k^2|x - y|^2) \end{aligned} \tag{5.2.1}$$

$$\text{R.H.S} = \max_{\lambda} \left\{ 4|x-y|^2, 4|kx-x|^2, 4|ky-y|^2, 4|ky-x|^2, 4|kx-y|^2 \right\} - \left(\frac{1}{2} \right) \max_{\lambda} \left\{ 4|x-y|^2, \lambda(4|kx-x|^2 + 4|ky-y|^2), \lambda(4|ky-x|^2 + 4|kx-y|^2) \right\} \quad (5.2.2)$$

Suppose $x > y$

$$\text{Thus R.H.S} = \max_{\lambda} \left\{ 4|x-y|^2, 4|kx-x|^2, 4|ky-y|^2, 4|ky-x|^2, 4|kx-y|^2 \right\} - \left(\frac{1}{2} \right) \max_{\lambda} \left\{ 4|x-y|^2, \lambda(4|kx-x|^2 + 4|ky-y|^2), \lambda(4|ky-x|^2 + 4|kx-y|^2) \right\}$$

Case (i) Suppose $4|x-y|^2 = \max \{ 4|x-y|^2, 4|kx-x|^2, 4|ky-y|^2, 4|ky-x|^2, 4|kx-y|^2 \}$

Thus $4|x-y|^2 = \max \{ 4|x-y|^2, \lambda(4|kx-x|^2 + 4|ky-y|^2), \lambda(4|ky-x|^2 + 4|kx-y|^2) \}$

So that $(3)(8)(4)k^2|x-y|^2 = (3)(8)k^2 \max \{ 4|x-y|^2, \lambda(4|kx-x|^2 + 4|ky-y|^2), \lambda(4|ky-x|^2 + 4|kx-y|^2) \}$

$$\begin{aligned} &\leq \left(\frac{1}{2} \right) 4|x-y|^2 \\ &= 4|x-y|^2 - \left(\frac{1}{2} \right) 4|x-y|^2 \\ &\leq \max \{ 4|x-y|^2, 4|kx-x|^2, 4|ky-y|^2, 4|ky-x|^2, 4|kx-y|^2 \} \\ &= -\left(\frac{1}{2} \right) \max \{ 4|x-y|^2, \lambda(4|kx-x|^2 + 4|ky-y|^2), \lambda(4|ky-x|^2 + 4|kx-y|^2) \} \end{aligned}$$

So that $(3)(8)(4)k^2|x-y|^2 \leq -\left(\frac{1}{2} \right) \max \{ 4|x-y|^2, \lambda(4|kx-x|^2 + 4|ky-y|^2), \lambda(4|ky-x|^2 + 4|kx-y|^2) \}$

$$\text{Consequently } k^2 \leq \frac{1}{(2)(3)(8)} \quad (5.2.3)$$

Clearly $\lambda(4|kx-x|^2 + 4|ky-y|^2) \leq \lambda(4|ky-x|^2 + 4|kx-y|^2)$

Case (ii) $\lambda(4|ky-x|^2 + 4|kx-y|^2) = \max \{ 4|x-y|^2, \lambda(4|kx-x|^2 + 4|ky-y|^2), \lambda(4|ky-x|^2 + 4|kx-y|^2) \}$

We observe that $4|kx-y|^2 \leq 4|ky-x|^2$

Here $4|x-y|^2 \leq \lambda(4|ky-x|^2 + 4|kx-y|^2) \leq 4|ky-x|^2$

$$\therefore (3)(8)(4)k^2|x-y|^2 = (3)(8)k^2 4|ky-x|^2$$

$$\begin{aligned} &\leq \left(\frac{1}{2} \right) 4|ky-x|^2 \\ &= 4|ky-x|^2 - \left(\frac{1}{2} \right) 4|ky-x|^2 \\ &\leq 4|ky-x|^2 - \left(\frac{1}{2} \right) \lambda(4|ky-x|^2 + 4|kx-y|^2) \\ &\leq -\left(\frac{1}{2} \right) \max \{ 4|x-y|^2, 4|kx-x|^2, 4|ky-y|^2, 4|ky-x|^2, 4|kx-y|^2 \} \end{aligned}$$

$$\therefore (3)(8)(4)k^2|x-y|^2 \leq -\left(\frac{1}{2} \right) \max \{ 4|x-y|^2, \lambda(4|kx-x|^2 + 4|ky-y|^2), \lambda(4|ky-x|^2 + 4|kx-y|^2) \}$$

$$\text{Hold if } \frac{1}{2}\lambda(4|kx-y|^2) \leq 4|ky-x|^2 - \left(\frac{1}{2} \right) \lambda(4|ky-x|^2 - (3)(8)k^2 4|ky-x|^2) \\ = \left(1 - \frac{\lambda}{2} - (3)(8)k^2 \right) 4|ky-x|^2 \quad (5.2.4)$$

Since $4|kx-y|^2 \leq 4|ky-x|^2$

$$(5.2.4) \text{ holds if } \frac{1}{2}\lambda \leq \left(1 - \frac{\lambda}{2} - (3)(8)k^2 \right)$$

$$\text{i.e., } \frac{1}{2}\lambda + \frac{1}{2}\lambda + (3)(8)k^2 \leq 1$$

$$\Rightarrow (3)(8)k^2 \leq 1$$

$$\therefore k^2 \leq \frac{1-\lambda}{(3)(8)} \quad (5.2.5)$$

$$\therefore k^2 \leq \frac{1}{(2)(3)(8)} \leq \frac{1-\lambda}{(3)(8)}$$

$$\therefore k^2 \leq \frac{1}{(2)(3)(8)}$$

$$\therefore k \leq \frac{1}{4\sqrt{3}}$$

Therefore if we choose $0 < k \leq \frac{1}{4\sqrt{3}}$, f is a $(\psi - \varphi - \lambda)$ contraction on (X, S_b) and by theorem 5.1, it follows that f has a unique fixed point.

Infact 0 is the fixed point of f .

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