An Analytical solution of (2+1)-dimensional Equal Width wave equation with diffusivity by HPM ,ADM and DTM

S. Padmasekaran[1], R. Asokan[2] and K. Alaguraja[3]
[1]Department of Mathematics, Periyar University, Salem, Tamil Nadu, India.
[2,3]Department of Mathematics, Madurai Kamaraj University, Madurai Tamil Nadu, India.

Abstract:
In this paper, we present the homotopy perturbation method (HPM) and Adomian Decomposition Method (ADM) to obtain a closed form solution of the (2+1)-dimensional Equal Width wave equation with diffusion. These methods consider the use of the initial or boundary conditions and find the solution without any discretization, transformation or restrictive conditions and avoid the round-off errors. Few numerical examples are provided to validate the reliability and efficiency of the three methods.
Keywords: Nonlinear PDE, Adomian Decomposition Method,Differential transform method,and homotopy perturbation method.

Introduction:
In the past few decades, traditional integral transform methods such as Fourier and Laplace transforms have commonly been used to solve engineering problems. These methods transform differential equations into algebraic equations which are easier to deal with. However, these integral transform methods are more complex and difficult when applying to nonlinear problems. The HPM, proposed first by He [2,3], for solving the differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences.

The HPM is applied to Volterra's integro-differential equation [4], to nonlinear oscillators [5], bifurcation of nonlinear problems [6], bifurcation of delay-differential equations [7], nonlinear wave equations [8], boundary value problems [9], quadratic Riccati differential equation of fractional order [1], and to other fields [10-20]. This HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions. Thus He's HPM is a universal one which can solve various kinds of nonlinear equations. Adomian decomposition method (ADM), which was introduced by Adomian [21], is a semi-numerical technique for solving linear and nonlinear differential equations by generating a functional series solution in a very efficient manner. The method has many advantages: it solves the problem directly without the need for linearization, perturbation, or any other transformation; it converges very rapidly and is highly accurate.

Differential transform method (DTM), which was first applied in the engineering field by Zhou [8], has many advantages: it solves the problem directly without the need for linearization, perturbation, or any other transformation. DTM is based on the Taylors series expansion. It constructs an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which needs symbolic computation of the necessary derivatives of the data functions. Taylor series method computationally takes a long time for larger orders while DTM method reduces the size of the computational domain, without massive computations and restrictive assumptions, and is easily applicable to various physical problems. The method and related theorems are well addressed in [9,10].

Let us consider the (2+1)-dimensional Equal Width wave equation with diffusivity term as,

$$u_t = uu_x + u_{xx} + u_{xxt} + u_{yyt}, \tag{1}$$

under the initial condition

$$u(x, y, 0) = u_0(x, y). \tag{2}$$

In this work, we have employed the Homotopy Perturbation Method (HPM),DTM and ADM to solve the (2+1)-dimensional Modified Equal Width Wave equation with diffusion term.

1.HOMOTOPY PERTURBATION METHOD (HPM):

To describe the HPM, consider the following general nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3}$$

under the boundary condition

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \partial\Omega, \tag{4}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and $\partial\Omega$ is a boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is a linear operator while N is a nonlinear operator. Then Eq. (3) can be rewrittten as

$$L(u) + N(u) - f(r) = 0, \tag{5}$$

Using the homotopy technique, we construct a homotopy:

$$V(r, p) : \Omega \times [0, 1] \to R$$

which satisfies

$$H(V, p) = (1 - p)[L(V) - L(u_0)] + p[A(V) - f(r)]$$

or

$$H(V,p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)], \ r \in \Omega, \quad p \in [0,1], \tag{6}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is the initial approximation of Eq. (3) which satisfies the boundary conditions. Obviously, considering Eq. (6), we will have

$$\begin{aligned} H(V,0) &= L(V) - L(u_0) = 0, \\ H(V,1) &= A(V) - f(r) = 0, \end{aligned} \tag{7}$$

changing the process of p from zero to unity is just that $V(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called the deformation also $A(V) - f(r)$ and $L(u)$ are called as homotopy. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [23-25] to obtain

$$V = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots = \sum_{n=0}^{\infty} p^n v_n. \tag{8}$$

$p \to 1$ results the approximate solution of eq (3) as

$$u = \lim_{p \to 1} V = v_0 + v_1 + v_2 + \ldots = \sum_{n=0}^{\infty} v_n. \tag{9}$$

A comparison of like powers of p gives the solutions of various orders.

Series (9) is convergent for most of the cases. However, convergence rate depends on the nonlinear operator,$N(V)$.

He [25] suggested the following opinions:

1. The second derivative of $N(V)$ with respect to v must be small as the parameter p may be relatively large.

2. The norm of $L^{-1}\frac{\partial N}{\partial u}$ must be smaller than one so that the series converges.

In this section, we describe the above method by the following example to validate the efficiency of the HPM.

Example : 1
Consider the (2+1)-dimensional Equal Width wave equation with diffusion as,

$$u_t = u_{xx} + uu_x + u_{xxt} + u_{yyt}, \tag{10}$$

under the initial condition

$$u(x,y,0) = u_0(x,y) = x + y. \tag{11}$$

Applying the homotopy perturbation method to Eq. (10), we have

$$u_t + p[(-u_{xx} - uu_x - u_{xxt} - u_{yyt}] = 0. \tag{12}$$

In the view of HPM, we use the homotopy parameter p to expand the solution

$$u(x,y,t) = u_0 + pu_1 + p^2u_2 + \ldots \tag{13}$$

The approximate solution can be obtained by taking $p = 1$ in Eq. (13) as

$$u(x,y,t) = u_0 + u_1 + u_2 + \ldots \tag{14}$$

Now substituting from Eq. (12) into Eq. (11) and equating the terms with identical powers of p, we obtain the series of linear equations, which can be easily solved. First few linear equations are given as

$$p^0 : \frac{\partial u_0}{\partial t} = 0. \tag{15}$$

$$p^1 : \frac{\partial u_1}{\partial t} = \frac{\partial^2 u_0}{\partial x^2} + u_0\frac{\partial u_0}{\partial x} + \frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t}. \tag{16}$$

$$p^2 : \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_1}{\partial x^2} + (\left(u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}\right) + \frac{\partial^3 u_1}{\partial x^2 \partial t} + \frac{\partial^3 u_1}{\partial y^2 \partial t}. \tag{17}$$

Using the initial condition (11), the solution of Eq. (15) is given by

$$u(x,y,0) = u_0(x,y) = (x+y). \tag{18}$$

Then the solution of Eq. (16) will be

$$u_1(x,y,t) = \int_0^t \left(\frac{\partial^2 u_0}{\partial x^2} + u_0\frac{\partial u_0}{\partial x} + \frac{\partial^3 u_0}{\partial x^2 \partial t} + \frac{\partial^3 u_0}{\partial y^2 \partial t}.\right) dt. \tag{19}$$

$$u_1(x,y,t) = (x+y)t. \tag{20}$$

Also, we can find the solution of Eq. (17) by using the following formula

$$u_2(x,y,t) = \int_0^t \left(\frac{\partial^2 u_1}{\partial x^2} + (\left(u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}\right) + \frac{\partial^3 u_1}{\partial x^2 \partial t} + \frac{\partial^3 u_1}{\partial y^2 \partial t}.\right) dt. \tag{21}$$

$$u_2(x,y,t) = (x+y)t^2. \tag{22}$$

etc. Therefore, from Eq. (18), the approximate solution of Eq.(10) is given as

$$u(x,y,t) = (x+y) + (x+y)t + (x+y)t^2 + \ldots \tag{23}$$

Hence the exact solution can be expressed as

$$u(x,y,t) = \frac{(x+y)}{1-t}, \tag{24}$$

provided that $0 \leq t < 1$.

2. ADOMAIN DECOMPOSITION METHOD(ADM):

Consider the following linear operator and their inverse operators:

$$L_t = \frac{\partial}{\partial t}; \; L_{x,x} = \frac{\partial^2}{\partial x^2}.$$

$$L_{x,x,t} = \frac{\partial^3}{\partial x^2 \partial t} \; Ł_{y,y,t} = \frac{\partial^3}{\partial y^2 \partial t}.$$

$$L_t^{-1} = \int_0^t (.) d\tau, \; L_{x,x} = \int_0^x \int_0^x (.) d\tau d\tau.$$

Using the above notations, Eq.(1) becomes

$$L_t(u) = L_{x,x}(u) + uu_x + L_{x,x,t}(u) + L_{y,y,t}(u), \tag{25}$$

operating the inverse operators L_t^{-1} to eqn. (25) and using the initial condition gives

$$u(x,y,t) = u_0(x,y,t) + L_t^{-1}(L_{x,x}(u)) + L_t^{-1}(L_{x,x,t}(u)) + L_t^{-1}(L_{y,y,t}(u)) + L_t^{-1}(uu_x). \tag{26}$$

The decomposition method consists of representing the solution $u(x,y,t)$ by the decomposition series

$$u(x,y,t) = \sum_{q=0}^{\infty} u_q(x,y,t). \tag{27}$$

The nonlinear term $u_x u$ is represented by a series of the so called Adomian polynomials, given by

$$u = \sum_{q=0}^{\infty} A_q(x,y,t). \tag{28}$$

The component $u_q(x,y,t)$ of the solution $u(x,y,t)$ is determined in a recursive manner. Replacing the decomposition series (27) and (28) for u into eqn. (26) gives

$$\begin{aligned} \sum_{q=0}^{\infty} u_q(x,y,t) &= u_0(x,y,t) + L_t^{-1}(L_{x,x}(u)) \\ &+ L_t^{-1}(L_{x,x,t}(u)) + L_t^{-1}(L_{y,y,t}(u)) + L_t^{-1}\sum_{q=0}^{\infty} A_q(x,y,t). \end{aligned} \tag{29}$$

According to ADM the zero-th component $u_0(x,y,t)$ is identified from the initial or boundary conditions and from the source terms. The remaining components of $u(x,y,t)$ are determined in a recursion manner as follows

$$u_0(x,y,t) = u_0(x,y), \tag{30}$$

$$u_k(x,y,t) = L_t^{-1}\left(L_{x,x}(u)\right) + L_t^{-1}\left(L_{x,x,t}(u)\right) + L_t^{-1}\left(L_{y,y,t}(u)\right) + L_t^{-1}(A_k),\ \ k \geq 0, \tag{31}$$

where the Adomian polynomials for the nonlinear term $u_x u$ are derived from the following recursive formulation,

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right)_{\lambda=0}, \ \ k = 0,1,2,\ldots. \tag{32}$$

First few Adomian polynomials are given as

$$A_0 = u_0\frac{\partial u_0}{\partial x_1},\ A_1 = u_0\frac{\partial u_1}{\partial x_1} + u_1\frac{\partial u_0}{\partial x_1},$$
$$A_2 = u_2\frac{\partial u_0}{\partial x_1} + u_1\frac{\partial u_1}{\partial x_1} + u_0\frac{\partial u_2}{\partial x_1}, \tag{33}$$

using eq.(31) for the adomian polynomials A_k, we get

$$u_0(x,y,t) = u_0(x,y), \tag{34}$$

$$u_1(x,y,t) = L_t^{-1}\left(L_{x,x}(u_0)\right) + L_t^{-1}\left(L_{x,x,t}(u_0)\right) + L_t^{-1}\left(L_{y,y,t}(u_0)\right) + L_t^{-1}(A_0), \tag{35}$$

$$u_2(x,y,t) = L_t^{-1}\left(L_{x,x}(u_1)\right) + L_t^{-1}\left(L_{x,x,t}(u_1)\right) + L_t^{-1}\left(L_{y,y,t}(u_1)\right) + L_t^{-1}(A_1), \tag{36}$$

and so on. Then the q-th term, u_q can be determined from

$$u_q = \sum_{0}^{q-1} u_k(x,y,t). \tag{37}$$

Knowing the components of u, the analytical solution follows immediately.

Computational Illustrations of ADM for (2+1)-dimensional Equal Width wave equaion with diffusivity:

Using Eqns.(32) and (33), first few components of the decomposition series are given by

$$u_0(x,y,t) = (x+y), \tag{38}$$
$$u_1(x,y,t) = (x+y)t, \tag{39}$$
$$u_2(x,y,t) = (x+y)t^2, \tag{40}$$
$$u_3(x,y,t) = (x+y)t^3, \tag{41}$$

$$\begin{aligned} u(x,y,t) &= \sum_{k=0}^{\infty} u_k(x,y,t), \\ &= u_0(x,y,t) + u_1(x,y,,t) + u_2(x,y,t) + \ldots, \\ &= (x+y) + (x+y)t + (x+y)t^2 + (x+y)t^3 + \ldots \end{aligned} \tag{42}$$

Hence, the exact solution can be expressed as

$$u(x,y,t) = \frac{(x+y)}{1-t}. \tag{43}$$

3. DIFFERENTIAL TRANSFORM METHOD(DTM):

We are using some basic definitions of the differential transformation. Let D denotes the differential transform operator and D^{-1} the inverse differential transform operator.

Basic Definition of DTM:

By the decomposition series, we get the solution:1 If $u(x_1, x_2, \ldots, x_n, t)$ is analytic in the domain Ω, then its $(n+1)-$ dimensional differential transform is given by

$$U(k_1, k_2, \ldots, k_n, k_{n+1}) = \left(\frac{1}{k_1!k_2!\ldots k_n!k_{n+1}!}\right) \times$$
$$\frac{\partial^{k_1+k_2+\ldots+k_n+k_{n+1}}}{\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\ldots\partial_{x_n}^{k_n}\partial_{x_t}^{k_{n+1}}}.u(x_1, x_2, \ldots, x_n, t)|_{x_1=0,x_2=0,\ldots,x_n=0,t=0}, \tag{44}$$

where

$$\begin{aligned} u(x_1, x_2, \ldots, x_n, t) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}\sum_{k_{n+1}=0}^{\infty} U(k_1, k_2, \ldots, k_n, k_{n+1}).x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}t^{k_{n+1}} \\ &= D^{-1}[U(k_1, k_2, \ldots, k_n, k_{n+1})]. \end{aligned} \tag{45}$$

Definition : 2

If $u(x_1, x_2, \ldots, x_n, t) = D^{-1}[U(k_1, k_2, \ldots, k_n, k_{n+1})]$, $v(x_1, x_2, \ldots, x_n, t)$ $= D^{-1}[V(k_1, k_2, \ldots, k_n, k_{n+1})]$, and $\otimes$ denotes convolution, then the fundamental operations of the differential transform are expressed as follows:

$$(a).D[u(x_1, x_2, \ldots, x_n, t)v(x_1, x_2, \ldots, x_n, t)] = U(k_1, k_2, \ldots, k_n, k_{n+1}) \otimes V(k_1, k_2, \ldots, k_n, k_{n+1})$$
$$= \sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\ldots\sum_{a_n=0}^{k_n}\sum_{a_{n+1}=0}^{k_{n+1}} U(a_1, k_2-a_2, \ldots, k_{n+1}-a_{n+1})V(k_1-a_1, a_2, \ldots, a_{n+1}). \tag{46}$$

$$(b).D[\alpha u(x_1, x_2, \ldots, x_n, t) \pm \beta v(x_1, x_2, \ldots, x_n, t)] = \alpha U(k_1, k_2, \ldots, k_n, k_{n+1}) \tag{47}$$

$$(c).D\left\{\frac{\partial^{r_1+r_2+\ldots+r_{n+1}}}{\partial_{x_1}^{r_1}\partial_{x_2}^{r_2}\ldots\partial_{x_n}^{r_n}\partial_t^{r_{n+1}}}u(x_1, x_2, \ldots, x_n, t)\right\}$$
$$= (k_1+1)(k_1+2)\ldots(k_1+r_1)(k_2+1)(k_2+2)\ldots(k_2+r_2)\ldots(k_{n+1}+1)(k_{n+1}+2)$$
$$\ldots(k_{n+1}+r_{n+1}).U(k_1+r_1, \ldots, k_{n+1}+r_{n+1}). \tag{48}$$

Computational Illustrations of (2+1)-dimensional Equal width wave equation with diffusivity:

Here we describe the method explained in the previous section, by the following to validate the efficiency of the DTM.
Consider the $(2+1)-$ dimensional equal width wave equation with diffusion as,

$$u_t = u_{xx} + uu_x + u_{xxt} + u_{yyt}, \tag{49}$$

subject to the initial condition

$$u(x, y, 0) = u_0(x, y) = x + y. \tag{50}$$

Taking the differential transform of eq.(49)., we have

$$\begin{aligned}
&(k_3+1)U(k_1,k_2,k_3+1) = (k_1+2)(k_1+1)U(k_1+1,k_2,k_3)\\
&+(k_1+2)(k_1+1)(k_3+1)U(k_1,k_2,k_3) + (k_2+2)(k_2+1)(k_3+1)U(k_1,k_2,k_3)\\
&+\sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\sum_{a_2=0}^{k_3}(k_1+1-a_1)U(k_1+1-a_1,a_2,a_3)\times U(a_1,k_2-a_2,k_3-a_3).
\end{aligned} \tag{51}$$

From the initial condition eq.(50), it can be seen that

$$u(x, y, 0) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} U(k_1, k_2, 0)(x^{k_1}.y^{k_2}) = x + y, \tag{52}$$

where

$$U(k_1, k_2, 0) = \begin{cases} 1 & if \quad k_i = 1, k_j = 0, i \neq j; i, j = 1, 2. \\ 0 & \qquad otherwise. \end{cases} \tag{53}$$

using eq.(53) into eq.(52) one can obtain the values of $U(k_1, k_2, k_3, k_4)$ as follows:

$$U(k_1, k_2, 1) = \begin{cases} 1 & if \quad k_i = 1, k_j = 0, i \neq j; i, j = 1, 2 \\ 0 & \qquad otherwise. \end{cases} \tag{54}$$

$$U(k_1, k_2, 2) = \begin{cases} 1 & if \quad k_i = 1, k_j = 0, i \neq j; i, j = 1, 2. \\ 0 & \qquad otherwise. \end{cases} \tag{55}$$

Then from eqn.(45) we have

$$\begin{aligned}
u(x, y, t) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} U(k_1, k_2, k_3)x^{k_1}y^{k_2}t^{k_3},\\
&= (x + y) + (x + y)t + (x + y)t^2 + \ldots
\end{aligned} \tag{56}$$

Hence, the exact solution can be expressed as

$$u(x, y, t) = \frac{(x + y)}{1 - t}. \tag{57}$$

Conclusion:

In this work, homotopy perturbation method,DTM and adomian decomposition method have been successfully applied for solving the (2+1)-dimensional Equal Width Wave equation with diffusivity term. The solutions obtained by these methods are an infinite power series for an appropriate initial condition, which can, in turn, be expressed in a closed form, the exact solution. The results reveal that the methods are very effective, convenient and quite accurate mathematical tools for solving (2+1)-dimensional Equal Width Wave equation with diffusivity. These methods, which can be used without any need to complex computations except simple and elementary operations, are also promising techniques for solving other nonlinear problems.

REFERENCES:

1. Z. Odibat, S. Momani, *Chaos Solitons Fractals*, in press.
2. J.H. He, *Comput. Methods Appl. Mech. Engrg.* 178,(1999),257.
3. J.H. He, *Int. J. Non-Linear Mech.*, 35 (1), (2000), 37.
4. M. El-Shahed, *Int. J. Nonlin. Sci. Numer. Simul.*, 6 (2),(2005),163.
5. J.H. He, *Appl. Math. Comput.*, 151, (2004), 287.
6. J.H. He, *Int. J. Nonlin. Sci. Numer. Simul.*, 6 (2),(2005),207.
7. J.H. He, *Phys. Lett. A* ,374, (4-6), (2005), 228.
8. J.H. He, *Chaos Solitons Fractals*, 26 (3), (2005), 695.
9. J.H. He, *Phys. Lett. A*, 350, (1-2), (2006), 87.
10. J.H. He,*Appl. Math. Comput.*, 135, (2003), 73.
11. J.H. He, *Appl. Math. Comput.*,156, (2004), 527.
12. J.H. He, *Appl. Math. Comput.*, 156,(2004), 591.
13. J.H. He, *Chaos Solitons Fractals*, 26, (3), (2005), 827.
14. A. Siddiqui, R. Mahmood, Q. Ghori, *Int. J. Nonlin. Sci. Numer. Simul.*, 7 (1), (2006), 7.
15. A. Siddiqui, M. Ahmed, Q. Ghori, *Int. J. Nonlin. Sci. Numer. Simul.*, 7 (1), (2006), 15.

16. J.H. He, *Int. J. Mod. Phys. B*, 20 (10), (2006), 1141.

17. S. Abbasbandy, *Appl. Math. Comput.*, 172, (2006), 485.

18. S. Abbasbandy, *Appl. Math. Comput.*, 173, (2006), 493.

19. S. Padmasekaran, S.Rajeswari and G. Sivagami,*Similarity Solution of Semilinear Parabolic Equations with Variable Coefficients*,IJMAA,Volume 4, Issue 3A (2016), 201209.

20. S.Padmasekaran and S.Rajeswari,*Solitons and Exponential Solutions for a Nonlinear (2+1)dim PDE*, IJPAM,Volume 115 No. 9 2017, 121-130.

21. S. Padmasekaran and S. Rajeswari, *Lies Symmetries of (2+1)dim PDE*IJMTT,Volume 51 No. 6,11-2017.

22. G. Adomian. *A review of the decomposition method in applied mathematics.*, J Math Anal Appl, (1988), 135, 501-544.

23. F.J. Alexander, J.L. Lebowitz . *Driven diffusive systems with a moving obstacle: a variation on the Brazil nuts problem. J Phys* (1990), 23, 375-382.

24. F.J. Alexander, J.L. Lebowitz.,*On the drift and diffusion of a rod in a lattice fluid. J Phys*, (1994), 27, 683-696.

25. J.D. Cole., *On a quasilinear parabolic equation occurring in aerodynamics. J Math Appl*, (1988), 135, 501-544.

26. He JH. *Homotopy perturbation technique. Comput Methods Appl Mech Eng*, (1999), 178, 257-262.

27. J.H. He.,*Homotopy perturbation method: a new nonlinear analytical technique.*, Appl Math Comput, (2003), 13, (2-3), 73-79.

28. J.H. He.,*A simple perturbation method to Blasius equation. Appl Math Comput*, (2003), (2-3), 217-222.