Computation of Various Queue Characteristics using Tri-Cum Biserial Queuing Model Connected with a Common Server

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Abstract: *The present paper deals with the investigation of various characteristics of a developed queuing model in which three servers are connected in parallel in tri-cum biserial way with a common server. The arrival and service pattern are assumed to follow the Poisson law. Queue length, variance, average waiting time and probabilities have been estimated using statistical tools, generating function technique and law of calculus. Extensive parametric study has been presented to show the efficacy of the present solution methodology. Present study demonstrates that the queuing theory provides a good approach to analyze the complicated probabilistic and deterministic systems.*

Keywords: *Queuing model, Poisson law, Variance, Probabilities.*

I. INTRODUCTION

Queuing theory has been widely used in the traditional service trade such as in supermarkets, restaurants, manufacturing processes, bus scheduling and hospital appointment bookings etc. One of the anticipated advantages from studying queuing systems is the assessment of the effectiveness of models in terms of utilization of the resources and waiting length. Consequently optimize the number of queues such that customers need not to wait longer when servers are too busy. Several literatures are available dealing with the realistic problems based on various queuing models. Jacksons [1] investigated the various characteristics of a queue system encompassing phase type service. Maggu [2] emphasized on phase type service queues with two servers in biseries to study the queue model. Singh *et al* [3] used parallel biserial queues to examine the transient behavior of a queuing network. Gupta *et al* [4] presented an extensive parametric study to explore the queuing model consist of biserial and parallel channels associated with a common server.

The state characteristics of a queue model having two subsystems with bi-serial channels connected with a common channel has been investigated by Kumar et al [5]. Gupta et al [6] presented a detailed investigation on the linkage of a flowshop scheduling model with a parallel biserial queue network. This work has been further explored by Seema et al [7] to optimize total flow time, waiting time and service time.

II. APPLICATION IN REAL TIME DOMAIN

Queuing theory can be implemented to a diversity of operational situations where it is not possible to precisely calculate the arrival rate (or time) of customers and service rate (or time) of service facilities. For example: In shopping mall, there are several sections such as electronic, garments, food etc. The customer entered in the mall can avail all the facilities available in the mall or can also enjoy some of them. It depends on the time available with the customer and the crowed in difference sections. Such problems can be dealt easily using present model.

III. MATHEMATICAL MODEL

In the present work, a queue model consist of three servers (Sr_a, Sr_b and Sr_c) are connected in parallel in tri-cum biserial way which are further connected to a common server Sr_d. Let Q_a, Q_b ,Q_c and Q_d are the queues associated with servers Sr_a, Sr_b, Sr_c and Sr_d respectively. The number of customers (n_a) coming at mean arrival rate (λ_a) after completion of service at server Sr_a, can avail the facility at server Sr_b or Sr_c (either of two or both) with the probabilities p_{ab} *and* p_{ac} or directly can avail the facility at server Sr_d with the probability p_{ad} such that $p_{ab} + p_{ac} + p_{ad} = 1$. The same criterion will be applicable to those customers who entered in servers Sr_b and Sr_c. The pictorial representation of the considered problem is illustrated in Fig 1.

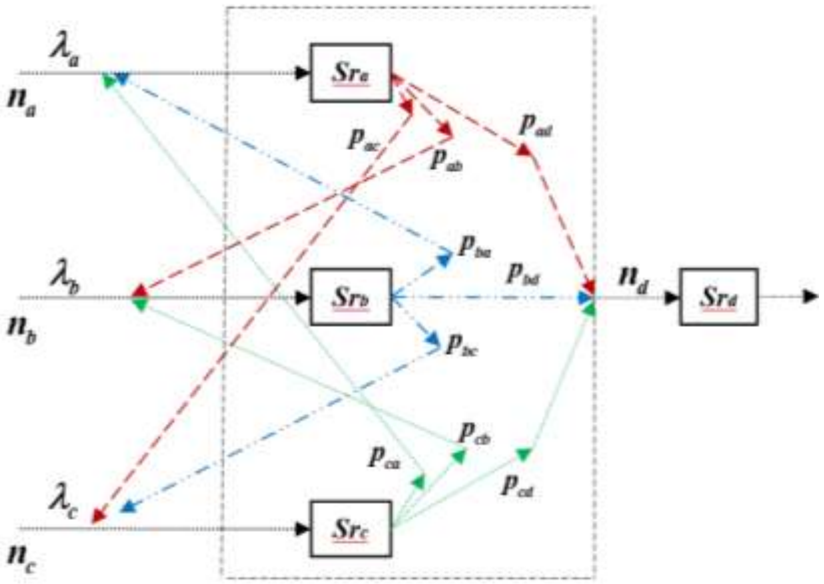

Fig 1**:** Linkage model of the queuing network

The governing differential difference equation in steady (transient) state of the model can be written as Eq. 1.

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c+\mu_d)P_{n_a,n_b,n_c,n_d} &= \lambda_a P_{n_a-1,n_b,n_c,n_d}+\lambda_b P_{n_a,n_b-1,n_c,n_d}+\lambda_c P_{n_a,n_b,n_c-1,n_d} \\
&+\mu_a p_{ab} P_{n_a+1,n_b-1,n_c,n_d}+\mu_a p_{ac} P_{n_a+1,n_b,n_c-1,n_d}+\mu_a p_{ad} P_{n_a+1,n_b,n_c,n_d-1} \\
&+\mu_b p_{ba} P_{n_a-1,n_b+1,n_c,n_d}+\mu_b p_{bc} P_{n_a,n_b+1,n_c-1,n_d}+\mu_b p_{bd} P_{n_a,n_b+1,n_c,n_d-1} \\
&+\mu_c p_{cb} P_{n_a,n_b-1,n_c+1,n_d}+\mu_c p_{ca} P_{n_a-1,n_b,n_c+1,n_d}+\mu_c p_{cd} P_{n_a,n_b,n_c+1,n_d-1} \\
&+\mu_d P_{n_a,n_b,n_c,n_d+1}
\end{aligned}
\quad (1)
$$

Considering $n_a = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_b+\mu_c+\mu_d)P_{0,n_b,n_c,n_d} &= \lambda_b P_{0,n_b-1,n_c,n_d}+\lambda_c P_{0,n_b,n_c-1,n_d} \\
&+\mu_a p_{ab} P_{1,n_b-1,n_c,n_d}+\mu_a p_{ac} P_{1,n_b,n_c-1,n_d}+\mu_a p_{ad} P_{1,n_b,n_c,n_d-1} \\
&+\mu_b p_{bc} P_{0,n_b+1,n_c-1,n_d}+\mu_b p_{bd} P_{0,n_b+1,n_c,n_d-1} \\
&+\mu_c p_{cb} P_{0,n_b-1,n_c+1,n_d}+\mu_c p_{cd} P_{0,n_b,n_c+1,n_d-1} \\
&+\mu_d P_{0,n_b,n_c,n_d+1}
\end{aligned}
\quad (2)
$$

Considering $n_b = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_c+\mu_d)P_{n_a,0,n_c,n_d} &= \lambda_a P_{n_a-1,0,n_c,n_d}+\lambda_c P_{n_a,0,n_c-1,n_d} \\
&+\mu_a p_{ac} P_{n_a+1,0,n_c-1,n_d}+\mu_a p_{ad} P_{n_a+1,0,n_c,n_d-1} \\
&+\mu_b p_{ba} P_{n_a-1,1,n_c,n_d}+\mu_b p_{bc} P_{n_a,1,n_c-1,n_d}+\mu_b p_{bd} P_{n_a,1,n_c,n_d-1} \\
&+\mu_c p_{ca} P_{n_a-1,0,n_c+1,n_d}+\mu_c p_{cd} P_{n_a,0,n_c+1,n_d-1} \\
&+\mu_d P_{n_a,0,n_c,n_d+1}
\end{aligned}
\quad (3)
$$

Considering $n_c = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_d)P_{n_a,n_b,0,n_d} = {} & \lambda_a P_{n_a-1,n_b,0,n_d} + \lambda_b P_{n_a,n_b-1,0,n_d} \\
& + \mu_a p_{ab} P_{n_a+1,n_b-1,0,n_d} + \mu_a p_{ad} P_{n_a+1,n_b,0,n_d-1} \\
& + \mu_b p_{ba} P_{n_a-1,n_b+1,0,n_d} + \mu_b p_{bd} P_{n_a,n_b+1,0,n_d-1} \\
& + \mu_c p_{cb} P_{n_a,n_b-1,1,n_d} + \mu_c p_{ca} P_{n_a-1,n_b,1,n_d} + \mu_c p_{cd} P_{n_a,n_b,1,n_d-1} \\
& + \mu_d P_{n_a,n_b,0,n_d+1}
\end{aligned} \tag{4}
$$

Considering $n_d = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b+\mu_c)P_{n_a,n_b,n_c,0} = {} & \lambda_a P_{n_a-1,n_b,n_c,0} + \lambda_b P_{n_a,n_b-1,n_c,0} + \lambda_3 P_{n_a,n_b,n_c-1,0} \\
& + \mu_a p_{ab} P_{n_a+1,n_b-1,n_c,0} + \mu_a p_{ac} P_{n_a+1,n_b,n_c-1,0} \\
& + \mu_b p_{ba} P_{n_a-1,n_b+1,n_c,0} + \mu_b p_{bc} P_{n_a,n_b+1,n_c-1,0} \\
& + \mu_c p_{cb} P_{n_a,n_b-1,n_c+1,0} + \mu_c p_{ca} P_{n_a-1,n_b,n_c+1,0} \\
& + \mu_d P_{n_a,n_b,n_c,1}
\end{aligned} \tag{5}
$$

Considering $n_a = 0$ & $n_b = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_c+\mu_d)P_{0,0,n_c,n_d} = {} & \lambda_c P_{0,0,n_c-1,n_d} \\
& + \mu_a p_{ac} P_{1,0,n_c-1,n_d} + \mu_a p_{ad} P_{1,0,n_c,n_d-1} \\
& + \mu_b p_{bc} P_{0,1,n_c-1,n_d} + \mu_b p_{bd} P_{0,1,n_c,n_d-1} \\
& + \mu_c p_{cd} P_{0,0,n_c+1,n_d-1} \\
& + \mu_d P_{0,0,n_c,n_d+1}
\end{aligned} \tag{6}
$$

Considering $n_a = 0$ & $n_c = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_b+\mu_d)P_{0,n_b,0,n_d} = {} & \lambda_b P_{0,n_b-1,0,n_d} \\
& + \mu_a p_{ab} P_{1,n_b-1,0,n_d} + \mu_a p_{ad} P_{1,n_b,0,n_d-1} \\
& + \mu_b p_{bd} P_{0,n_b+1,0,n_d-1} \\
& + \mu_c p_{cb} P_{0,n_b-1,1,n_d} + \mu_c p_{cd} P_{0,n_b,1,n_d-1} \\
& + \mu_d P_{0,n_b,0,n_d+1}
\end{aligned} \tag{7}
$$

Considering $n_a = 0$ & $n_d = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_b+\mu_c)P_{0,n_b,n_c,0} = {} & \lambda_b P_{0,n_b-1,n_c,0} + \lambda_c P_{0,n_b,n_c-1,0} \\
& + \mu_a p_{ab} P_{1,n_b-1,n_c,0} + \mu_a p_{ac} P_{1,n_b,n_c-1,0} \\
& + \mu_b p_{bc} P_{0,n_b+1,n_c-1,0} \\
& + \mu_c p_{cb} P_{0,n_b-1,n_c+1,0} \\
& + \mu_d P_{0,n_b,n_c,1}
\end{aligned} \tag{8}
$$

Considering $n_b = 0$ & $n_c = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_d)P_{n_a,0,0,n_d} = {} & \lambda_a P_{n_a-1,0,0,n_d} \\
& + \mu_a p_{ad} P_{n_a+1,0,0,n_d-1} \\
& + \mu_b p_{ba} P_{n_a-1,1,0,n_d} + \mu_b p_{bd} P_{n_a,1,0,n_d-1} \\
& + \mu_c p_{ca} P_{n_a-1,0,1,n_d} + \mu_c p_{cd} P_{n_a,0,1,n_d-1} \\
& + \mu_d P_{n_a,0,0,n_d+1}
\end{aligned} \tag{9}
$$

Considering $n_b = 0$ & $n_d = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_c)P_{n_a,0,n_c,0} = {} & \lambda_a P_{n_a-1,0,n_c,0}+\lambda_c P_{n_a,0,n_c-1,0} \\
& +\mu_a p_{ac} P_{n_a+1,0,n_c-1,0} \\
& +\mu_b p_{ba} P_{n_a-1,1,n_c,0}+\mu_b p_{bc} P_{n_a,1,n_c-1,0} \\
& +\mu_c p_{ca} P_{n_a-1,0,n_c+1,0} \\
& +\mu_d P_{n_a,0,n_c,1}
\end{aligned} \tag{10}
$$

Considering $n_c = 0$ & $n_d = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a+\mu_b)P_{n_a,n_b,0,0} = {} & \lambda_a P_{n_a-1,n_b,0,0}+\lambda_b P_{n_a,n_b-1,0,0} \\
& +\mu_a p_{ab} P_{n_a+1,n_b-1,0,0} \\
& +\mu_b p_{ba} P_{n_a-1,n_b+1,0,0} \\
& +\mu_c p_{cb} P_{n_a,n_b-1,1,0}+\mu_c p_{ca} P_{n_a-1,n_b,1,0} \\
& +\mu_d P_{n_a,n_b,0,1}
\end{aligned} \tag{11}
$$

Considering $n_a = 0, n_b = 0$ & $n_c = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_d)P_{0,0,0,n_d} = {} & \mu_a p_{ad} P_{1,0,0,n_d-1} \\
& +\mu_b p_{bd} P_{0,1,0,n_d-1} \\
& +\mu_c p_{cd} P_{0,0,1,n_d-1} \\
& +\mu_d P_{0,0,0,n_d+1}
\end{aligned} \tag{12}
$$

Considering $n_a = 0, n_b = 0$ & $n_d = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_c)P_{0,0,n_c,0} = {} & \lambda_c P_{0,0,n_c-1,0} \\
& +\mu_a p_{ac} P_{1,0,n_c-1,0} \\
& +\mu_b p_{bc} P_{0,1,n_c-1,0} \\
& +\mu_d P_{0,0,n_c,1}
\end{aligned} \tag{13}
$$

Considering $n_b = 0, n_c = 0$ & $n_d = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_a)P_{n_a,0,0,0} = {} & \lambda_a P_{n_a-1,0,0,0} \\
& +\mu_b p_{ba} P_{n_a-1,1,0,0} \\
& +\mu_c p_{ca} P_{n_a-1,0,1,0} \\
& +\mu_d P_{n_a,0,0,1}
\end{aligned} \tag{14}
$$

Considering $n_a = 0, n_c = 0$ & $n_d = 0$; the Eq (1) can be given as

$$
\begin{aligned}
(\lambda_a+\lambda_b+\lambda_c+\mu_b)P_{0,n_b,0,0} = {} & \lambda_b P_{0,n_b-1,0,0} \\
& +\mu_a p_{ab} P_{1,n_b-1,0,0} \\
& +\mu_c p_{cb} P_{0,n_b-1,1,0} \\
& +\mu_d P_{0,n_b,0,1}
\end{aligned} \tag{15}
$$

Considering $n_a = 0, n_b = 0, n_c = 0$ & $n_d = 0$; the Eq (1) can be given as

$$
(\lambda_a+\lambda_b+\lambda_c)P_{0,0,0,0} = \mu_d P_{0,0,0,1} \tag{16}
$$

IV. SOLUTION METHODOLOGY

To solve the governing Equations, Generating function is assumed as

$$f(X_1, X_2, X_3, X_4) = \sum_{n_a=0}^{\infty} \sum_{n_b=0}^{\infty} \sum_{n_c=0}^{\infty} \sum_{n_d=0}^{\infty} X_1^{n_a} X_2^{n_b} X_3^{n_c} X_4^{n_d} \quad (17)$$

such that $|X_1| = |X_2| = |X_3| = |X_4| = 1$

Also, taking partial generating function as

$$f_{n_b, n_c, n_d}(X_1) = \sum_{n_a=0}^{\infty} P_{n_a, n_b, n_c, n_d} X_1^{n_a} \quad (18)$$

$$f_{n_c, n_d}(X_1, X_2) = \sum_{n_b=0}^{\infty} f_{n_b, n_c, n_d}(X_1).X_2^{n_b} \quad (19)$$

$$f_{n_d}(X_1, X_2, X_3) = \sum_{n_c=0}^{\infty} f_{n_c, n_d}(X_1, X_2).X_3^{n_c} \quad (20)$$

$$f(X_1, X_2, X_3, X_4) = \sum_{n_d=0}^{\infty} f(X_1, X_2, X_3).X_4^{n_d} \quad (21)$$

On solving equations from (1) to (16) with the help of (17), (18), (19), (20), (21), we get [2, 3]

$$f(X_1, X_2, X_3, X_4) = \frac{\begin{array}{r} \mu_a \left\{1 - \frac{p_{ab} X_2}{X_1} - \frac{p_{ac} X_3}{X_1} - \frac{p_{ad} X_4}{X_1}\right\} f(X_2, X_3, X_4) + \mu_b \left\{1 - \frac{p_{ba} X_1}{X_2} - \frac{p_{bc} X_3}{X_2} - \frac{p_{bd} X_4}{X_2}\right\} f(X_1, X_3, X_4) \\ + \mu_c \left\{1 - \frac{p_{ca} X_1}{X_3} - \frac{p_{cb} X_2}{X_3} - \frac{p_{cd} X_4}{X_3}\right\} f(X_1, X_2, X_4) + \mu_d \left\{1 - \frac{1}{X_4}\right\} f(X_1, X_2, X_3) \end{array}}{\begin{array}{r} \lambda_a (1 - X_1) + \lambda_b (1 - X_2) + \lambda_c (1 - X_3) + \mu_a \left\{1 - \frac{p_{ab} X_2}{X_1} - \frac{p_{ac} X_3}{X_1} - \frac{p_{ad} X_4}{X_1}\right\} \\ + \mu_b \left\{1 - \frac{p_{ba} X_1}{X_2} - \frac{p_{bc} X_3}{X_2} - \frac{p_{bd} X_4}{X_2}\right\} + \mu_c \left\{1 - \frac{p_{ca} X_1}{X_3} - \frac{p_{cb} X_2}{X_3} - \frac{p_{cd} X_4}{X_3}\right\} + \mu_d \left\{1 - \frac{1}{X_4}\right\} \end{array}}$$

Assuming $f(X_2, X_3, X_4) = f_a$, $f(X_1, X_3, X_4) = f_b$, $f(X_1, X_2, X_4) = f_c$, $f(X_1, X_2, X_3) = f_d$

$$f(X_1, X_2, X_3, X_4) = \frac{\begin{array}{r} \mu_a \left\{1 - \frac{p_{ab} X_2}{X_1} - \frac{p_{ac} X_3}{X_1} - \frac{p_{ad} X_4}{X_1}\right\} f_a + \mu_b \left\{1 - \frac{p_{ba} X_1}{X_2} - \frac{p_{bc} X_3}{X_2} - \frac{p_{bd} X_4}{X_2}\right\} f_b \\ + \mu_c \left\{1 - \frac{p_{ca} X_1}{X_3} - \frac{p_{cb} X_2}{X_3} - \frac{p_{cd} X_4}{X_3}\right\} f_c + \mu_d \left\{1 - \frac{1}{X_4}\right\} f_d \end{array}}{\begin{array}{r} \lambda_a (1 - X_1) + \lambda_b (1 - X_2) + \lambda_c (1 - X_3) + \mu_a \left\{1 - \frac{p_{ab} X_2}{X_1} - \frac{p_{ac} X_3}{X_1} - \frac{p_{ad} X_4}{X_1}\right\} \\ + \mu_b \left\{1 - \frac{p_{ba} X_1}{X_2} - \frac{p_{bc} X_3}{X_2} - \frac{p_{bd} X_4}{X_2}\right\} + \mu_c \left\{1 - \frac{p_{ca} X_1}{X_3} - \frac{p_{cb} X_2}{X_3} - \frac{p_{cd} X_4}{X_3}\right\} + \mu_d \left\{1 - \frac{1}{X_4}\right\} \end{array}}$$

Since $f(1, 1, 1, 1) = 1$, the total probability. Considering $X_1 = 1$ as $X_2 \to 1, X_3 \to 1, X_4 \to 1$, $f(X_1, X_2, X_3, X_4)$ is of (0/0) indeterminate form. Therefore, using L- Hospital rule, we get

$$1 = \frac{\mu_a (p_{ab} + p_{ac} + p_{ad}) f_a + \mu_b (-p_{ba}) f_b + \mu_c (-p_{ca}) f_c}{-\lambda_a + \mu_a (p_{ab} + p_{ac} + p_{ad}) + \mu_b (-p_{ba}) + \mu_c (-p_{ca})}$$

Where, $p_{ab} + p_{ac} + p_{ad} = 1$

$$\mu_a f_a - \mu_b p_{ba} f_b - \mu_c p_{ca} f_c = -\lambda_a + \mu_a - \mu_b p_{ba} - \mu_c p_{ca} \qquad (22)$$

Again differentiating numerator and denominator separately w.r.t. X_2 by taking $X_2 = 1$ as $X_1 \to 1,\ X_3 \to 1,\ X_4 \to 1$, we get

$$1 = \frac{\mu_a(-p_{ab}) f_a + \mu_b(p_{ba} + p_{bc} + p_{bd}) f_b + \mu_c(-p_{cb}) f_c}{-\lambda_b + \mu_a(-p_{ab}) + \mu_b(p_{ba} + p_{bc} + p_{bd}) + \mu_c(-p_{cb})}$$

where $p_{ba} + p_{bc} + p_{bd} = 1$

$$-\mu_a p_{ab} f_a + \mu_b f_b - \mu_c p_{cb} f_c = -\lambda_b - \mu_a p_{ab} + \mu_b - \mu_c p_{cb} \qquad (23)$$

Again differentiating numerator and denominator separately w.r.t. X_3 by taking $X_3 = 1$ as $X_1 \to 1,\ X_2 \to 1,\ X_4 \to 1$, we get

$$1 = \frac{\mu_a(-p_{ac}) f_a + \mu_b(-p_{bc}) f_b + \mu_c(p_{ca} + p_{cb} + p_{cd}) f_c}{-\lambda_c + \mu_a(-p_{ac}) + \mu_b(-p_{bc}) + \mu_c(p_{ca} + p_{cb} + p_{cd})}$$

where $p_{ca} + p_{cb} + p_{cd} = 1$

$$-\mu_a p_{ac} f_a - \mu_b p_{bc} f_b + \mu_c f_c = -\lambda_c - \mu_a p_{ac} - \mu_b p_{bc} + \mu_c \qquad (24)$$

Again differentiating numerator and denominator separately w.r.t. X_4 by taking $X_4 = 1$ as $X_1 \to 1,\ X_2 \to 1,\ X_3 \to 1$, we get

$$1 = \frac{\mu_a(-p_{ad}) f_a + \mu_b(-p_{bd}) f_b + \mu_c(-p_{cd}) f_c + \mu_d f_d}{\mu_a(-p_{ad}) + \mu_b(-p_{bd}) + \mu_c(-p_{cd}) + \mu_d}$$

$$-\mu_a p_{ad} f_a - \mu_b p_{bd} f_b - \mu_c p_{cd} f_c + \mu_d f_d = -\mu_a p_{ad} - \mu_b p_{bd} - \mu_c p_{cd} + \mu_d \qquad (25)$$

On solving (22) , (23), (24) and (25), we get:

$$f_a = 1 - \frac{\lambda_a(1 - p_{bc} p_{cb}) + \lambda_b(p_{ba} + p_{bc} p_{ca}) + \lambda_c\{p_{ca}(1 - p_{bc} p_{cb}) + p_{cb}(p_{ba} + p_{bc} p_{ca})\}}{\mu_a\{(1 - p_{ac} p_{ca})(1 - p_{bc} p_{cb}) - (p_{ab} + p_{ac} p_{cb})(p_{ba} + p_{bc} p_{ca})\}}$$

$$f_b = 1 - \frac{\lambda_a\{p_{ab}(1 - p_{ca} p_{ac}) + p_{ac}(p_{cb} + p_{ca} p_{ab})\} + \lambda_b(1 - p_{ca} p_{ac}) + \lambda_c(p_{cb} + p_{ca} p_{ab})}{\mu_b\{(1 - p_{ba} p_{ab})(1 - p_{ca} p_{ac}) - (p_{bc} + p_{ba} p_{ac})(p_{cb} + p_{ca} p_{ab})\}}$$

$$f_c = 1 - \frac{\lambda_a(p_{ac} + p_{ab} p_{bc}) + \lambda_b\{p_{bc}(1 - p_{ab} p_{ba}) + p_{ba}(p_{ac} + p_{ab} p_{bc})\} + \lambda_c(1 - p_{ab} p_{ba})}{\mu_c\{(1 - p_{cb} p_{bc})(1 - p_{ab} p_{ba}) - (p_{ca} + p_{cb} p_{ba})(p_{ac} + p_{ab} p_{bc})\}}$$

$$\begin{aligned} f_d = 1 &- \mu_a \frac{p_{ad}}{\mu_d}\left[\frac{\lambda_a(1 - p_{bc} p_{cb}) + \lambda_b(p_{ba} + p_{bc} p_{ca}) + \lambda_c\{p_{ca}(1 - p_{bc} p_{cb}) + p_{cb}(p_{ba} + p_{bc} p_{ca})\}}{\mu_a\{(1 - p_{ac} p_{ca})(1 - p_{bc} p_{cb}) - (p_{ab} + p_{ac} p_{cb})(p_{ba} + p_{bc} p_{ca})\}}\right] \\ &- \mu_b \frac{p_{bd}}{\mu_d}\left[\frac{\lambda_a\{p_{ab}(1 - p_{ca} p_{ac}) + p_{ac}(p_{cb} + p_{ca} p_{ab})\} + \lambda_b(1 - p_{ca} p_{ac}) + \lambda_c(p_{cb} + p_{ca} p_{ab})}{\mu_b\{(1 - p_{ba} p_{ab})(1 - p_{ca} p_{ac}) - (p_{bc} + p_{ba} p_{ac})(p_{cb} + p_{ca} p_{ab})\}}\right] \\ &- \mu_c \frac{p_{cd}}{\mu_d}\left[\frac{\lambda_a(p_{ac} + p_{ab} p_{bc}) + \lambda_b\{p_{bc}(1 - p_{ab} p_{ba}) + p_{ba}(p_{ac} + p_{ab} p_{bc})\} + \lambda_c(1 - p_{ab} p_{ba})}{\mu_c\{(1 - p_{cb} p_{bc})(1 - p_{ab} p_{ba}) - (p_{ca} + p_{cb} p_{ba})(p_{ac} + p_{ab} p_{bc})\}}\right] \end{aligned}$$

The solution (Joint Probability) is written as

$$P_{n_a, n_b, n_c, n_d} = (1 - f_a)^{n_a}(1 - f_b)^{n_b}(1 - f_c)^{n_c}(1 - f_d)^{n_d}\ f_a f_b f_c f_d$$

$$P_{n_a,n_b,n_c,n_d} = \rho_a^{\,n_a}\ \rho_b^{\,n_b}\ \rho_c^{\,n_c}\ \rho_d^{\,n_d}\ (1-\rho_a)(1-\rho_b)(1-\rho_c)(1-\rho_d)$$

where $\rho_a = 1 - f_a,\ \rho_b = 1 - f_b,\ \rho_c = 1 - f_c,\ \rho_d = 1 - f_d$

$$\rho_a = \frac{\lambda_a (1-p_{bc}p_{cb}) + \lambda_b (p_{ba} + p_{bc}p_{ca}) + \lambda_c \{p_{ca}(1-p_{bc}p_{cb}) + p_{cb}(p_{ba}+p_{bc}p_{ca})\}}{\mu_a \{(1-p_{ac}p_{ca})(1-p_{bc}p_{cb}) - (p_{ab}+p_{ac}p_{cb})(p_{ba}+p_{bc}p_{ca})\}}$$

$$\rho_b = \frac{\lambda_a \{p_{ab}(1-p_{ca}p_{ac}) + p_{ac}(p_{cb}+p_{ca}p_{ab})\} + \lambda_b (1-p_{ca}p_{ac}) + \lambda_c (p_{cb}+p_{ca}p_{ab})}{\mu_b \{(1-p_{ba}p_{ab})(1-p_{ca}p_{ac}) - (p_{bc}+p_{ba}p_{ac})(p_{cb}+p_{ca}p_{ab})\}}$$

$$\rho_c = \frac{\lambda_a (p_{ac}+p_{ab}p_{bc}) + \lambda_b \{p_{bc}(1-p_{ab}p_{ba}) + p_{ba}(p_{ac}+p_{ab}p_{bc})\} + \lambda_c (1-p_{ab}p_{ba})}{\mu_c \{(1-p_{cb}p_{bc})(1-p_{ab}p_{ba}) - (p_{ca}+p_{cb}p_{ba})(p_{ac}+p_{ab}p_{bc})\}}$$

$$\rho_d = \frac{p_{ad}}{\mu_d}\left[\frac{\lambda_a (1-p_{bc}p_{cb}) + \lambda_b (p_{ba} + p_{bc}p_{ca}) + \lambda_c \{p_{ca}(1-p_{bc}p_{cb}) + p_{cb}(p_{ba}+p_{bc}p_{ca})\}}{\{(1-p_{ac}p_{ca})(1-p_{bc}p_{cb}) - (p_{ab}+p_{ac}p_{cb})(p_{ba}+p_{bc}p_{ca})\}}\right]$$
$$+ \frac{p_{bd}}{\mu_d}\left[\frac{\lambda_a \{p_{ab}(1-p_{ca}p_{ac}) + p_{ac}(p_{cb}+p_{ca}p_{ab})\} + \lambda_b (1-p_{ca}p_{ac}) + \lambda_c (p_{cb}+p_{ca}p_{ab})}{\{(1-p_{ba}p_{ab})(1-p_{ca}p_{ac}) - (p_{bc}+p_{ba}p_{ac})(p_{cb}+p_{ca}p_{ab})\}}\right]$$
$$+ \frac{p_{cd}}{\mu_d}\left[\frac{\lambda_a (p_{ac}+p_{ab}p_{bc}) + \lambda_b \{p_{bc}(1-p_{ab}p_{ba}) + p_{ba}(p_{ac}+p_{ab}p_{bc})\} + \lambda_c (1-p_{ab}p_{ba})}{\{(1-p_{cb}p_{bc})(1-p_{ab}p_{ba}) - (p_{ca}+p_{cb}p_{ba})(p_{ac}+p_{ab}p_{bc})\}}\right]$$

The solution of this model exist if $\rho_a,\ \rho_b,\ \rho_c,\ \rho_d < 1$ (26)

V. SYSTEM CHARECTERSTICS

(1) Mean queue length (average number of customers)

$$L_Q = L_a + L_b + L_c + L_d$$

$$L_Q = \frac{\rho_a}{1-\rho_a} + \frac{\rho_b}{1-\rho_b} + \frac{\rho_c}{1-\rho_c} + \frac{\rho_d}{1-\rho_d}$$

Where $L_a = \frac{\rho_a}{1-\rho_a},\quad L_b = \frac{\rho_b}{1-\rho_b},\quad L_c = \frac{\rho_c}{1-\rho_c},\quad L_d = \frac{\rho_d}{1-\rho_d}$

(2) Fluctuation (Variance) in queue length

$$V_{ar} = V_a + V_b + V_c + V_d$$

$$V_{ar} = \frac{\rho_a}{(1-\rho_a)^2} + \frac{\rho_b}{(1-\rho_b)^2} + \frac{\rho_c}{(1-\rho_c)^2} + \frac{\rho_d}{(1-\rho_d)^2}$$

Where $V_a = \frac{\rho_a}{(1-\rho_a)^2},\quad V_b = \frac{\rho_b}{(1-\rho_b)^2},\quad V_c = \frac{\rho_c}{(1-\rho_c)^2},\quad V_d = \frac{\rho_d}{(1-\rho_d)^2}$

(3) Average waiting time for customer

$$E_{wt} = \frac{L_Q}{\lambda_{sum}},\quad where\ \ \lambda_{sum} = \lambda_a + \lambda_b + \lambda_c$$

VI. PARAMETRIC STUDY

After developing the governing equations using mathematical model as discussed in section 3 and 4, following input parameters given in Table 1, have been used in the calculation of the results. In Table 2, various utilization of servers and joint probabilities have been shown for various values of mean arrival rate (λ_a ,λ_b and λ_c). It is evident from the computed results that the values utilization of servers is less than 1 which also satisfies Eq. (26). In Table 3 and Table 4, the average waiting time, queue lengths and variances have been calculated keeping different mean arrival rate (λ_a ,λ_b and λ_c) same as considered in Table 2.

Table 1: Various input parameters considered in computation of results

μ_a	μ_b	μ_c	μ_d	p_{ab}	p_{ac}	p_{ad}	p_{ba}	p_{bc}	p_{bd}	p_{ca}	p_{cb}	p_{cd}	n_a	n_b	n_c	n_d
13	12	12	14	0.3	0.3	0.4	0.3	0.3	0.4	0.3	0.3	0.4	2	3	4	9

Table 2: Joint Probability and utilization of servers for various mean arrival rates

λ_a	λ_b	λ_c	ρ_a	ρ_b	ρ_c	ρ_d	P
1	3	4	0.414201	0.576923	0.641026	0.571429	1.377×10^{-6}
2			0.517751	0.625	0.689103	0.642857	5.556×10^{-6}
3			0.621302	0.673077	0.737179	0.714286	1.564×10^{-5}
4			0.724852	0.721154	0.785256	0.785714	3.018×10^{-5}
5			0.828402	0.769231	0.833333	0.857143	3.546×10^{-5}
6			0.931953	0.817308	0.88141	0.928571	1.546×10^{-5}
λ_a	λ_b	λ_c	ρ_a	ρ_b	ρ_c	ρ_d	P
2	1	4	0.428994	0.400641	0.592949	0.5	1.99×10^{-7}
	2		0.473373	0.512821	0.641026	0.571429	1.308×10^{-6}
	3		0.517751	0.625	0.689103	0.642857	5.556×10^{-6}
	4		0.56213	0.737179	0.737179	0.714286	1.563×10^{-5}
	5		0.606509	0.849359	0.785256	0.785714	2.667×10^{-5}
	6		0.650888	0.961538	0.83333	0.57143	1.450×10^{-5}
λ_a	λ_b	λ_c	ρ_a	ρ_b	ρ_c	ρ_d	P
2	3	1	0.384615	0.480769	0.352564	0.428571	1.464×10^{-8}
		2	0.428994	0.528846	0.464744	0.5	1.785×10^{-7}
		3	0.473373	0.576923	0.576923	0.571429	1.251×10^{-6}
		4	0.517751	0.625	0.689103	0.642857	5.556×10^{-6}
		5	0.56213	0.673077	0.801282	0.714286	1.562×10^{-5}
		6	0.606509	0.721154	0.913462	0.785714	2.230×10^{-5}

Table 3: Average waiting time and queue lengths for various mean arrival rates

λ_a	λ_b	λ_c	L_a	L_b	L_c	L_d	L_Q	E_{wt}
1	3	4	0.707071	1.363636	1.785714	1.333333	5.189755	0.576639
2			1.07362	1.666667	2.216495	1.8	6.756781	0.750753
3			1.640625	2.058824	2.804878	2.5	9.004327	1.000481
4			2.634409	2.586207	3.656716	3.666667	12.544	1.393778
5			4.827586	3.333333	5	6	19.16092	2.128991
6			13.69565	4.473684	7.432432	13	38.60177	4.289086
λ_a	λ_b	λ_c	L_a	L_b	L_c	L_d	L_Q	E_{wt}
2	1	4	0.751295	0.668449	1.456693	1	3.876437	0.430715
	2		0.898876	1.052632	1.785714	1.333333	5.070556	0.563395
	3		1.07362	1.666667	2.216495	1.8	6.756781	0.750753
	4		1.283784	2.804878	2.804878	2.5	9.39354	1.043727
	5		1.541353	5.638298	3.656716	3.666667	14.50303	1.611448
	6		1.8644	25	5	6	37.86441	4.207157

λ_a	λ_b	λ_c	L_a	L_b	L_c	L_d	L_Q	E_{wt}
2	3	1	0.625	0.925926	0.544554	0.75	2.84548	0.316164
		2	0.751295	1.122449	0.868263	1	3.742008	0.415779
		3	0.898876	1.363636	1.363636	1.333333	4.959482	0.551054
		4	1.07362	1.666667	2.216495	1.8	6.756781	0.750753
		5	1.283784	2.058824	4.032258	2.5	9.874865	1.097207
		6	1.541353	2.586207	10.55556	3.666667	18.34978	2.038864

Table 4: Variances for various mean arrival rates

λ_a	λ_b	λ_c	V_a	V_b	V_c	V_d	V_{ar}
1	3	4	1.20702	3.22314	4.97449	3.111111	12.51576
2			2.226279	4.444444	7.129344	5.04	18.84007
3			4.332275	6.297578	10.67222	8.75	30.05207
4			9.574517	9.274673	17.02829	17.11111	52.98859
5			28.13317	14.44444	30	42	114.5776
6			201.2665	24.48753	62.67348	182	470.4276
λ_a	λ_b	λ_c	V_a	V_b	V_c	V_d	V_{ar}
2	1	4	1.31574	1.115274	3.578647	2	8.009661
	2		1.706855	2.160665	4.97449	3.111111	11.95312
	3		2.226279	4.444444	7.129344	5.04	18.84007
	4		2.931885	10.67222	10.67222	8.75	33.02632
	5		3.917124	37.4287	17.02829	17.11111	75.48523
	6		5.350419	650	30	42	727.34
λ_a	λ_b	λ_c	V_a	V_b	V_c	V_d	V_{ar}
2	3	1	1.015625	1.783265	0.841094	1.3125	4.952484
		2	1.31574	2.382341	1.622145	2	7.320226
		3	1.706855	3.22314	3.22314	3.111111	11.26425
		4	2.226279	4.444444	7.129344	5.04	18.84007
		5	2.931885	6.297578	20.29136	8.75	38.27083
		6	3.917124	9.274673	121.9753	17.11111	152.2782

VII. CONCLUSION

In the present article a queuing model has been proposed and implemented to find the various characteristics of the queue network. Queue length, variance, Joint probability and average waiting time for customers have been computed using proposed mathematical model. The present model can be implemented in various stochastic and deterministic real time situations to have the accurate prediction of the systems.

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Appendix

Symbol	Notations
Servers	Sr_a, Sr_b, Sr_c, Sr_d
No. of Customers	n_a, n_b, n_c, n_d
Mean arrival rates	$\lambda_a, \lambda_b, \lambda_c$
Mean Service Rates	$\mu_a, \mu_b, \mu_c, \mu_d$
Probabilities	p_{ab}, p_{ac}, p_{ad}
	p_{ba}, p_{bc}, p_{bd}
	p_{ca}, p_{cb}, p_{cd}
Traffic intensity or utilization of servers	$\rho_a, \rho_b, \rho_c, \rho_d$
Queue lengths	L_a, L_b, L_c, L_d, L_Q
Variances	$V_a, V_b, V_c, V_d, V_{ar}$
Average waiting time for customers	E_{wt}